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LOUIS NIRENBERG AND KLAUS SCHMITT: THE JOY OF DIFFERENTIAL EQUATIONS

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Abstract. A tribute to Louis Nirenberg for his 80th birthday anniversary and to Klaus Schmitt for his 65th birthday anniversary.

1. Introduction

The life of a research mathematician is like the one of a gold digger: a constant and often harassing work, from time to time the discovery of a nugget, and for mathematicians like for diggers, the size of nuggets may be somewhat different. But, for mathematicians like for diggers, the joy lies not only in the discovery, but mostly in the quest.

The two mathematicians we celebrate today have devoted their life to this quest, and I know them enough to say that this quest gave them a lot of joy. Their successes, their beautiful results, have been in turn a source of joy and inspiration for many other mathematicians, who have climbed on their shoulders to find their own nuggets. The common denominator of the amazingly diverse production of Louis Nirenberg and of the many elegant results of Klaus Schmitt is differential equations, giving me the title of this afterdinner speech.

2. Louis Nirenberg

A tout seigneur tout honneur, as we say in French, let me start by the inhuman task of describing for you the achievements of Louis Nirenberg, the results of about fifty-five years of cruising in the mathematical sky at the highest altitude.

Louis Nirenberg was born February 28 1925. Can you do anything else than mathematics when your birth place is Hamilton. Louis refused taking any advantage of this, and decided to study physics at McGill. But nobody can escape to his predestination. The young physicist started to work, on the atomic bomb, at the National Research Council in Montréal, and had for colleagues Richard Courant's older son Ernst and his Canadian wife. He asked her to consult her father-in-law about the best place in New York to study theoretical physics. Courant's answer was an interview at New York University, followed by an assistantship in mathematics. This is the reason why Canada does not have today the world leadership

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in nuclear weapons, and why its military threatening of United States only exists in the imagination of Michael Moore.

Then comes a first keyword in Louis's career: **fidelity**. He entered the Mathematics Department of New York University as a graduate student in 1945 and he never left it since for another institution. There should be big celebrations in Courant this year. In addition, more than one quarter of his papers have appeared in the *Communications in Pure and Applied Mathematics*, with a record of four in 1953, his first year of publication. They include his PhD thesis, defended in 1949, with Jim Stoker as adviser and Kurt Friedrichs as model. The thesis contains the complete positive solution of *Weyl's problem in differential geometry* : given a Riemannian metric on the unit sphere, with positive Gauss curvature, can you embed this 2-sphere isometrically into 3-space as a convex surface ? So we must retain **geometry** as another keyword for Nirenberg's contributions, an area that he revisited regularly during his life with results on surfaces with curvature of fixed sign, on the rigidity of closed surfaces, on the intrinsic norms on complex manifolds (with Chern and Levine) and on the characterization of convex bodies.

Many historians claim that the whole career of a scientist can be read in his first publications like in a cristal ball : Louis Nirenberg's proof of Weyl's problem consists in a reduction to a partial differential equation, solved, by a continuity method, through the obtention of new a priori estimates. This equation is elliptic, and **ellipticity** is definitely another keyword of Nirenberg's mathematical work. More than one third of his articles contain the work 'elliptic' just in their title, and a much larger fraction deals with elliptic equations or systems. There is hardly one aspect of those equations he has not considered.

The **regularity** problem for example. Royal paths exist to prove the existence and uniqueness of weak solutions of elliptic problems, but a much more difficult question is to know how regular is a weak solution. Today, no graduate student ignores Nirenberg's *method of differences* for proving interior and boundary regularity, or the Agmon-Douglis-Nirenberg's extension to general elliptic systems! For *Navier-Stokes equation*, essential questions about regularity are still open seventy years after Leray pioneering work (and the nugget worth one million dollars !). Among the best results today is Caffarelli-Kohn-Nirenberg's estimate of the Hausdorf measure of the set of singularities. It is the study, with Joseph Kohn, of *regularity problems in several complex variables* connected with boundary regularity for the $\overline{\partial}$-Neumann problem, which led Kohn and Nirenberg to create and developed the fruitful concept of *pseudo-differential operator*, generalizing and unifying the singular integral and partial differential operators. Their importance in analysis needs not to be emphasized. Incidentally, Friedrichs found the name for the new baby.

It is hard to be a democrat when you are an analyst. Your success depends much more on creating and using **inequalities** than equalities. In this respect, Louis is one of the tycoons of this mathematical wild liberalism. We all know, teach and use the *Gagliardo-Nirenberg*, the *John-Nirenberg* and the *Caffarelli-Kohn-Nirenberg* inequalities, which extend and unify so many fundamental ones. Louis has expressed himself, at several occasions, his love of inequalities :

> Friedrichs was a great lover of inequalities and that affected me very much. The point of view was that the inequalities are more interesting than the equalities.

> I love inequalities. If somebody shows me a new inequality, I say: Oh, that's beautiful, let me think about it, and I may have some ideas connected to it.

Those inequalities play a fundamental role in partial differential equations for the obtention of *a priori estimates* for the possible solutions or through some *operator inequalities*. Descartes is famous, in his philosophy of existence, for his sentence

> Je pense, donc je suis – I am thinking, hence I do exist.

The philosophy of the solutions of partial differential equations could be summarized by something like

> I am limited, hence I do exist.

You can now wonder how such an strong advocate of inequalities was able to build so many tight professional and personal relations with French people. You all know the motto of France:

> Liberté, égalité, fraternité – Liberty, equality, fraternity

His cult of inequalities did not prevent Louis to be elected a foreign member of the French Academy of Science and promoted Doctor *honoris causa* of the University of Paris ! Much more, Louis has won over the heart of Nanette, a charming French lady! I bed that his fights for liberty and his cult of fraternity have won over his unlimited taste for inequality. When you observe that ninety percents of Louis's papers are written in collaboration, you can add fraternity to the keywords describing his personality. Any afterdinner speech being uncomplete without one bad play upon words, Liberty gives me the opportunity to mention the *free boundary value problems* and Louis's fundamental contributions, with David Kinderlehrer, to the regularity of their solutions.

Positivity is another feature of Louis's character : almost each of us has received from him, one day or another, a positive encouragement or a stimulating remark. His forty-five PhD students (several ones are here) would all testify in this direction. In elliptic and parabolic differential equations, positivity is an essential manifestation of the *maximum principle*, and Louis's virtuosity in using this beautiful instrument is unequalled. Do I need to recall the *Gidas-Ni-Nirenberg symmetry theorem* and its variations, in particular in joint work with Henri Berestycki and others. Louis Nirenberg is the Paganini of the maximum principle, and the fact is caught for ever in an AMS movie, a deserved reward to a movies lover. Positivity can also occurs in variational problems, for example in Brezis-Nirenbergs discussion of positive solutions of *nonlinear elliptic equations of the Yamabe type*, which have inspired so many other mathematicians by opening the way to problems with lack of compactness. This is only one of the many contributions of Nirenberg in critical point theory, advocated and developed in inspiring survey papers.

Critical point theory is one of the keys to attack **nonlinearity**. A pioneer in nonlinear elliptic equations (the topics of his first published paper), Louis has returned, at various stages of his career, to *fully nonlinear elliptic equations* to make striking breakthroughs, like the ones in a series of papers with Caffarelli and Spruck on *Monge-Ampère and related equations*. His invited lecture at the International Congress of Mathematicians of 1962 in Stockholm, contains two sentences that I always offer as a guide to my students dealing with nonlinear problems. The first one is:

> Most results for nonlinear problems are still obtained via linear ones, i.e. despite the fact that the problems are nonlinear not because of it.

The second one comments a result of Moser:

> The nonlinear character of the equations is used in an essential way, indeed he obtains results because of the nonlinearity not despite it.

Both aspects have been masterfully explored by Louis Nirenberg, but there is some Dr Jekyll and Mr Hyde aspect we cannot hide: this master of nonlinearity never hesitated to betray the club in making striking contributions to linear problems, forcing us to retain **linearity** as another keyword. Louis probably could argue that some of his fundamental contributions to linear elliptic equations, in particular his *generalized Schauder and Sobolev estimates*, were motivated by solving new nonlinear problems. But how can he justify the necessary and sufficient conditions for *local solvability of general linear partial differential equations* obtained with François Treves ?

Another keyword of Louis's mathematical Wonderland is **holomorphy**. One important question is the study of *almost complex structures* : how to recognize the Cauchy-Riemann operators when given in some arbitrary coordinate system. For higher dimensions, necessary integrability conditions are needed. Newlander and Nirenberg have shown them to be sufficient as well. Another question is the existence of *deformations of complex structures*, families of diffeomorphic complex manifolds, differentiable in the parameter. The answer is given by a theorem of Kodaira-Nirenberg-Spencer. Needless to say that some exciting partial differential equations are hidden in the proof.

Topology must be retained as an other keyword of Louis's mathematical activity. In recent work with Haim Brezis, he created a big scandal among topologists when developing – motivated by some nonlinear models of physics – a *degree theory for mappings which need not be continuous* ! They live instead in the VMO space of functions with *vanishing mean oscillation*, a close relative of the BMO space of functions with *bounded mean oscillation*, invented, for other purposes, by Fritz John and Louis Nirenberg, and widely used in many parts of analysis. The BMO space is an enlargement of the space of essentially bounded measurable functions, and the VMO space a subspace of BMO, the closure of the space of continuous functions. But Louis's links with topology are somewhat older. Already in 1970, he introduced the use of *stable homotopy* in generalizing Landesman-Lazer conditions for bounded nonlinear perturbations of elliptic operators with positive index. In terms of classical degree, instead of questioning the continuity of the map, he was questioning the equal dimensions of the underlying spaces. This paper has had a smaller lineage as most other ones, because, as Louis frankly observed :

> So far, not natural example has come up in which it has been used.

When I am depressed, I like to remember that I gave one, neither natural nor elliptic, just a simple system of two second order ordinary differential equations with three nonlinear boundary conditions. Other important contributions of Brezis-Nirenberg were inspired by Landesman-Lazer's paper.

Incidentally, I am not sure everybody is aware of the role played by Louis in advertising, within the PDE world, Landesman-Lazer's result. In French language, we like to oppose the *savoir faire*, the ability, to the *faire savoir*, the advertising. I think that the marvelous saga of the Landesman-Lazer problem is a wonderful combination of Landesman-Lazer "savoir faire" with Louis Nirenberg's "faire savoir". And this is far from an unique example of the **generosity** (another keyword) and enthusiasm of Louis for presenting, in his marvellous lectures and in survey papers, the results of other mathematicians. This is often an opportunity for a silver nugget to be changed in a golden one. In such a lecture, Louis always starts by apologizing that he is not really an expert in the area he is presenting. Each evening, before sleeping, I ask God to make me a non-expert like Louis Nirenberg. To punish me for this arrogance, God has changed me into an afterdinner speaker, transforming this personal punishment into a collective one.

The world would not turn round if this combination of exceptional personality and outstanding achievements had not been recognized and crowned by distinctions and awards. I mention only a few, to avoid starting another lecture and becoming a tormentor for Louis's simplicity. The Bocher, Crafoord and Steele prizes, the National Medal of Science, fellowship from all important academies in the world and honorary degrees from many prestigious universities. They show the universal appreciation of a man whose human qualities are at the level of his mathematical accomplishments.

3. Klaus Schmitt

If my feelings for Louis Nirenberg are much like the ones of a son for a father (although he by no means needs to recognize any paternity), my feelings for Klaus Schmitt are the ones to a brother, because of the proximity of our ages and the many memories we share. Klaus Schmitt is born on May 14 1940, in Rimbach (Germany), near Heidelberg, and I do not know of any mathematician whose name is Rimbach. Klaus has discovered America when he was still a teenager and, in terms of fidelity, he has followed the example of Louis, just replacing New York by Salt Lake City. Klaus is a member of the Department of Mathematics of the University of Utah since 1967.

Even if he looks quite healthy and is a true sportman, one must describe Klaus' professional life in terms of a impressive series of mathematical diseases. In each case, using the strength of his scientific vocation and the force of his character, Klaus has been able, by observing and analyzing his own case (an interesting fixed point problem) to contribute to a better knowledge of the diseases he was suffering. Let me tell you the hight lights of this fight.

Klaus got the first disease is a city of Nebraska, officially called Lincoln but better known locally as Jackson city. He was trapped into some type of sect or gang, usually called a research group to escape the local police and FBI. This first disease is characterized by an immodest use of **upper and lower solutions (or sub- and super solutions)**, to study nonlinear boundary value problems. In the language of western movies, the method can be described by saying that if you are able to miss your target to the left and to miss it to the right, then you are able to catch it. It is a technique widely used by militars, and I would not be surprised that Klaus got some financial support from them. The target here is the solution of an ordinary, retarded or partial differential equation of elliptic or parabolic type. The symptoms of the illness, as observed on Klaus, can take very different forms, with periods of invariance, crises of convexity, or periodic returns. This is why we owe to Klaus elegant and fruitful extensions of the method to systems, based upon *convexity properties*. As the method of upper and lower solutions is connected to the maximum principle and the maximum principle to positivity, Klaus proved also in this period a few striking results on the existence and multiplicity of *positive solutions*, a topics to which he has remained faithful all his life. He has been very early a distinguished ambassador in the West for *Krasnosel'skii's fixed point theorem in cones*, and proved several extensions.

The first signs of (uncomplete) recovering appeared in the late seventies, unfortunately due to the emergence of another disease, called **bifurcation**. The main symptom there is the apparition of continuous branches at some of your eigenvalues, and the main question about their evolution is to know if they will grow to infinity or connect another eigenvalue (the famous *Rabinowitz dilemma* in the diagnostics). The second case is of course better if you want to continue to travel or if you live in a small country. In the case of Klaus, the apparition of the symptoms was detected even in the absence of *differentiability properties*, but he was able to use the bifurcation branches in the most efficient way to prove the existence and multiplicity of solutions for some nonlinear elliptic partial differential equations which had escaped the upper and lower solutions butchery. A result which remain unsurpassed in this area is a beautiful discussion, with Renate Schaaf, of how the multiplicity of the solutions depends upon the space dimension for *nonlinear periodic perturbations of the Laplacian* with Dirichlet boundary conditions. During some period of fever, Klaus has even discovered some *ghost solutions*, which occur when you discretize a continuous problem without enough care. It was a joint work with Heinz-Otto Peitgen, who never recovered and became the apostle of mathematics seen as a fine art.

A very invasive virus, called the **p-Laplacian**, can affect visitors to Chile, in particular in Santiago, or people who have close scientific contacts with the Manásevich family. This variant of the well known Laplacian virus is characterized by degenerescence properties, which make difficult or impossible the use of the classical variational or topological treatments prescribed for the Laplacian. Again, this illness has left traces both in Klaus' body and in mathematical knowledge, in the form of *variational identities* and *mountain pass solutions*.

About the same period, Klaus also suffered from **variational inequalities**, a disease he probably got from contacts with Vietnamese colleagues or students. Be careful, some of them are here. Those variational inequalities are innocent looking symptoms of severe *obstacle problems*, which can become particularly serious when they come in contact with p-Laplacians. Klaus striking observations during this most recent period of activity has shed much light to this very active domain of research, and his book with Le is on the shelve of all practitioners in this area. One can be especially concerned by the fact that Klaus has taken the risk of combining upper and lower solutions, p-Laplacians and variational inequalities. Such a coalition can create new mutations and could overcome the strongest patients, but I remain convinced that Klaus once more, and the mathematical community, will take benefit of this attack.

Klaus' generosity is well known among the mathematical community, and corroborated by the large number of collaborators and students (many of them are here). Klaus has shared with them, without the slightest restriction, his successive favorite diseases, and enjoyed with proudness and pleasure the apparition and development of new forms of illnesses in those young organisms. Klaus has tried various remedies to escape to other diseases, even some interest in *ecology*. You will not be surprised that he has developed a strong interest for questions of *permanence*.

But I must tell you a few words about another disease which may look less professional than the previous ones, but is far to be unrelated. The **Hirschen disease** is a mysterious illness spread in the German Black Forest, but concentrated in an obscure hamlet called Oberwolfach Walke. Its main symptom is a permanent and intense feeling of thirst, especially in the evening. The only known remedy is to alternate white wine with kirchwasser during a sufficiently long period. Klaus' reason of catching the Hirschen disease is of course the many conferences he has co-organized in Oberwolfach, and the many ones he was invited to. Needless to say that the consequences of the Hirschen disease are specially dramatic if you live in Utah, where Joseph Smiths questionable ideas on alcoholic beverages are still dominant. But, once more and for our delight, Klaus has successfully solved this problem in the form of a striking mathematical result, already called in literature the *Schmitt-Smith theorem*, whose statement is as deep as beautiful:

A dry state needs not to be locally dry.

Many proofs by pictures exist for this theorem, and experimental proofs have been given during gorgeous hikes in the mountains or friendly parties in Klaus and Claudia's house.

I first met Klaus, in Salt Lake City, in 1973 and cherish a friendship and collaboration of more than thirty years. It started through some reprints request in the early seventies, when Klaus was running in Utah a pioneering seminar on the use of degree in nonlinear ordinary differential equations, a most unusual topics in those years among ode people. Klaus had been pleased to discover that some young Belgian mathematician had published a few papers on similar topics. Having always be responsive to some forms of beauty in mathematics, Klaus was anxious to meet this Belgian whose first name was Jean. He enthusiastically arranged a visit to Salt Lake City at the occasion of Jean's first trip to the US. Unverified rumors claim that a romantic dinner with candles had even been planned. Klaus had enough self-control to hide his surprise and disappointment when Jean came out of the plane with his wife and children, but he never reproached Jean to be Jean. On the contrary, at each of Jean's many visits to Salt Lake, Klaus initiated him, besides beautiful new mathematics, to new areas, like cross-country ski, mountain biking or cow-boy bars. Each time Jean was so excited that, back home, he immediately bought a pair of skis (forgetting that Belgium has little or no snow), then a mountain-bike (forgetting that Belgium has no mountains), then a pair of guns (forgetting that Belgium has no Wild West; only Flemishs in the North and Walloons in the South).

Klaus of course was always happy to visit Germany and, as a Humboldt Prize Winner, has spent several sabbaticals in his native country. He has been a visiting professor in most important German universities, and one can say that he is a legend there. He visited many other places in the world, very often with his charming wife Claudia, who succeeded like no other one in distracting Klaus from his mathematical diseases. She is a wonderful host if you happen to visit Utah and her beautiful smile shines like the sun on the Rocky Mountains.

Conclusion

Louis Nirenberg said, in a recent interview:

> One of the wonders of mathematics is you go somewhere in the world and you meet other mathematicians, and it's like one big family. This large family is a wonderful joy.

The joy of such a meeting was offered to all of us in Mississipi State, at the occasion of this double celebration. We unanimously ask Louis and Klaus to give us many further opportunities of sharing this wonderful joy with them.

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