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GLOBAL STABILITY, PERIODIC SOLUTIONS, AND OPTIMAL CONTROL IN A NONLINEAR DIFFERENTIAL DELAY MODEL

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Abstract. A nonlinear differential equation with delay serving as a mathematical model of several applied problems is considered. Sufficient conditions for the global asymptotic stability and for the existence of periodic solutions are given. Two particular applications are treated in detail. The first one is a blood cell production model by Mackey, for which new periodicity criteria are derived. The second application is a modified economic model with delay due to Ramsey. An optimization problem for a maximal consumption is stated and solved for the latter.

1. Introduction

This article has two principal components. The first one is a theoretical part dealing with the global asymptotic stability and the existence of periodic solutions in a class of essentially nonlinear differential equations with delay. The second part concerns two particular applications where those equations appear as mathematical models of several real life phenomena.

The class of equations can be represented in the form

$$x'(t) = F(x(t-\tau)) - G(x(t)) \tag{1.1}$$

where F and G are continuous real-valued functions. In section 2 we first introduce and discuss necessary preliminaries and definitions related to the equation. We then state several results on the global asymptotic stability and the existence of periodic solutions. One of the new elements in our considerations is that both functions F and G are generally assumed to be nonlinear. In most of the available literature on equation (1.1) function G is linear of the form $G(x) = bx$, $b > 0$. However, many recent applications involve cases where function G is essentially nonlinear. One of such applications is a blood cell production model due to Mackey [12, 13].

Section 3 deals with two instances of application for equation (1.1). The first one is the above mentioned physiological model by M.C. Mackey, considered in subsection 3.1. We present explicit sufficient conditions for the global asymptotic stability and for the existence of periodic solutions in this equation, in terms of the

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parameters defining the nonlinearities F and G. Our results for the Mackey model are new and complementary to those recently obtained in [13].

Subsection 3.2 is devoted to a generalized economic model in the form of equation (1.1) and to its partial case in the form of a modified Ramsey equation with delay. The Ramsey model was originally introduced in paper [17], initially as a system of ordinary differential equations. A modified version in the form of differential delay equation (1.1) was proposed in [8] where natural delay effects due to production/investment cycles are taken into account. In subsection 3.2 we consider an optimal control problem for the generalized economic model subject to specific control functions, which involves maximizing a consumption functional. As a consequence, we present a complete solution of the problem for the Ramsey model.

2. Preliminaries and Mathematical Results

We assume that for every initial function $\phi \in \mathcal{C} := C([-\tau, 0], \mathbb{R}^+)$, $\mathbb{R}^+ := \{x : x > 0\}$, there exists a unique solution $x = x(t, \phi)$ of equation (1.1) defined for all $t \geq 0$. We do not address in detail this question of global existence of solutions of equation (1.1). We only note that the results are well-known and readily available in the literature (see e.g. [2, 5] and further references therein). One of such conditions of global existence can be the assumption that G is uniformly Lipschitz continuous, that is
$$|G(x) - G(y)| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}^+,$$
for some constant L.

In this section we present some basic properties and principal mathematical results on differential delay equation (1.1) that are needed in the sequel. They are used to analyze the two applied models considered in section 3. Some of the stated results can be derived from analogous results for equation (1.1) with $G(x) = x$, which were proved in [7]. Other results require certain new considerations and developments which are somewhat outside the scope of this paper. Detailed proofs of all statements in this section are rather long; some of them are given in the forthcoming paper [6].

The following hypotheses on the nonlinearities F and G will be assumed in different combinations throughout the paper.

- (H1) F and G are defined and continuous on the positive semiaxis $\mathbb{R}^+$, $F, G \in C(\mathbb{R}^+, \mathbb{R}^+)$, and $G(0) = 0$, $F(0) \geq 0$.
- (H2) F and G satisfy (H1) and there exists $M_0 \geq 0$ such that $G(x) > F(x)$ and $G(x)$ is increasing in $[M_0, \infty)$. In addition, either (i) $\lim_{x\to\infty} G(x) = +\infty$ or (ii) $\lim_{x\to\infty} G(x) = G_\infty < \infty$ and $\sup\{F(x), x \in (0, M_0)\} < G_\infty$.
- (H3) F and G satisfy (H1) and there exists a unique value $x = x_* > 0$ such that $F(x_*) = G(x_*)$. In addition, $F(x) > G(x)$ for $x \in (0, x_*)$ and $F(x) < G(x)$ for $x \in (x_*, \infty)$.

Hypothesis (H1) is a standard assumption of general type which will be assumed to hold throughout the remainder of the paper. Its importance is seen from the following basic property of solutions of equation (1.1).

Propostion 2.1 (Positive invariance)**.** *Assume* (H1) *and let $\phi \in \mathcal{C}$ be arbitrary. Then the corresponding solution $x = x(t, \phi)$ of equation* (1.1) *satisfies $x(t) \geq 0$ for all $t \geq 0$* ($\forall\, \tau > 0$).

The importance of assumption (H2) is seen from the following statement.

Propostion 2.2 (Boundedness). *Assume* (H2) *to hold and let* $\phi \in \mathcal{C}$ *be arbitrary. Then there exists a positive constant* K *such that the corresponding solution* $x = x(t, \phi)$ *of equation* (1.1) *satisfies*

$$\limsup_{t\to\infty} x(t) \le K.$$

The following statement is useful when $F(0) > 0$ or when $F(0) = 0$ and the steady state $x(t) \equiv 0$ is unstable. We note that the assumption $F(0) = 0$ and $F(x) > G(x)$ for all $x \in (0, \delta_0)$ implies the instability of the trivial solution $x(t) \equiv 0$; while $F(x) < G(x)$ for all $x \in (0, \delta_0)$ means it is locally asymptotically stable ($\forall\, \tau > 0$).

Propostion 2.3 (Persistence). *Assume* (H2). *Suppose in addition that* $F(x) > G(x)$ *for all* $x \in (0, \delta_0)$ *and some* $\delta_0 > 0$. *Then there exists* $k > 0$ *such that for arbitrary* $\phi \in \mathcal{C}$ *the corresponding solution* $x = x(t, \phi)$ *of equation* (1.1) *satisfies* $\liminf_{t\to\infty} x(t) \ge k$.

As an easy consequence of Propositions 2.2 and 2.3 one has the following property.

Corollary 2.4 (Permanence). *Assume* (H2). *Suppose in addition that* $F(x) > G(x)$ *for all* $x \in (0, \delta_0)$ *and some* $\delta_0 > 0$. *There exist positive constants* k *and* K *such that for arbitrary initial function* $\phi \in \mathcal{C}$ *the corresponding solution* $x = x(t, \phi)$ *of equation* (1.1) *satisfies*

$$k \le \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) \le K.$$

Under assumption (H3) the constant solution $x(t) \equiv x_*$ is the only positive equilibrium of equation (1.1). Note that Corollary 2.4 is valid in this case too. When $F(0) = 0$ equation (1.1) also admits the trivial equilibrium $x(t) \equiv 0$. The latter will be the case in some actual models from applications considered in this paper.

Note that there is a trivial possibility for the nonlinearities F and G satisfying assumption (H2) that $F(0) = G(0) = 0$ and $x \equiv 0$ is the only equilibrium of equation (1.1). The dynamical behavior in equation (1.1) is rather simple then, as the following statement shows.

Propostion 2.5. *Assume* (H2) *with* $M_0 = 0$. *Then the trivial solution* $x(t) \equiv 0$ *of* (1.1) *is globally asymptotically stable. That is, for arbitrary* $\phi \in \mathcal{C}$ *the corresponding solution* $x = x(t, \phi)$ *satisfies* $\lim_{t\to\infty} x(t) = 0$ *(for all* $\tau > 0$*).*

Notice that the uniqueness of the zero solution ($F(0) = G(0) = 0$) and the assumption $F(x) > G(x)$ for all $x \in \mathbb{R}^+$ result in the fact that $\lim_{t\to\infty} x(t) = +\infty$ for all solutions of equation (1.1). This is a trivial case which does not represent an interest in real applications.

Equation (1.1) can be transformed, via the change of the independent variable $t = \tau\, s$, to the form $\mu y'(s) = F(y(s-1)) - G(y(s))$, where $\mu = 1/\tau$ and $y(s) = x(\tau s)$. It is a standard form of singularly perturbed differential delay equations with the normalized delay $\tau = 1$ [7]. Therefore, we will also be considering the differential delay equation

$$\mu x'(t) = F(x(t-1)) - G(x(t)), \quad \mu = \frac{1}{\tau} \tag{2.1}$$

as an equivalent form of equation (1.1).

The limiting case $\mu \to 0+$ ($\tau \to \infty$) in equation (2.1) corresponds to the implicit difference equation

$$F(x(t-1)) - G(x(t)) = 0,$$

which can also be written in the form

$$F(x_n) - G(x_{n+1}) = 0. \tag{2.2}$$

Note that in the case of monotone G, when the inverse function G^{-1} exists, the latter can be explicitly resolved for x_{n+1}

$$x_{n+1} = G^{-1}(F(x_n)). \tag{2.3}$$

In the case of non-monotone G equation (2.2) implicitly defines a multi-valued difference equation or inclusion. We shall denote it by

$$x_{n+1} \in \Phi(x_n), \tag{2.4}$$

where the scalar function Φ is generally multi-valued. In this paper we shall restrict our considerations to the case when Φ can assume only a finite number of values. This restriction results from the case of G being piecewise monotone in $\mathbb{R}^+$ with a finite number of the monotonicity branches.

As usual, a sequence $\{x_n\}$ will be called a solution of difference inclusion (2.4) if $G(x_{n+1}) = F(x_n)$ for all $n \in \mathbb{Z}^+ := \{0, 1, 2, 3, \dots\}$. Therefore, the solution $\{x_n\}$ satisfies all three equations (2.2), (2.4), and (2.3) (if G^{-1} exists for the latter). Given x_n, due to the non-monotonicity of G, there can be several values of x_{n+1} which satisfy equation (2.2). They all are incorporated in (2.4) as images of x_n under the multi-valued map Φ.

A fixed point $x = x_*$ of map Φ ($G(x_*) = F(x_*)$) is called attracting if there exists its neighborhood $\mathcal{U}$ such that $\Phi(x) \in \mathcal{U}$ and $\lim_{n\to\infty} \Phi^n(x) = x_*$ for all $x \in \mathcal{U}$. Here $\Phi^k(x) = \Phi(\Phi^{k-1}(x))$ is the k^{th} iteration of the map Φ. Fixed point x_* is called globally attracting (on a set S) if the above limit is valid for all x (in S).

As usual, a closed bounded interval $I \subset \mathbb{R}^+$ is called invariant under map Φ if for every $x \in I$ all values $\Phi(x)$ satisfy: $\Phi(x) \in I$.

Assume that map Φ has an invariant interval $I \subset \mathbb{R}^+$, and introduce a subset $\mathcal{C}_I := C([-\tau, 0], I) \subseteq \mathcal{C}$ of initial functions which range is within the interval I. The following invariance principle holds for solutions of differential delay equation (1.1) with the initial values in $\mathcal{C}_I$.

Propostion 2.6 (Invariance Property). *Let $I := [a, b]$ be a closed bounded invariant interval of the multi-valued map Φ such that $G'(a) > 0$ and $G'(b) > 0$. For arbitrary $\phi \in \mathcal{C}_I$ the corresponding solution $x = x(t, \phi)$ of equation* (1.1) *satisfies $x(t) \in I$ $\forall t \geq 0$ and every $\tau > 0$.*

This proposition shows that the set $\mathcal{C}_I$ is invariant under the action of semiflow S^t defined by the differential delay equation (1.1).

Note that the assumption of the differentiability and positiveness of $G'(a)$ and $G'(b)$ is made in Proposition 2.6 for the sake of simplicity. This can be relaxed to the requirement that $G(x)$ is increasing in a small vicinity of both points a and b.

Theorem 2.7 (Global Asymptotic Stability). *Assume* (H3) *holds. Suppose that the fixed point x_* of map Φ is globally attracting. Then the constant solution $x(t) = x_*$ of differential delay equation* (1.1) *is globally asymptotically stable for all values of $\tau > 0$.*

In section 3 we will use a version of this theorem when the nonlinearity G is monotone increasing. It is given by the following statement.

Propostion 2.8. *Assume* (H3) *to hold and let function G be monotone increasing on $\mathbb{R}^+$. Suppose x_* is a globally attracting fixed point of map Φ. Then the constant solution $x(t) \equiv x_*$ of differential delay equation* (1.1) *is globally asymptotically stable for all values $\tau > 0$.*

The proof of Proposition 2.8 is essentially based on the following statement, which represents an independent interest on its own.

Propostion 2.9. *Assume that functions F and G satisfy* (H3) *and G is increasing in $\mathbb{R}^+$. Let $I_0 := [a, b]$ be arbitrary interval such that $x_* \in (a, b)$. Set $I_1 := \Phi(I_0) := [a_1, b_1]$. Then for every $\phi \in \mathcal{C}_{I_0}$ there exists a time t_0 such that the corresponding solution $x(t)$ of equation* (1.1) *satisfies $x(t) \in I_1$ for all $t \geq t_0$.*

From Proposition 2.9 one immediately deduces the following

Corollary 2.10. *Assume* (H3) *to hold and let functions F and G be monotone on $\mathbb{R}^+$. Then the constant solution $x(t) \equiv x_*$ of equation* (1.1) *is globally asymptotically stable for all values $\tau > 0$.*

As usual, the linearization of differential delay equation (1.1) about $x(t) \equiv x_*$ is given by

$$x'(t) = p\, x(t - \tau) - q\, x(t), \tag{2.5}$$

where $p = F'(x_*)$, $q = G'(x_*)$, while

$$\mu x'(t) = p\, x(t - 1) - q\, x(t), \quad \mu = \frac{1}{\tau}, \tag{2.6}$$

is the linearization of equation (2.1).

Let I be an interval containing point x_*, $I \ni x_*$. We say that equation (1.1) has a negative feedback (about x_*) on I if the nonlinearities F and G are such that

$$[F(x) - F(x_*)] \cdot [G(x) - G(x_*)] < 0 \quad \text{for all } x \in I, x \neq x_*. \tag{2.7}$$

A solution $x(t)$ of equation (1.1) is called slowly oscillating about the constant solution x_* if the distance between any two zeros of the function $x(t) - x_*$ is greater than the delay τ. The main result on the existence of periodic solutions in equation (1.1) which will be used in section 3 is the following

Theorem 2.11 (Existence of periodic solutions)**.** *Assume* (H3) *and that the multivalued map Φ has a closed bounded invariant interval $I \ni x_*$ such that the negative feedback condition* (2.7) *is satisfied for all $x \in I, x \neq x_*$. Let in addition the linearized equation* (2.6) *be unstable. Then differential delay equation* (1.1) *has a slowly oscillating period solution.*

The theorem is essentially due to Kuang [9, 10]. It uses the standard techniques of the ejective fixed point theory [2, 5] along the approach developed by Chow and Hale [1]. The assumptions in [9] are that G is increasing and F is decreasing in $\mathbb{R}^+$. However, the reasoning there can easily be modified to cover the case of non-monotone F and G in the presence of the negative feedback. An alternative approach to prove the existence of periodic solutions when $G(x) = bx$, $b > 0$ has been developed in the original paper [4]. It can also be slightly modified to prove the periodicity in our case. We refer the reader to both works for the relevant details of the proofs. See also paper [14] for more of related results.

3. Applied Models

In this section we apply the theoretical results from the previous section to several cases of real life models. The first one is a physiological model of Mackey [12, 13] which describes the blood cell production in human body. The model fits the differential delay equation (1.1) with essentially nonlinear functions F and G. We derive sufficient conditions for the global asymptotic stability of its unique positive equilibrium and for the existence of a periodic solution slowly oscillating about the equilibrium. The latter complements a recent periodicity result on this model derived in paper [13]. The second application is an optimization problem of maximum consumption for an economic model with delay of Ramsey type subject to control.

3.1. **Blood Cell Production Model of Mackey.** An essentially nonlinear differential equation with delay of form (1.1) was proposed in [12, 13] as a mathematical model of blood cell production for the case of chronic myelogenous leukemia. The equation reads

$$\frac{dx}{dt} = k\beta(x(t-\tau))x(t-\tau) - [\beta(x(t)) + \delta]x(t), \tag{3.1}$$

where the nonlinear function β is a monotone Hill function

$$\beta(x) = \beta_0 \frac{1}{1+x^n} \tag{3.2}$$

and $\beta_0, k = 2e^{-\gamma\tau}$, n, δ are all positive constants defined by the physiological process behind. In this subsection we provide a detailed analysis of model (3.1) based on the given nonlinearities F and G

$$F(x) = k\beta_0 \frac{x}{1+x^n}, \quad G(x) = x\Big[\beta_0 \frac{1}{1+x^n} + \delta\Big] \tag{3.3}$$

and values of the parameters β_0, k, n, δ. We establish sufficient conditions for the global asymptotic stability of the equilibria and for the existence of slowly oscillating period solutions. Our results are complementary to those recently obtained in [13].

We first make several simple observations about the involved nonlinearities F and G.

For $0 < n \le 1$ function F is increasing with $\lim_{x\to\infty} F(x) = \infty$ when $n < 1$ and $\lim_{x\to\infty} F(x) = k\beta_0$ when $n = 1$. For $n > 1$ function F is unimodal with the only critical point $x_{cr} = 1/(\sqrt[n]{n-1})$ and the absolute maximum value $F_{cr} := F(x_{cr}) = k\beta_0 n/(n-1)$. Also, $\lim_{x\to\infty} F(x) = 0$ when $n > 1$.

An easy calculation shows that $G(x)$ is either monotone increasing for all $x \in \mathbb{R}^+$ or it has two local extreme values x_1 and x_2 such that $G(x)$ in increasing in $[0, x_1] \cup [x_2, \infty)$ and decreasing in $[x_1, x_2]$. Function G is monotone increasing in $\mathbb{R}^+$ if and only if $\beta_0(n-1)^2 \le 4n\delta$. When $\beta_0(n-1)^2 > 4n\delta$ it has the two local extreme points x_1 and x_2. The values of x_1 and x_2 are given by

$$x_{2,1} = \Big[\frac{(n-1)\beta_0 \pm \sqrt{(n-1)^2\beta_0^2 - 4n\delta\beta_0}}{2\delta} - 1\Big]^{1/n}. \tag{3.4}$$

We shall also need to refer to the respective values of function $G : G_1 = G(x_1), G_2 = G(x_2)$ (these expressions are easily found but are somewhat lengthy to write down explicitly in terms of the parameters).

Later in this subsection we shall be referring to the respective branches of $y = G(x)$ (its graph). The first branch is defined on the interval $[0, x_1]$ where $G(x)$ is monotone increasing with the range $[0, G_1]$. $G(x)$ is decreasing on its second branch with the domain $x \in [x_1, x_2]$ and the range $[G_2, G_1]$. The third branch is defined for $x \in [x_2, \infty)$ where it is increasing with the range $[G_2, \infty)$. x_1 is the only local maximum and x_2 is the only local minimum of $G(x)$ for $x \in \mathbb{R}^+$.

Depending on the parameter values model (3.1) admits either one or two steady states, $x(t) \equiv 0$ and $x(t) \equiv x_*$, where

$$x_* = \Big(\beta_0 \frac{k-1}{\delta} - 1\Big)^{1/n}. \tag{3.5}$$

Propostion 3.1. *The nontrivial equilibrium x_* exists if and only if $k > 1 + \delta/\beta_0$. When $k \leq 1 + \delta/\beta_0$ equation* (3.1) *has the trivial equilibrium $x(t) \equiv 0$ only which is globally asymptotically stable.*

The equilibrium x_* is found from solving the equation $F(x) = G(x)$, and it is given by formula (3.5). It is easy to check that the condition $k \leq 1 + \delta/\beta_0$ is equivalent to $F'(0) \leq G'(0)$, and therefore, $F(x) < G(x)$ for all $x \in \mathbb{R}^+$. The second part of Proposition 3.1 follows from Proposition 2.5.

In view of Proposition 3.1, for the remainder of this subsection, we will be considering only the case when the non-trivial equilibrium x_* exists.

Global asymptotic stability. We describe first the possibilities when the positive equilibrium $x(t) \equiv x_*$ of equation (3.1) is globally asymptotically stable.

Propostion 3.2. *The positive equilibrium x_* is globally asymptotically stable if either one of the following two conditions is satisfied:*

(1) *F and G are increasing for all $x \in \mathbb{R}^+$;*
(2) *$x_* \leq x_{cr}$.*

Proof. The proof in all possible subcases follows from Proposition 2.7. We shall show that the fixed point x_* of the underlying one-dimensional map Φ is globally attracting. Indeed, in the case of G being monotone it is given by $\Phi(x) = G^{-1}(F(x))$. When F is also increasing, the map Φ is monotone increasing on $\mathbb{R}^+$ with the fixed point x_* being globally attracting. The global stability follows from Corollary 2.10. When F is unimodal and $x_* \leq x_{cr}$, both functions are monotone on $[0, x_{cr}]$, and the above monotonicity arguments apply there too. For every $x > x_{cr}$, since F is decreasing there, one has $\Phi(x) \in [0, x_{cr}]$. Therefore, x_* is globally attracting under Φ.

The subcase $x_* \leq x_{cr}$ allows for two additional possibilities when G is not monotone: (i) $x_* \in [0, x_1]$ or (ii) $x_* \in [x_2, \infty)$.

In case (i), if $G(x_2) > F_{cr}$ then $\Phi([0, x_{cr}]) \subset [0, x_{cr}]$ and $\Phi([x_{cr}, \infty) \subset [0, x_{cr}]$. Therefore $\Phi(\mathbb{R}^+) \subset [0, x_{cr}]$ and x_* is globally attracting. If $G(x_2) < F_{cr}$ then there exists a positive integer $N = N(F, G)$ such that $\Phi^N(x) \in [0, x_{cr}]$ for every $x \in [x_{cr}, \infty)$. Therefore, $\Phi^N([x_{cr}, \infty)) \subset [0, x_{cr}]$. As before, $\Phi([0, x_{cr}]) \subset [0, x_{cr}]$. Thus, x_* is globally attracting fixed point for map Φ.

In case (ii), G is non-monotone on the interval $(0, x_*)$ but F is monotone there with $F(x) > G(x)$. Both functions F and G are monotone on the interval $[x_2, x_{cr}]$. Therefore, like in the monotonicity case above, the fixed point x_* is globally attracting on the interval $[x_2, x_{cr}]$. For every $x \in [x_{cr}, \infty)$ its image satisfies $\Phi(x) \in (0, c_{cr})$ and $\Phi^i(x) \in (0, x_{cr})$ for all $i \geq 1$. It is easily seen that for every $x \in (0, x_1)$ there exists positive integer N that $\Phi^N(x) \in [x_2, x_*]$. Therefore, x_* is globally attracting in this subcase too. □

Existence of periodic solutions. The existence of nontrivial slowly oscillating periodic solutions is deduced by applying Theorem 2.11. The following statement describes possible cases for this to happen.

Propostion 3.3. *Equation* (3.1) *has a slowly oscillating periodic solution if any one of the following conditions is satisfied:*

(1) *G is increasing for all $x \in \mathbb{R}^+$, $x_* > x_{cr}$, $F(\Phi^2(x_{cr})) > F(x_*)$, and $x(t) \equiv x_*$ is unstable;*
(2) *$x_* > x_{cr}$, $G_2 > G(x_*)$, $F(\Phi^2(x_{cr})) > F(x_*)$, and $x(t) \equiv x_*$ is unstable;*
(3) *$x_* > x_{cr}$, $G_1 < G(x_*)$, $F(\Phi^2(x_{cr})) > F(x_*)$, and $x(t) \equiv x_*$ is unstable.*

Proof. For each of the listed cases (1)-(3) we shall indicate an invariant interval I_0 on which the negative feedback condition (2.7) holds. Together with the instability assumption of the steady state $x(t) \equiv x_*$, and in view of Theorem 2.11, this implies the existence of periodic solutions.

In case (1), given x_{cr} and $F_{cr} = F(x_{cr})$, let $u_1 > x_*$ be such value of x that $G(x) = F(x_{cr})$. Thus $u_1 = G^{-1}F(x_{cr}) = \Phi(x_{cr})$. Let $u_2 < x_*$ be such value of x that $G(x) = F(u_1)$. Therefore, $u_2 = G^{-1}(F(u_1)) = \Phi^2(x_{cr})$. One now can see that if $F(u_2) > F(x_*)$ then $F(x) > F(x_*)$ for all $x \in [u_2, x_*)$ and $F(x) < F(x_*)$ for all $x \in (x_*, u_1]$. Since G is increasing in $[u_2, u_1]$ the negative feedback condition (2.7) holds for all $x \in [u_2, u_1] := I_0$. Interval I_0 is also invariant under $\Phi = G^{-1} \circ F$. The other two cases are treated similar. We leave the details to the reader.

All three cases assume the instability of the constant solution $x(t) \equiv x_*$ of equation (1.1). It follows from the instability of the zero solution of the linearized about x_* equation (2.5) (or (2.6)) [2, 5]:

$$x'(t) = p\,x(t-\tau) - q\,x(t), \quad \text{where} \quad p = F'(x_*), q = G'(x_*).$$

Since $F'(x_*)$ and $G'(x_*)$ are readily found from the value of x_* given by (3.5) the coefficients p and q are easily evaluated in terms of the parameters defining functions F and G (they are too large and cumbersome, however, to be written explicitly here). We note that $G'(x_*) > 0$ in case (1), since G is increasing. In case (2), x_* belongs to the first branch of G. In case (3), x_* belongs to the third branch of G. Therefore, $G'(x_*) > 0$ for both. $F'(x_*) < 0$ in all three cases since $x_* > x_{cr}$. Thus, $p < 0$ and $q > 0$ for the linearized equation in all three cases.

The exact stability/instability conditions for the linear equation (2.5) in terms of the coefficients p, q and delay τ are well known. We refer the reader to the four references [2, 4, 5, 9] on our list, in addition to many others not included here. □

We note that our periodicity results supplement those recently obtained in paper [13]. The latter treats the case when the equilibrium $x(t) \equiv x_*$ belongs to the second branch of $y = G(x)$. The authors in particular consider the case when $n \to \infty$ (while the other parameters of F and G are fixed). It can be verified that $G'(x_*) < 0$ in this case. In the limiting case the nonlinearity F is given by $F(x) = 0$ for $x \geq 1$ and $F(x) = \beta_0$ for $x < 1$.

Open cases. There are several remaining cases for the parameter values defining F and G when our results do not apply to make a conclusion on either the global asymptotic stability or the existence of periodic solutions. The first such case is when G is monotone increasing in $\mathbb{R}^+$, $c_{cr} > x_*$, and $x(t) \equiv x_*$ is locally asymptotically stable. The other cases are when x_* belongs to either branch one or branch three of function G and the equilibrium $x(t) \equiv x_*$ is also locally asymptotically stable. In the general situation of arbitrary F and G the global dynamics of equation (1.1) in any of the three cases can be complicated. However, for the particular nonlinearities F and G of the Mackey model (3.1) it looks like the corresponding one-dimensional map Φ can have x_* as a globally attracting fixed point. This would imply the global asymptotic stability for the differential delay equation (3.1). Therefore, we come up with the following

Conjecture 3.4. *The positive equilibrium $x(t) \equiv x_*$ of equation* (3.1) *is globally asymptotically stable whenever it is locally asymptotically stable.*

The other case when our approaches and results cannot be applied is when the equilibrium $x(t) \equiv x_*$ belongs to the interval $[x_1, x_2]$ (i.e., the second branch of G). As it was mentioned above, this case was treated in paper [13] for a piece-wise constant nonlinearity F. The case of general F represents a difficult challenge for which new related approaches need to be developed.

3.2. **An Optimal Control Problem.** Many economic models lead to differential delay equations of the form (1.1). We refer the reader to a partial list of economic applications given in papers [3, 11, 15]. In this subsection we consider an optimization problem for equation (1.1) as a general model of several economical processes, which in particular includes the modified Ramsey model with delay [8].

We study the global dynamics of the following optimal control model described by the differential equation (1.1) with delay and control

$$x'(t) = u(t)F(x(t-\tau)) - G(x(t)), \tag{3.6}$$

where $x(t)$ is the capital, $u(t)$ is a control with values within some interval $[\alpha, 1]$, and $\tau > 0$ is the length of the production (investment) cycle. The component $F(x(t-\tau))$ describes a general commodity being produced at time t and the part $G(x(t))$ stands for the "amortization" of the capital. After each cycle of production a certain part of the commodity (capital) is used for the investment while the remaining part is consumed. We shall assume that, at any time $t \geq 0$, the part $u(t) \cdot F(x(t-\tau))$ is assigned for the production purposes (investment) while the part

$$C(t) = [1 - u(t)] \cdot F(x(t-\tau)) \tag{3.7}$$

is consumed. The optimality is defined by the following functional:

$$J(x(\cdot)) \doteq \liminf_{t\to\infty} C(t) \Longrightarrow \max . \tag{3.8}$$

This functional aims to maximize the minimal possible consumption when $t \to \infty$. It can be considered as an analogue of the terminal functional for infinite time horizon. We refer to [16] for more information about the results on the stability of optimal solutions in terms of this functional.

As before, both nonlinearities F and G satisfy the hypothesis (H1). However, instead of (H3) the following modified hypothesis will be used.

(H3′)
1. G and F are strictly increasing in $\mathbb{R}^+$;
2. For each $u \in [\alpha, 1], \alpha \geq 0$, there exists a unique point $x_u \geq 0$ such that $uF(x_u) = G(x_u)$;
3. $\alpha \geq 0$ is the minimal point satisfying (2), and $x_u = 0$ if $u = \alpha$;
4. $uF(x) > G(x)$ if $x \in (0, x_u)$ and $uF(x) < G(x)$ if $x > x_u$.

These assumptions are justified by economic's interpretations of the involved nonlinearities [3, 17]. In particular, it is clear that the hypothesis (H3′) holds for the generalized Ramsey model (3.13) considered below.

Note that the generic case $F'(0) > G'(0) > 0$ results in the range $[\alpha, 1]$ for the values of control $u(t)$, where $\alpha := G'(0)/F'(0)$. If $F'(0) = \infty$ and $G'(0)$ is finite, which are the commonly used assumptions in the literature, then we have $\alpha = 0$.

When $\alpha > 0$, the zero solution of equation (3.6) is globally asymptotically stable in the class of solutions corresponding to control $u < \alpha$.

Note that $x_{u_1} < x_{u_2}$ when $u_1 < u_2$. The point x_u that corresponds to $u = 1$ will be denoted by M; that is,

$$F(M) - G(M) = 0 \quad \text{and} \quad F(x) - G(x) < 0, \forall x > M. \tag{3.9}$$

It is assumed that $x_u = 0$ if $u = \alpha$. The interval $[0, M]$ will be refereed to as the set of stationary points.

Introduce the notation

$$x^* = \operatorname{argmax}\{F(x) - G(x) : \ x \geq 0\} \tag{3.10}$$

and

$$c^* = F(x^*) - G(x^*). \tag{3.11}$$

The following hypothesis will also be used in this subsection.

(H4) $c^* > 0$ and x^* is unique; that is,

$$F(x) - G(x) < c^*, \quad \text{for all} \quad x \neq x^*. \tag{3.12}$$

Note that this hypothesis implies $M > x^*$, as it can be seen from (3.9).

It is easy to see that (H4) holds for monotone functions F and G which do not have inflection points. In particular, this applies to the modified Ramsey model with delay (3.13) where $G(x) = bx, b > 0$ and $F(x) = Bx^p$, $0 < p < 1$.

3.3. **Fixed/steady controls.** In this subsection we consider the case when the proportion between investment and consumption is taken fixed for all $t \geq 0$. That is, we consider scalar control functions $u(t) \equiv u$, $t \geq 0$, where $u \in [\alpha, 1]$.

Theorem 3.5. *Assume (H3′). There exists an optimal control u_* to the problem* (3.6)-(3.7)-(3.8) *in the class of scalar control functions. In addition if (H4) holds then the control u_* is unique.*

Proof. Let a control $u(t) \equiv u_0$ be given, and consider equation (3.6). Since both F and G are increasing, and in view of Corollary 2.10, its constant solution $x(t) = x_{u_0}$ is globally asymptotically stable. That is, for arbitrary initial function $\phi \in \mathcal{C}$ one has $\lim_{t\to\infty} x(t, \phi) = x_{u_0}$. Therefore, the corresponding value of the functional $J(x(\cdot))$ is given by

$$J(x(\cdot), u_0) = (1 - u_0) \cdot F(x_{u_0}) := J(u_0),$$

which is dependent on u_0 only (and independent of the choice of the initial function ϕ).

Since x_{u_0} is continuous in u_0, and $x_{u_0} = 0$ at $u_0 = \alpha$ one has that $J(u_0)$ is also continuous in u_0 and satisfies

$$J(\alpha) = J(1) = 0 \quad \text{and} \quad J(u_0) > 0, u_0 \in (\alpha, 1).$$

Therefore, there exists a point $u_* \in (\alpha, 1)$ where the maximum value is achieved: $J(u_*) = \max\{J(u), u \in [0, 1]\}$. Then $u(t) \equiv u_*$ is an optimal control. Note that in general u_* does not have to be unique (appropriate non-uniqueness examples are readily constructed).

Suppose next that F and G are monotone and satisfy (H4). We claim that the above optimal control $u(t) \equiv u_*$ is unique then. Indeed, the value of the functional J with the constant control u is

$$J(x(\cdot), u) = (1-u)F(x_u) = \big[1 - \frac{G(x_u)}{F(x_u)}\big] \cdot F(x_u) = F(x_u) - G(x_u),$$

which assumes the unique maximum value at x_{u_*} when $u = u_*$. □

Example 3.6. Controlled Ramsey model with delay.

The differential equation

$$\frac{dK(t)}{dt} = BK^p(t-\tau) - bK(t)\,. \tag{3.13}$$

was proposed as a modified Ramsey economic model with delay [8, 17]. Consider here the respective control problem (3.6)-(3.7)-(3.8)

$$\frac{dK(t)}{dt} = u(t)BK^p(t-\tau) - bK(t)\,,$$

where $B > 0$, $b > 0$, $0 < p < 1$ and $u(t) \equiv u \in [0,1]$ is a constant control. It is easy to check that assumptions (H3 $'$) and (H4) hold with $\alpha = 0$.

One readily finds the steady state x_u and the respective value of the functional $J(x_u)$ as

$$x_u = \Big(\frac{B}{b}\Big)^{1/(1-p)} \cdot u^{1/(1-p)}, \quad J(u) = \Big(\frac{B}{b}\Big)^{p/(1-p)} \cdot (1-u) \cdot u^{p/(1-p)}.$$

The unique maximum value of $J(u)$ is achieved when $u = p$.

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