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FIRST INTEGRALS FOR THE DUFFING-VAN DER POL TYPE OSCILLATOR

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Abstract. In this article, under certain parametric conditions, we study the first integrals of the Duffing-van der Pol-type oscillator equations which include the van der Pol and the Duffing oscillator systems, as particular cases. After making a series of variable transformations and applying the Preller-Singer method, we find the first integrals of the simplified equations without complicated calculations. Through the inverse transformations we obtain the first integrals of the original equations. Some statements in the literature are indicated and clarified.

1. Introduction

Many nonlinear differential equations arise in physical, chemical and biological contexts. Finding innovative methods to solve and analyze these equations has been an interesting subject in the field of differential equations and dynamical systems [10, 13]. In these problems, it is not always possible and sometimes not even advantageous to express exact solutions of nonlinear differential equations explicitly in terms of elementary functions, but it is possible to find elementary functions that are constant on solution curves, that is, elementary first integrals. These first integrals allow us to occasionally deduce properties that an explicit solution would not necessarily reveal. In the pioneering work [12], Prelle and Singer introduced a procedure to find the first integrals of first-order ordinary differential equations (ODEs) of the form $y = P(x, y)/Q(x, y)$, with both $P(x, y)$ and $Q(x, y)$ polynomials whose coefficients lie in the field of complex numbers $\mathbb{C}$. Duarte et al. [5] extended this procedure to second-order ODEs which is based on a conjecture that if the given second-order ODE has an elementary solution, then there exists at least one elementary first integral $I(x, y, y')$ whose derivatives are all rational functions of x, y and y'. Recently, much attention has been received to various nonlinear oscillator systems for finding the first integrals and obtaining exact solutions [3, 4, 10, 15]. Special types of first integrals and exact solutions are of fundamental importance

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to our understanding of physical, chemical and biological phenomena modelled differential equations.

In this article, we consider a more general nonlinear oscillator system of the form

$$\ddot{u} + (\delta + \beta u^m)\dot{u} - \mu u + \alpha u^n = 0. \tag{1.1}$$

where an over-dot represents differentiation with respect to the independent variable ξ, and all coefficients δ, β, μ and α are real. It is referred as to the Duffing–van der Pol–type oscillator, since the choices $\alpha = 0$ and $m = 2$ lead equation (1.1) to the van der Pol oscillator

$$\ddot{u} + (\delta + \beta u^2)\dot{u} - \mu u = 0, \tag{1.2}$$

which was originally discovered by the Dutch electrical engineer van der Pol in electrical circuits [16, 17]. The choices $\beta = 0$ and $n = 3$ lead equation (1.1) to the damped Duffing equation [6, 8]

$$\ddot{u} + \delta\dot{u} - \mu u + \alpha u^3 = 0. \tag{1.3}$$

When $\beta = 0$ and $n = 2$, equation (1.1) becomes the damped Helmholtz oscillator [1, 14]

$$\ddot{u} + \delta\dot{u} - \mu u + \alpha u^2 = 0. \tag{1.4}$$

It is well known that there are a great number of theoretical works to deal with equations (1.2)–(1.4) [8, 10], and applications of these three equations and the related equations can be seen in quite a few scientific areas [7].

In the present paper, we wish to show that under certain parametric conditions some first integrals of oscillator system (1.1) are established. The paper is organized as follows. In the next section, in order to make this paper well self-contained, we summarize the Prelle–Singer procedure developed by Duarte et al. [5] for constructing the first integrals of second-order ODEs. In Section 3, we will show that after a series of nonlinear transformations, we simplify equation (1.1), then by means of the Preller–Singer method we derive the first integral of the simplified equation without complicated calculations. Through the inverse transformations we obtain the first integral of the original oscillator equation. Some statements in the literature are indicated and clarified, and some exact solutions of equation (1.1) in the parametric forms are obtained accordingly. Section 4 is a brief conclusion.

2. Prelle–Singer Method for Solving Second-Order ODEs

In this section, in order to present our results in a straightforward way, we start our attention by briefly reviewing the Prelle–Singer procedure for solving second-order ODEs developed by Duarte et al. [5] and Chandrasekar et al. [3].

Consider the second-order ODE of the rational form

$$\frac{d^2y}{dx^2} = \phi(x, y, y') = \frac{P(x, y, y')}{Q(x, y, y')}, \quad P, Q \in \mathbb{C}[x, y, y']. \tag{2.1}$$

where y' denotes differentiation with respect to x, P and Q are polynomials in x, y and y' with coefficients in the complex field. Suppose that equation (2.1) admits a first integral $I(x, y, y') = C$, with C constant on the solutions, so we have the total differential

$$dI = I_x\, dx + I_y\, dy + I_{y'}\, dy' = 0, \tag{2.2}$$

where the subscript denotes partial differentiation with respect to the corresponding variable. On the solution, since $y'\,dx = dy$ and equation (2.1) is equivalent to $\frac{P}{Q}\,dx = dy'$, adding a null term $S(x,y,y')y'\,dx - S(x,y,y')\,dy$ to both sides yields

$$\Big(\frac{P}{Q} + Sy'\Big)\,dx - S\,dy - dy' = 0. \tag{2.3}$$

From (2.2) and (2.3), one can see that on the solutions, the corresponding coefficients of (2.2) and (2.3) should be proportional. There exists a proper integrating factor $R(x,y,y')$ for expression (2.3), such that on the solutions

$$dI = R(\phi + Sy')\,dx - SR\,dy - R\,dy' = 0. \tag{2.4}$$

Comparing the corresponding terms in (2.2) and (2.4), we have

$$\begin{aligned} I_x &= R(\phi + Sy'), \\ I_y &= -SR, \\ I_{y'} &= -R, \end{aligned} \tag{2.5}$$

and the compatibility conditions $I_{xy} = I_{yx}$, $I_{xy'} = I_{y'x}$ and $I_{yy'} = I_{y'y}$. Using these three compatibility conditions respectively, we obtain three equivalent equations as follows:

$$\begin{aligned} D[S] &= -\phi_y + S\phi_{y'} + S^2, \\ D[R] &= -R(S + \phi_{y'}), \\ R_y &= R_{y'}S + S_{y'}R, \end{aligned} \tag{2.6}$$

where D is an differential operator

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial y'}.$$

For the given expression of ϕ, one can solve the first equation of (2.6) for S. Substituting S into the second equation of (2.6) one can get an explicit form for R by solving it. Once a compatible solution R and S satisfying the extra constraint (the third equation of (2.6)) is derived, integrating (2.5), from (2.2) one may obtain a first integral of motion as follows

$$\begin{aligned} &I(x,y,y') \\ &= \int R(\phi + Sy')\,dx - \int \Big[RS + \frac{\partial}{\partial y}\int R(\phi + Sy')\,dx\Big]\,dy \\ &\quad - \int \Big\{R + \frac{\partial}{\partial y'}\Big(\int R(\phi + Sy')\,dx - \int \Big[RS + \frac{\partial}{\partial y}\int R(\phi + Sy')\,dx\Big]\,dy\Big)\Big\}\,dy'. \end{aligned} \tag{2.7}$$

3. First Integrals of Nonlinear Oscillator Systems

3.1. **Nonlinear Transformations.** In this subsection, in order to avoid doing complicated computations, we will make a series of nonlinear transformations to equation (1.1). For our convenience, we assume $\alpha = 1$ in equation (1.1) (this can be easily obtained by re-scaling parameters of equation (1.1)). Namely, we consider the oscillator equation:

$$\ddot{u} + (\delta + \beta u^m)\dot{u} - \mu u + u^n = 0. \tag{3.1}$$

Firstly, we make the natural logarithm transformation:

$$\xi = -\frac{1}{\delta}\ln\tau; \tag{3.2}$$

that is,

$$\frac{\partial\tau}{\partial\xi} = -\delta e^{-\xi\delta} = -\delta\tau.$$

After substituting the following two derivatives into (3.1):

$$\frac{\partial u}{\partial\xi} = \frac{\partial u}{\partial\tau} * \frac{\partial\tau}{\partial\xi} = -\delta\tau\frac{\partial u}{\partial\tau},$$
$$\frac{\partial^2 u}{\partial\xi^2} = \delta^2\tau\frac{\partial u}{\partial\tau} + \delta^2\tau^2\frac{\partial^2 u}{\partial\tau^2},$$

then it becomes

$$\delta^2\tau^2\frac{\partial^2 u}{\partial\tau^2} - \beta\delta\tau u^m\frac{\partial u}{\partial\tau} - \mu u + u^n = 0. \tag{3.3}$$

Further, we take the variable transformation as:

$$q = \tau^\kappa, \quad u = \tau^{-\frac{1}{2}(\kappa-1)}H(q), \tag{3.4}$$

A direction calculation gives

$$\frac{\partial u}{\partial\tau} = -\frac{1}{2}(\kappa-1)q^{-\frac{\kappa+1}{2\kappa}}H(q) + \kappa q^{\frac{\kappa-1}{2\kappa}}\frac{\partial H}{\partial q},$$
$$\frac{\partial^2 u}{\partial\tau^2} = \frac{1}{4}(\kappa^2-1)q^{-\frac{\kappa+3}{2\kappa}}H(q) + \kappa^2 q^{\frac{3(\kappa-1)}{2\kappa}}\frac{\partial^2 H}{\partial q^2}.$$

After substituting the above equalities into (3.3), we obtain

$$\frac{\partial^2 H}{\partial q^2} = \frac{\beta}{\delta\kappa}q^{\frac{m-\kappa(m+2)}{2\kappa}}H^m\frac{\partial H}{\partial q} - \frac{1}{\delta^2\kappa^2}q^{\frac{-(3+n)\kappa+n-1}{2\kappa}}H^n - \frac{1}{2}\frac{(\kappa-1)\beta}{\delta\kappa^2}H^{m+1}q^{\frac{m-\kappa(m+4)}{2\kappa}}, \tag{3.5}$$

where an over-dot represents differentiation with respect to the independent variable q, and

$$\kappa^2 = \frac{4\mu}{\delta^2} + 1. \tag{3.6}$$

3.2. **Force-Free Duffing-van der Pol Oscillator.** We know that the choices $m = 2$ and $n = 3$ lead equation (1.1) to the standard form of the Duffing-van der Pol oscillator equation, whose autonomous version (force-free) is:

$$\ddot{u} + (\delta + \beta u^2)\dot{u} - \mu u + u^3 = 0. \tag{3.7}$$

Equation (3.7) arises in a model describing the propagation of voltage pulses along a neuronal axon and has recently received much attention from many authors. A vast amount of literature exists on this equation; for details and applications, see [9, 11] and references therein.

From (3.5), one can see that if we take $n = 3$ and $m = 2$, then equation (3.5) can be reduced to a simple form

$$\frac{\partial^2 H}{\partial q^2} = Aq^p H^2\frac{\partial H}{\partial q} + Bq^{p-1}H^3, \tag{3.8}$$

where

$$p = \frac{1}{\kappa} - 2, \quad A = \frac{\beta}{\delta\kappa},$$
$$B = -\frac{1}{\delta^2\kappa^2} - \frac{(\kappa-1)\beta}{2\delta\kappa^2}.$$

Choosing $\phi(q, H, H') = Aq^pH^2\frac{\partial H}{\partial q} + Bq^{p-1}H^3$ and following the procedure in Section 2, we obtain three determining equations:

$$S_q + \dot{H}S_H + \phi S_{\dot{H}} = -2Aq^pH\dot{H} + (ASq^p - 3Bq^{p-1})H^2 + S^2, \tag{3.9}$$

$$R_q + R_H\dot{H} + \phi R_{\dot{H}} = -RS - RAq^pH^2, \tag{3.10}$$

$$R_H = R_{\dot{H}}S + S_{\dot{H}}R. \tag{3.11}$$

In general, it is not easy to solve system (2.6) and get exact solutions (S, R) in the explicit forms. But in our case of (3.9)-(3.11) we may seek an ansatz for S and R of the forms as suggested in [5]:

$$S = \frac{a(q,H) + b(q,H)\dot{H}}{c(q,H) + d(q,H)\dot{H}}, \quad R = e(q,H) + f(q,H)\dot{H}, \tag{3.12}$$

where a, b, c, d, e, f are functions of q, H to be determined. Substituting S into equation (3.9), we get the equation system

$$\begin{aligned}
&[\dot{H}]^0 : -3Bc^2H^2q^{p-1} + Aacq^pH^2 + a^2 = a_qc - ac_q + bcBH^3q^{p-1} - adBH^3q^{p-1},\\
&[\dot{H}]^1 : -2Ac^2q^pH - 6BcdH^2q^{p-1} + 2Aadq^pH^2 + 2ab\\
&\qquad = a_qd + b_qc - ad_q - bc_q + a_Hc - ac_H,\\
&[\dot{H}]^2 : -4Acdq^pH - 3Bd^2H^2q^{p-1} + Abdq^pH^2 + b^2\\
&\qquad = b_qd - bd_q + a_Hd + b_Hc - ad_H - bc_H,\\
&[\dot{H}]^3 : -2Ad^2q^pH = b_Hd - bd_H.
\end{aligned}$$

Substituting S and R into equation (3.10), we obtain another equation system:

$$\begin{gathered}
[\dot{H}]^0 : e_qc + BcfH^3q^{p-1} = -ae - Aceq^pH^2,\\
[\dot{H}]^1 : f_qc + e_Hc + 2Afcq^pH^2 + e_qd + BfdH^3q^{p-1} = -be - Adeq^pH^2 - af,\\
[\dot{H}]^2 : f_Hc + f_qd + e_Hd + 2Afdq^pH^2 = -bf,\\
[\dot{H}]^3 : f_Hd = 0.
\end{gathered}$$

Under the parametric condition

$$\delta = \frac{3}{\beta} - \frac{\mu\beta}{3}, \tag{3.13}$$

we solve the above two nonlinear systems for a nontrivial solution with the aid of Maple, and the corresponding forms of S and R read

$$S = -\frac{1}{q} - \frac{\beta}{\delta\kappa}q^{\frac{1-2\kappa}{\kappa}}H^2, \quad R = e^{\ln q}, \tag{3.14}$$

which also satisfy equation (3.11).

Substituting the solution set (3.14) into (2.7), we can obtain the first integral of equation (3.8) immediately:

$$\kappa\delta H - \kappa\delta q\dot{H} + \frac{2}{\delta(1-\kappa)}q^{\frac{(1-\kappa)}{\kappa}}H^3 = I. \tag{3.15}$$

Using the inverse transformations (3.2) and (3.4), and changing to the original variables, we obtain that under the parametric condition (3.13), the Duffing-van der Pol equation (3.7) has the first integral of the form

$$\left[\dot{u} + \left(\delta - \frac{3}{\beta}\right) u + \frac{\beta}{3} u^3\right] e^{\frac{3\xi}{\beta}} = I_1. \tag{3.16}$$

It is remarkable that in [3, p. 2467], [4, p.4528], and [15, p. 1936], the authors studied the first integral of the oscillator equation (3.7) by the Lie symmetry method and claimed that the nontrivial first integral exists only for the parametric choice

$$\delta = \frac{4}{\beta}, \quad \mu = -\frac{3}{\beta^2}. \tag{3.17}$$

However, in view of our condition (3.13) and formula (3.16), it shows that our parametric constraint (3.13) is weaker than the corresponding ones described in the literature [3, 4, 15], and the first integral presented in [3, 4, 15] is just a particular case of (3.16).

3.3. **Duffing–van der Pol–Type Oscillator.** In this subsection, we extend the technique used in the preceding subsection to a more general oscillator equation in the case of $n = m + 1$; that is,

$$\ddot{u} + (\delta + \beta u^m)\dot{u} - \mu u + u^{m+1} = 0, \tag{3.18}$$

where an over-dot still denotes differentiation with respect to ξ. Note that the choice $n = m + 1$ leads equation (3.5) to a simple form

$$\frac{\partial^2 H}{\partial q^2} = \frac{\beta}{\delta\kappa} q^p H^m \frac{\partial H}{\partial q} + \left(-\frac{1}{\delta^2\kappa^2} - \frac{(\kappa-1)\beta}{2\delta\kappa^2}\right) H^{m+1} q^{p-1}, \tag{3.19}$$

where

$$p = \frac{m - \kappa(m+2)}{2\kappa}.$$

For the notational convenience, we denote that

$$A = \frac{\beta}{\delta\kappa}, \quad B = -\frac{1}{\delta^2\kappa^2} - \frac{(\kappa-1)\beta}{2\delta\kappa^2},$$

then equation (3.19) becomes

$$\ddot{H} = Aq^p H^m \dot{H} + BH^{m+1} q^{p-1}. \tag{3.20}$$

Choosing $\phi(q, H, H') = Aq^p H^m \frac{\partial H}{\partial q} + Bq^{p-1} H^{m+1}$ and following the procedure in Section 2, we obtain three determining equations:

$$S_q + \dot{H} S_H + \phi S_{\dot{H}} = -mAq^p H^{m-1}\dot{H} + (ASq^p - (m+1)Bq^{p-1})H^m + S^2, \tag{3.21}$$

$$R_q + R_H \dot{H} + \phi R_{\dot{H}} = -RS - RAq^p H^m, \tag{3.22}$$

$$R_H = R_{\dot{H}} S + S_{\dot{H}} R. \tag{3.23}$$

Here we use the same ansatz for S and R as given in (3.12). Substituting S into (3.21), we get the system

$$
\begin{aligned}
[\dot{H}]^0 &: -(m+1)Bc^2H^mq^{p-1} + Aacq^pH^m + a^2 \\
&\quad = a_qc - ac_q + bcBH^{m+1}q^{p-1} - adBH^{m+1}q^{p-1}, \\
[\dot{H}]^1 &: -mAc^2q^pH^{m-1} - 2(m+1)BcdH^mq^{p-1} + Aadq^pH^m + Aq^pH^mbc + 2ab \\
&\quad = a_qd + b_qc - ad_q - bc_q + a_Hc - ac_H + bcAq^pH^m - adAq^pH^m, \\
[\dot{H}]^2 &: -2mAcdq^pH^{m-1} - (m+1)Bd^2H^mq^{p-1} + Abdq^pH^m + b^2 \\
&\quad = b_qd - bd_q + a_Hd + b_Hc - ad_H - bc_H, \\
[\dot{H}]^3 &: -mAd^2q^pH^{m-1} = b_Hd - bd_H.
\end{aligned} \tag{3.24}
$$

Substituting S and R into (3.22), we obtain another system,

$$
\begin{gathered}
[\dot{H}]^0 : e_qc + BcfH^{m+1}q^{p-1} = -ae - Aceq^pH^m, \\
[\dot{H}]^1 : f_qc + e_Hc + 2Afcq^pH^m + e_qd + BfdH^{m+1}q^{p-1} = -be - Adeq^pH^m - af, \\
[\dot{H}]^2 : f_Hc + f_qd + e_Hd + 2Afdq^pH^m = -bf, \\
[\dot{H}]^3 : f_Hd = 0.
\end{gathered} \tag{3.25}
$$

We solve the nonlinear systems (3.24) and (3.25), for a nontrivial solution, with the aid of Maple and find that under the parametric conditions

$$
m = \frac{(1-\kappa)\beta\delta}{2} - 1, \quad \kappa^2 = \frac{4\mu}{\delta^2} + 1, \tag{3.26}
$$

the three determining equations (3.21)–(3.23) have the solution of the form

$$
S = -\frac{1}{q} - \frac{\beta}{\delta\kappa}q^{\frac{m(1-\kappa)}{2\kappa}-1}H^m, \quad R = e^{\ln q}. \tag{3.27}
$$

After substitution of the solution set (3.27) into (2.7), we derive the first integral of (3.20) as follows

$$
\kappa\delta H - \kappa\delta q\dot{H} + \frac{2}{\delta(1-\kappa)}q^{\frac{m(1-\kappa)}{2\kappa}}H^{m+1} = I,
$$

where I is an arbitrary integration constant. By virtue of the inverse transformations (3.2) and (3.4), and changing to the original variables, we obtain that under the parametric condition (3.26), the Duffing-van der Pol–type equation (3.18) has the first integral of the form

$$
\Big[\dot{u} + \frac{\delta(\kappa+1)}{2}u + \frac{2}{\delta(1-\kappa)}u^{m+1}\Big]e^{\frac{1}{2}\delta(1-\kappa)\xi} = I_2. \tag{3.28}
$$

It is remarkable that the first integral of the Duffing-van der Pol oscillator equation (3.7) obtained in Section 3.2 is just a particular case of formula (3.28). In the recently published Handbooks of ODEs such as [2, 13, 18], there are quite a few first integrals (conservation laws) collected for ordinary differential equations of the type $y'' = c_1x^{l_1}y^{m_1}(y')^{k_1} + c_1x^{l_2}y^{m_2}(y')^{k_2}$, but our formulas of first integrals of equation (3.18) or (3.19) described herein are not presented there.

3.4. **Solutions in the Parametric Forms.** In this subsection, by virtue of the first integral (3.28), we may choose a proper value for I_2 and consider three particular cases where exact solutions of the oscillator equation (3.18) can be expressed in the parametric forms.

Case 1: Assume that $m \neq -1$ and $\kappa \neq -1$, and

$$\begin{gathered} m = -\frac{2\kappa}{\kappa+1}, \\ \frac{\beta}{\delta\kappa} = \frac{1}{\delta^2\kappa^2} + \frac{(\kappa-1)\beta}{2\delta\kappa^2}, \end{gathered} \tag{3.29}$$

where

$$\kappa^2 = \frac{4\mu}{\delta^2} + 1.$$

In this case, (3.19) takes the form

$$\ddot{H} = Aq^{-m-2}H^m\dot{H} - AH^{m+1}q^{-m-3}. \tag{3.30}$$

From the first integral (3.28), taking $I_2 = 0$, we know that the solution of equation (3.30) can be expressed in the parametric form [13]:

$$\begin{gathered} q = aC_1^m\Big(\int \frac{dt}{1 \pm t^{m+1}} + C_2\Big)^{-1}, \\ H = bC_1^{m+1}t\Big(\int \frac{dt}{1 \pm t^{m+1}} + C_2\Big)^{-1}, \end{gathered} \tag{3.31}$$

where C_1 and C_2 are arbitrary constants, a and b are also arbitrary but satisfy

$$\frac{\beta}{\delta\kappa} = \mp(m+1)a^{m+1}b^{-m}. \tag{3.32}$$

Applying the inverse transformation of (3.4) to formula (3.31), namely

$$\tau = q^{\frac{1}{\kappa}}, \quad H = u\tau^{\frac{1}{2}(\kappa-1)},$$

we have

$$\begin{gathered} \tau = a^{\frac{1}{\kappa}}C_1^{m/\kappa}\Big(\int \frac{dt}{1 \pm t^{m+1}} + C_2\Big)^{-1/\kappa}, \\ u = \tau^{-(\kappa-1)/2}bC_1^{m+1}t\Big(\int \frac{dt}{1 \pm t^{m+1}} + C_2\Big)^{-1}. \end{gathered} \tag{3.33}$$

Further, applying the inverse transformation of (3.2) to formula (3.33), under the given parametric condition (3.29), we obtain the solution for equation (3.18) in the parametric form as follows:

$$\begin{gathered} \xi = \frac{-\ln\Big(a^{\frac{1}{\kappa}}C_1^{m/\kappa}\Big(\int \frac{dt}{1 \pm t^{m+1}} + C_2\Big)^{-1/\kappa}\Big)}{\delta}, \\ u = e^{\delta(\kappa-1)/2\xi}bC_1^{m+1}t\Big(\int \frac{dt}{1 \pm t^{m+1}} + C_2\Big)^{-1}, \end{gathered} \tag{3.34}$$

where a and b are arbitrary constants, and satisfy condition (3.32).

Case 2: Assume that

$$m = -2, \quad \kappa = -2, \quad \beta\delta = -2. \tag{3.35}$$

So (3.19) takes the form

$$\ddot{H} = Aq^{1/2}H^{-2}\dot{H} - AH^{-1}q^{-1/2}, \tag{3.36}$$

where $A = -\beta/(2\delta)$.

Using the first integral (3.28) again, we know that the solution of equation (3.36) can be expressed in the parametric form:

$$\begin{gathered} q = aC_1^4 F^{-2}, \\ H = bC_1^3 t^{-1} E F^{-2}, \end{gathered} \tag{3.37}$$

where a and b are also arbitrary but satisfy

$$\frac{\beta}{2\delta} = a^{-3/2} b^2, \tag{3.38}$$

and

$$E = \sqrt{t(t+1)} - \ln(\sqrt{t} + \sqrt{t+1}) + C_2, \quad F = E\sqrt{\frac{t+1}{t}} - t. \tag{3.39}$$

Applying the inverse transformation of (3.4) to formula (3.37), namely

$$\tau = q^{-1/2}, \quad H = u\tau^{-3/2},$$

we have

$$\begin{gathered} \tau = a^{-1/2} C_1^{-2} F, \\ u = \tau^{\frac{3}{2}} b C_1^3 t^{-1} E F^{-2}. \end{gathered} \tag{3.40}$$

Further, applying the inverse transformation of (3.2) to formula (3.40), under the given parametric condition (3.35), we obtain the solution for equation (3.18) in the parametric form as follows:

$$\begin{gathered} \xi = \frac{\ln\left(aC_1^4 F^{-2}\right)}{2\delta}, \\ u = e^{-\frac{3}{2}\delta\xi} b C_1^3 t^{-1} E F^{-2}, \end{gathered}$$

where a and b are arbitrary constants, and satisfy condition (3.38).

Case 3: Assume that

$$m = -3, \quad \kappa = -3, \quad \beta\delta = -1. \tag{3.41}$$

In this case, (3.19) takes the form

$$\ddot{H} = AqH^{-3}\dot{H} - AH^{-2}, \tag{3.42}$$

where $A = -\frac{\beta}{3\delta}$.

We know that the solution of equation (3.42) can be expressed in the parametric form

$$\begin{gathered} q = aC_1^3 F^{-1}\sqrt{\frac{t+1}{t}}, \\ H = bC_1^2 F^{-1}, \end{gathered} \tag{3.43}$$

where F is the same as that in (3.39), C_1 and C_2 are arbitrary constants, a and b are also arbitrary but satisfy

$$\frac{\beta}{3\delta} = 2a^{-2} b^3. \tag{3.44}$$

Applying the inverse transformation of (3.4) to formula (3.43), namely

$$\tau = q^{-1/3}, \quad H = u\tau^{-2},$$

we have

$$\begin{gathered} \tau = a^{-1/3} C_1^{-1} F^{1/3} \big(\frac{t+1}{t}\big)^{-1/6}, \\ u = \tau^2 b C_1^2 F^{-1}. \end{gathered} \tag{3.45}$$

Further, applying the inverse transformation of (3.2) to formula (3.45), under the given parametric condition (3.41), we obtain the solution for equation (3.18) in the parametric form as follows:

$$\xi = \frac{\ln\left(aC_1^3F^{-1}\sqrt{\frac{t+1}{t}}\right)}{3\delta},$$
$$u = e^{-2\delta\xi}bC_1^2F^{-1},$$

where a and b are arbitrary constants, and satisfy condition (3.44).

4. Conclusion

Finding first integrals (conservation laws) and exact solutions for various nonlinear differential equations has been an interesting subject in mathematical and physical communities. Since 1983, Prelle and Singer presented a deductive method for solving first–order ODEs that presents a solution in terms of elementary functions if such a solution exists. This technique has attracted many researchers from diverse groups and has been extended to autonomous systems of ODEs of higher dimensions for finding the first integrals and exact solutions under certain assumptions. From illustrative examples in these works, the obtained first integrals of autonomous systems are usually of rational or quasi-rational forms and searching for solution sets (S, R) usually involves complicated calculations. However, the generalization of this procedure to autonomous/nonautonomous systems of higher dimensions to find elementary first integrals in an effective manner is still an interesting and important subject.

In this paper, we showed that under certain parametric conditions, some new first integrals of the Duffing–van der Pol–type oscillator equation (1.1) could be established. To reach our goal, we first made a series of nonlinear transformations to simplify equation (1.1) to a simple form, then by means of the Preller–Singer method we derived the first integral of the resultant equation. Through the inverse transformations we obtain the first integrals of the original oscillator equations. Finally, using the established first integral, we obtain exact solutions of equation (1.1) in the parametric forms. Some statements in the literature are corrected and clarified.

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