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NONLINEAR STOCHASTIC HEAT EQUATIONS WITH CUBIC NONLINEARITIES AND ADDITIVE Q-REGULAR NOISE IN $\mathbb{R}^1$

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Abstract. Semilinear stochastic heat equations perturbed by cubic-type nonlinearities and additive space-time noise with homogeneous boundary conditions are discussed in $\mathbb{R}^1$. The space-time noise is supposed to be Gaussian in time and possesses a Fourier expansion in space along the eigenfunctions of underlying Lapace operators. We follow the concept of approximate strong (classical) Fourier solutions. The existence of unique continuous L^2-bounded solutions is proved. Furthermore, we present a procedure for its numerical approximation based on nonstandard methods (linear-implicit) and justify their stability and consistency. The behavior of related total energy functional turns out to be crucial in the presented analysis.

1. Introduction

Consider semilinear stochastic heat equations with cubic-type nonlinearities

$$\begin{gathered}\frac{du}{dt} = \sigma^2 \Delta u + B(u) + G(u)\frac{dW(t,x)}{dt} \\ u = u(t,x), \quad 0 < x < L,\ t \ge 0\end{gathered} \tag{1.1}$$

perturbed by additive space-time random noise W which is supposed to be Gaussian in time and possesses a Fourier expansion in terms of the eigenfunctions of the Laplace operator Δ in $\mathbb{R}^1$. The objective of this paper is to discuss properties of its strong Fourier-type solutions $u = u(t,x)$ and its numerical approximations by appropriate truncation of its Fourier series and nonstandard methods to integrate them numerically in time.

Analytical aspects of solvability of equations (1.1) with Lipschitz-continuous B and G are discussed by several authors. For example, see Bensoussan & Temam (1972), Pardoux (1975/79), Walsh (1984/86), DaPrato & Zabzcyk (1992), Greksch & Tudor (1996), among many others. Moreover, equations with monotone B are treated in Pardoux (1979), Bessaih & S. (2005, JCAM), S. (2007, JMAA). Not so much known is for equations with cubic-type $B(u) = u(a_1 - a_2\|u\|^2_{L^2})$ with real

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parameters $a_2 > 0$ and a_1. Such equations occur in neurophysiological modeling of large nerve cell systems with action potential B in mathematical biology (see also remarks in Walsh (1984/86)). For example, there are biochemical models of the form (1.1) to calculate the flow of the electric current and voltage along active neuronal fibres (neurites) in computational neurosciences (Recall that neuronal fibres are composed of segments with dendritic membranes with voltage-dependent capacitances and resistance, equipped with voltage-gated ion channels). For more details, see Hodgkin and Rushton (1946), Koch (1999), Koch and Segev (1998), Stuart and Sakmann (1994), Tuckwell and Walsh (1983). Especially, we shall treat here the most biologically relevant one-dimensional special case

$$du = [\sigma^2 \frac{\partial^2 u}{\partial x^2} + u\Big(a_1 - a_2 \|u\|^2_{L^2}\Big)]dt + b\, dW(t,x) \tag{1.2}$$

where the norm $\|u\|_{L^2}$ is taken with respect to the L^2-space $L^2(0,L)$ and $b \in \mathbb{R}^1$ is an overall noise intensity parameter. Homogenous boundary conditions (BC)

$$u(t,0) = u(t,L) = 0 \quad \forall t \geq 0 \tag{1.3}$$

and $L^2(0,L)$-integrable initial conditions (IC)

$$u(0,x) = u_0(x) \quad \forall x \in (0,L) \tag{1.4}$$

are opposed on the solutions u throughout the paper. Moreover, the equation (1.2) is driven by space-time Q-regular noise

$$W(t,x) = \sum_{n=1}^{+\infty} \alpha_n W_n(t) \underbrace{\sqrt{\frac{2}{L}} \sin\big(\frac{n\pi x}{L}\big)}_{=e_n(x)} \tag{1.5}$$

with i.i.d. Wiener processes W_n with $W_n(t) \in \mathcal{N}(0,t)$, where

$$\mathrm{trace}(Q) = \sum_{n=1}^{+\infty} \alpha_n^2 < +\infty. \tag{1.6}$$

Note that

$$e_n(x) = \sqrt{\frac{2}{L}} \sin\big(\frac{n\pi x}{L}\big), \quad n \geq 1 \tag{1.7}$$

are the eigenfunctions of the Laplace operator Δ in $\mathbb{R}^1$, $\Delta e_n = -(n^2\pi^2/L^2)e_n$, and they form an orthonormal system in $L^2(0,L)$; i.e.,

$$\langle e_n, e_k \rangle_{L^2(0,L)} = \int_0^L e_n(x) e_k(x)\, dx = \delta_{n,k} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

where $\delta_{n,k}$ is the Kronecker symbol. Moreover, it is not too restrictive that the noise W has an eigenfunction expansion (1.5) in the separable Hilbert space $L^2([0,+\infty) \times [0,L])$ with respect to the same eigenfunctions as the underlying Laplace operator with homogeneous boundary conditions (1.3). This is due to the perturbations by additive space-time noise (Gaussian in time), the specific Dirichlet boundary conditions (1.3) and the other part of the eigenbasis determined by $\cos(n\pi x/L)$ and spanning the space $L^2(0,L)$ is orthogonal to $e_n, n \geq 1$ (while forming together a complete orthonormal system in $L^2(0,L)$).

The paper is organized as follows. After this introduction, we begin with the verification of the unique existence of strong global solutions with not more than exponentially increasing second moments in time in Section 2. Section 3 provides a truncation procedure of Fourier series solutions approximating those strong solutions. There a finite-dimensional system of nonlinear stochastic ODEs determining its Fourier coefficients $c_k(t)$ is derived. The unique existence of strong solutions of those systems is justified by estimating the truncated total energy. Section 4 reports on the total expected energy of the original infinite-dimensional stochastic system (1.2). We are going to show that the energy functional is linearly bounded in time in the mean sense, provided that the initial Fourier coefficients $c_k(0)$ are mean square summable. In the final Section 5 we suggest 3 numerical methods (explicit and implicit difference methods) to find those Fourier coefficients.

2. Existence of Unique Approximate Strong Solutions

Indeed we may verify the existence of a.s. unique, approximately strong global solution with finite second moments. For this purpose, we exploit the technique of monotonicity of semilinearities. Recall the concept of approximate strong solution from [37].

To be more self-explanatory, we consider the following definition of strong solution concepts. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a complete probability space equipped with a nondecreasing filtration $(\mathcal{F}_t)_{0\le t\le T})$. Suppose that H is a Hilbert space and A a linear operator of H with domain $D(A)$. Then, an H-valued stochastic process $u = (u(t))_{0\le t\le T}$ is said to be a **strong solution** of the SPDE

$$du = [A(t)u + B(u)]dt + G(u)dW \tag{2.1}$$

on $([0,T] \times H \times \Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ if and only if

(a) u is an element of the class of progressively measurable processes with values in H (which is also closed with respect to progressively measurable versions),

(b) $u(t) \in D(A(t)) \cap D(B(t,\cdot)) \cap D(G(t,\cdot))$ ($\mathbb{P}$-almost surely) for all $t \in [0,T]$ (almost everywhere) and $A(.)u(.) \in L^1_{\rm loc}([0,T], H)$,

(c) and, for every $0 \le s \le t \le T$, we have ($\mathbb{P}$-almost surely)

$$u(t) = u(s) + \int_s^t \Big[A(r)u(r) + B(r,u)\Big]dr + \int_s^t G(r,u)dW(r).$$

Moreover, an H-valued stochastic process $u = (u(t))_{0\le t\le T}$ is called an **approximate strong solution** of (2.1) on $([0,T] \times H \times \Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ if there is a sequence of stopping times $\tau_r(t)$ with $\lim_{r\to+\infty} \tau_r(t) = t$ ($\mathbb{P}$-almost surely) such that $u_r = (u(\tau_r(t)))_{0\le t\le T}$ is a strong solution of (2.1) on $([0,\tau_r(T)] \times H \times \Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ for all $r > 0$ and $u = \lim_{r\to+\infty} u_r \in H$ ($\mathbb{P}$-almost surely). Besides, the process $u_r = (u_r(t))_{0\le t\le T}$ is said to be a **localized** (strong) solution of (2.1). There are other solution concepts such as mild, weak and evolution solutions. For more details and relations between those concepts, see Grecksch and Tudor [12]. We shall devote our studies to the concept of approximate strong solutions here.

The existence and uniqueness of strong solutions of (2.1) is well-known when all operators are globally Lipschitz-continuous on H. In this case, a stochastic localization procedure is not needed. For example, see Bensoussan and Temam [4], Da Prato and Zabzcyk [7, 8], Grecksch and Tudor [12], Rozovskii [27] or Pardoux [22, 23]. Their main results imply the existence of local pathwise unique continuous (strong) solutions $u_r \in H$ of (2.1) on balls

$$K_r = \{u \in H : \|u\|_H < r\}. \tag{2.2}$$

Thus, the remaining important question is how we can guarantee that u cannot explode as r tends to $+\infty$ and stays in H, i.e. our aim is to establish an existence and uniqueness result of global pathwise unique continuous (strong) solutions u of (1.1) under conditions weaker than global Lipschitz-continuity such as local Lipschitz-continuity of nonlinearities B on the Hilbert-space $H = L^2([0,T]\times[0,L])$.

Let $\mathcal{B}(S)$ be the σ-algebra of all Borel sets of inscribed set S and $\mathcal{F}_t = \sigma(W_j(s) : s \le t, j \in \mathbb{N})$ the naturally generated σ-algebra belonging to the Wiener processes W_j and forming the underlying filtration.

Theorem 2.1. *Assume that the assumptions in Section 1 are satisfied together with*

$$\mathbb{E}\|u(0,\cdot)\|_H^2 < +\infty$$

for $\mathcal{B}(0,L)\times\mathcal{F}_0$-measurable initial data $u(0,\cdot)\in H$, where $H = L^2([0,L])$. Then the approximate strong, global solution of (1.2) *exist and has uniformly bounded second moments on any finite-time interval $t\in[0,T]$. More precisely,*

$$\forall T < +\infty\ \exists K_0, K_1 \ge 0\ \forall 0 \le t \le T\ :\ \mathbb{E}\|u(t,\cdot)\|_H^2 \le (\mathbb{E}\|u(0,\cdot)\|_H^2 + K_0)\exp(K_1 T).$$

Remark 2.2. In fact, if $\sigma^2\pi^2 > L^2 a_1$, we shall be able to improve qualitatively these estimates of second moments to linearly bounded ones (in time)

$$\forall T < +\infty\ \exists c \ge 0\ \forall 0 \le t \le T\ :\ \mathbb{E}\|u(t,\cdot)\|_H^2 \le \mathbb{E}\|u(0,\cdot)\|_H^2 + ct$$

with universal constant c (depending on diverse parameters) by using the energy estimates from Section 4.

Proof. First, note that the unique localized (strong) solution u_r of SPDE (1.2) with local Lipschitzian coefficients exists. This fact we know from [7], [5] or [12]. Now, apply Lemma 2.3 from below and check that the conditions of Theorem 3 from [37] (p. 339) are fulfilled. Thus, the unique, approximate strong, continuous solution u to SPDE (1.2) exists and its second moments $\mathbb{E}\|u(t,\cdot)\|_H^2$ are exponentially bounded in time. This confirms the conclusion. $\square$

Lemma 2.3. *Let H be a Hilbert space equipped with the real-valued scalar product $\langle .,. \rangle_H$ and naturally induced norm $\|u\|_H = \sqrt{\langle u,u\rangle_H}$. Then, for all $a_2 \ge 0$, the mapping $u \in H \longmapsto B(u) = (a_1 - a_2\|u\|_H^2)u$ satisfies the angle condition on H, i.e., for all $\gamma \ge 0$ and all $u, v \in H$, we have*

$$\begin{aligned} F(u,v) &:= \langle B(u)-B(v), u-v\rangle_H \\ &\le \Big(a_1 - a_2\frac{\|u\|_H^2+\|v\|_H^2}{2}\Big)\|u-v\|_H^2 \le a_1\|u-v\|_H^2 \end{aligned}$$

and

$$\langle B(u),u\rangle_H \le \Big(a_1 - a_2\frac{\|u\|_H^2}{2}\Big)\|u\|_H^2 \le a_1\|u\|_H^2.$$

Proof. For $u, v \in H$, define $f(u) := \|u\|_H^2 u$ and

$$g(u,v) := < f(u)-f(v), u-v\rangle_H \ge \frac{\|u\|_H^2+\|v\|_H^2}{2}\|u-v\|_H^2.$$

First, note that the above defined g is symmetric, i.e. $g(u,v) = g(v,u)$ for all $u, v \in H$. Thus, $2g(u,v) = g(u,v) + g(v,u)$. Second, we find that

$$\begin{aligned} g(u,v) &= \langle \|u\|_H^2 u - \|u\|_H^2 v + \|u\|_H^2 v - \|v\|_H^2 v, u-v\rangle_H \\ &= \|u\|_H^2\langle u-v,u-v\rangle_H + \Big(\|u\|_H^2 - \|v\|_H^2\Big)\langle v, u-v\rangle_H. \end{aligned}$$

for all $u, v \in H$. Third, both findings imply that

$$2g(u,v) = \Big(\|u\|_H^2 + \|v\|_H^2\Big)\|u-v\|_H^2 + \Big(\|u\|_H^2 - \|v\|_H^2\Big) \cdot \Big(\|u\|_H^2 - \|v\|_H^2\Big).$$

Note that the last product term is always positive-definite. Consequently, we have

$$g(u,v) \geq \frac{\|u\|_H^2 + \|v\|_H^2}{2}\|u-v\|_H^2 \tag{2.3}$$

for all $u, v \in H$. Hence, f is increasing (In fact, $g(u,v) = 0$ or $g(u,v)$ is equal to the right side of last inequality if and only if $u = v$ in H). Now, we find that

$$B(u) = a_1 u - a_2 f(u).$$

Hence, we have

$$F(u,v) = \langle B(u) - B(v), u - v\rangle_H = a_1\|u-v\|_H^2 - a_2 g(u,v).$$

Finally, applying the estimate (2.3) to the above expression of F confirms that

$$F(u,v) \leq \Big(a_1 - a_2\frac{\|u\|_H^2 + \|v\|_H^2}{2}\Big)\|u-v\|_H^2 \leq a_1\|u-v\|_H^2, \tag{2.4}$$

$$\langle B(u), u\rangle_H \leq \Big(a_1 - a_2\frac{\|u\|_H^2}{2}\Big)\|u\|_H^2 \leq a_1\|u\|_H^2 \tag{2.5}$$

since $a_2 \geq 0$. In passing, we note that the relation (2.5) is obtained directly from (2.4) by setting $v = 0$. Thus, the proof of Lemma 2.3 is complete. □

3. Fourier-Series Solutions

By the principle of linear superposition (LSP), it is clear that the Fourier series

$$u(t,x) = \sum_{n=1}^{+\infty} c_n(t)e_n(x), \quad t \geq 0,\ 0 \leq x \leq L \tag{3.1}$$

forms a strong solution of (1.2), provided that this series converges and $c_n(0)$ are chosen such that the initial conditions (IC) are satisfied. This series is truncated as

$$u_N(t,x) = \sum_{n=1}^{N} c_n(t)e_n(x), \quad t \geq 0,\ 0 \leq x \leq L \tag{3.2}$$

which also form strong solutions of (1.2).

Theorem 3.1. *The Fourier coefficients of* (3.1) *satisfy* ($\mathbb{P}$*-a.s.*) *the infinite-dimensional system of ordinary SDEs*

$$dc_k = \Big[-\sigma^2\frac{k^2\pi^2}{L^2} + a_1 - a_2\sum_{n=1}^{+\infty} c_n^2\Big]c_k dt + b_k dW_k \tag{3.3}$$

for $k = 1, 2, \dots,$ *where* $b_k = b\alpha_k$.

Proof. First, plug the Fourier series (3.1) into the SPDE (1.2). So, one arrives at

$$du(t,x) = \sum_{n=1}^{\infty} c_n(t)e_n(x)\Big[-\sigma^2\frac{n^2\pi^2}{L^2} + a_1 - a_2\sum_{k=1}^{\infty}[c_k(t)]^2\Big]dt + b\sum_{n=1}^{\infty}\alpha_n e_n(x)dW_n(t)$$

for $0 \leq t \leq T$, $0 \leq x \leq L$. Second, multiply this differential identity by the eigenfunctions $e_k(x)$. Third, integrate the obtained identity with respect to the space-coordinate x over $[0, L]$. Thus, for all $k \in \mathbb{N}$, we encounter

$$
\begin{aligned}
&\int_0^L du(t,x) e_k(x) dx \\
&= \sum_{n=1}^{\infty} dc_n(t) \int_0^L e_n(x) e_k(x) dx \\
&= \sum_{n=1}^{\infty} dc_n(t) \delta_{n,k} = dc_k(t) \\
&= \sum_{n=1}^{\infty} c_n(t) \int_0^L e_n(x) e_k(x) dx \Big[-\sigma^2 \frac{n^2 \pi^2}{L^2} + a_1 - a_2 \sum_{k=1}^{\infty} [c_k(t)]^2 \Big] dt \\
&\quad + b \sum_{n=1}^{\infty} \int_0^L e_n(x) e_k(x) dx \alpha_n dW_n(t) \\
&= c_k \Big[-\sigma^2 \frac{n^2 \pi^2}{L^2} + a_1 - a_2 \sum_{k=1}^{\infty} [c_k(t)]^2 \Big] dt + b \alpha_k dW_k(t)
\end{aligned}
$$

for $0 \leq t \leq T$. Note that we may exchange differentiation and integration in the above computations since we know that the unique strong solution u of (1.2) with

$$
\|u(t, \cdot)\|_H^2 = \sum_{k=1}^{\infty} [c_k(t)]^2 < +\infty
$$

and continuous Fourier coefficients $c_k(t)$ exists for all $0 \leq t \leq T$ (which implies that all terms are finite and mean square summable). Consequently, Theorem 3.1 is proven. ☐

Remark 3.2. The truncated Fourier solutions u_N have Fourier coefficients c_k which can be approximated by the truncated finite-dimensional system of ordinary SDEs

$$
dc_k = \Big[-\sigma^2 \frac{k^2 \pi^2}{L^2} + a_1 - a_2 \sum_{n=1}^{N} c_n^2 \Big] c_k dt + b_k dW_k \tag{3.4}
$$

for $k = 1, 2, \dots$, where $b_k = b\alpha_k$. Notice also that, for stochastic systems with additive noise, the stochastic integration leads to the same type of stochastic integral (i.e. Itô, Stratonovich, $\alpha-$ and quadrature-integrals are all the same, see [35], [39]). That is why we have not mentioned earlier in which sense we interpret the stochastic integration (as it does not matter in our calculations).

4. Total Energy Evolution

For the case of sufficiently strong diffusion with $\sigma^2 \pi^2 > L^2 a_1$, we investigate the behavior of related energy functional. The total energy $\mathcal{E}$ of system (1.2) at time $t \geq 0$ is defined

$$
\mathcal{E}(t) = \frac{\sigma^2}{2} \|u_x(t, \cdot)\|_{L^2}^2 - \frac{a_1}{2} \|u(t, \cdot)\|_{L^2}^2 + \frac{a_2}{4} \|u(t, \cdot)\|_{L^2}^4. \tag{4.1}
$$

This energy functional is indeed nonnegative and finite (a.s.) as one can see from the following theorem. For its proof, we express this functional in terms of its Fourier coefficients c_k by

$$V(t) := V(c_k(t) : k \in \mathbb{N}) = \frac{1}{2} \sum_{n=1}^{+\infty} [\sigma^2 \frac{n^2\pi^2}{L^2} - a_1] c_n^2(t) + \frac{a_2}{4} \Big(\sum_{n=1}^{+\infty} c_n^2(t) \Big)^2 \tag{4.2}$$

for $t \geq 0$. Note that $V \geq 0$ for all sequences $(c_k(t))_{k\in\mathbb{N}}$ under $\sigma^2\pi^2 > L^2 a_1$. Moreover, under $\sigma^2\pi^2 > a_1 L^2$ and $a_2 \geq 0$, V acts as a Lyapunov functional. Besides, $\mathcal{E}(t) = V(t)$ for all $t \geq 0$. Furthermore, this energy functional directly relates to the total temperature distribution absorbed (and stored) by the underlying physical system over time $t \in [0, T]$.

Theorem 4.1. *Assume that* $e(0) = \mathbb{E}V(c_k(0) : k \in \mathbb{N}) < +\infty$, $\sigma^2\pi^2 \geq L^2 a_1$ *and* $\mathrm{trace}(Q) = \sum_{n=1}^{\infty} \alpha_n^2 < +\infty$. *Then, the total expected energy of the original system* (1.2) *is linearly bounded in time by*

$$\begin{aligned} e(t) &= \mathbb{E}V(c_k(t) : k \in \mathbb{N}) \\ &\leq e(0) + \Big[b^2 \sum_{n=1}^{\infty} \alpha_n^2 \Big(\frac{\sigma^2 n^2 \pi^2}{L^2} - a_1 \Big) + a_2 (b^2\beta^2)^{3/2} \Big(\frac{1}{12a_2} \Big)^{1/2} \frac{5}{6} \Big] t \end{aligned}$$

where

$$\beta^2 = \sum_{n=1}^{\infty} \alpha_n^2 + 2 \max_{n\in\mathbb{N}} \alpha_n^2.$$

Remark 4.2. Therefore, the quadratic magnitude of the temperature u averaged in space cannot grow faster than a linear curve in time t.

Proof of Theorem 4.1. Consider the energy of the truncated system (3.4) given by

$$V_N(t) := V_N(c_k(t) : k = 0, 1, \dots, N) = \frac{1}{2} \sum_{n=1}^{N} [\sigma^2 \frac{n^2\pi^2}{L^2} - a_1] c_n^2(t) + \frac{a_2}{4} \Big(\sum_{n=1}^{N} c_n^2(t) \Big)^2 \tag{4.3}$$

for $t \geq 0$. Now, apply Dynkin formula (see [9], [18], cf. also Itô Formula in [2]) to the functional $e_N(t) = \mathbb{E}[V_N(t)]$ with coefficients c_k satisfying (3.4). For this purpose, compute its infinitesimal generator

$$\mathcal{L}V_N = \Big(\sum_{n=1}^{N} \Big[-\frac{\sigma^2 n^2 \pi^2}{L^2} + a_1 - a_2 \sum_{n=1}^{N} c_k^2 \Big] c_n \frac{\partial}{\partial c_n} + \frac{b^2}{2} \sum_{n=1}^{N} \alpha_n^2 \frac{\partial^2}{\partial c_n^2} \Big) V_N.$$

Thus, one arrives at the estimate

$$\mathcal{L}V_N \quad \leq \quad b^2 \sum_{n=1}^{\infty} \alpha_n^2 \Big(\frac{\sigma^2 n^2 \pi^2}{L^2} - a_1 \Big) + a_2 (b^2 \beta_N^2)^{3/2} \big(\frac{1}{12a_2} \big)^{1/2} \frac{5}{6}$$

where

$$\beta_N^2 = \sum_{n=1}^{N} \alpha_n^2 + 2 \max_{n=1,2,\dots,N} \alpha_n^2.$$

Consequently, Dynkin formula says that

$$\begin{aligned} e_N(t) &= \mathbb{E}[V_N(c_k(t):k=1,2,\dots,N)] \\ &= \mathbb{E}[V_N(c_k(0):k=1,2,\dots,N)] + \mathbb{E}\Big[\int_0^t \mathcal{L}V_N(c_k(s):k=1,2,\dots,N)ds\Big] \\ &\le e(0) + \Big[b^2\sum_{n=1}^N \alpha_n^2\Big(\frac{\sigma^2 n^2\pi^2}{L^2}-a_1\Big) + a_2(b^2\beta_N^2)^{3/2}\left(\frac{1}{12a_2}\right)^{1/2}\frac{5}{6}\Big]t \end{aligned}$$

for $t \ge 0$. Since $e_N \ge 0$ is increasing in N and uniformly bounded in time t for any $t \in [0,T]$, we know that the limit $\lim_{N\to+\infty} e_N(t)$ exists, $e(t) = \lim_{N\to+\infty} e_N(t)$ and

$$0 \le e(t) \le e(0) + \Big[b^2\sum_{n=1}^\infty \alpha_n^2\Big(\frac{\sigma^2 n^2\pi^2}{L^2}-a_1\Big) + a_2(b^2\beta^2)^{3/2}\big(\frac{1}{12a_2}\big)^{1/2}\frac{5}{6}\Big]t$$

for $t\in[0,T]$, as long as $e(0) < +\infty$, $\sigma^2\pi^2 \ge L^2 a_1$, and $\mathrm{trace}(Q) = \sum_{n=1}^\infty \alpha_n^2 < +\infty$. This completes the proof of Theorem 4.1. □

5. Numerical Methods for Fourier Coefficients c_k

Recall the form of Fourier solutions u and its approximate Fourier solutions u_N given by

$$u_N(t,x) = \sum_{k=1}^N c_k(t)\sqrt{\frac{2}{L}}\sin\big(\frac{k\pi x}{L}\big)$$

with its coefficients c_k satisfying (3.4) (see Remark 3.2). An explicit solution of the system of nonlinear equations for c_k is not known under the presence of nonlinearities with $a_2 > 0$. Thus, one has to resort to numerical approximations. For $k \in \mathbb{N}$, set

$$b_k = b\alpha_k.$$

Along partitions

$$t_0 = 0 < t_1 < t_2 < \dots < t_{n_T} = T$$

of time-intervals $[0,T]$ with current step sizes $h_n = t_{n+1} - t_n > 0$, consider the forward Euler method (FEM) for c_k,

$$c_k(n+1) = c_k(n) + h_n c_k(n)\Big(-\sigma^2\frac{k^2\pi^2}{L^2} + a_1 - a_2\sum_{l=1}^N [c_l(n)]^2\Big) + b_k\Delta W_n^k \quad (5.1)$$

where

$$\Delta W_n^k = W_k(t_{n+1}) - W_k(t_n) \in \mathcal{N}(0,h_n), \quad h_n = t_{n+1}-t_n.$$

An alternative to is given by the backward Euler method (BEM)

$$c_k(n+1) = c_k(n) + h_n c_k(n+1)\Big(-\sigma^2\frac{k^2\pi^2}{L^2} + a_1 - a_2\sum_{l=1}^N [c_l(n+1)]^2\Big) + b_k\Delta W_n^k \quad (5.2)$$

where

$$\Delta W_n^k = W_k(t_{n+1}) - W_k(t_n) \in \mathcal{N}(0,h_n), \quad h_n = t_{n+1}-t_n.$$

Our favorite choice is the linear-implicit Euler-type method (LIM)

$$c_k(n+1) = c_k(n) + h_n c_k(n+1)\Big(-\sigma^2\frac{k^2\pi^2}{L^2} + a_1 - a_2\sum_{l=1}^N [c_l(n)]^2\Big) + b_k\Delta W_n^k \quad (5.3)$$

where

$$\Delta W_n^k = W_k(t_{n+1}) - W_k(t_n) \in \mathcal{N}(0, h_n), \quad h_n = t_{n+1} - t_n.$$

The disadvantage of FEM (5.1) is their lack of stability (in fact substability) (see [28, 29, 30, 31, 32]) and monotonicity deficits. Moreover, global convergence and its rates have not been shown for nonlinear equations with nonLipschitzian coefficients. The advantage of methods (5.2) and (5.3) is seen with respect to their good stability and moment dissipativity behavior, and they keep some monotonicity properties (see [28, 29, 30, 31, 32]). Besides, convergence has been shown for some nonlinear equations with nonLipschitzian coefficients (e.g. see [15, 34]). A slight disadvantage of methods (5.2) is given by their superstable behavior and by the necessity to solve locally implicit algebraic equations at each iteration step n. The latter problem is more computationally efficiently solved by our methods (5.3) where no implicit algebraic equations need to be solved due to their linear-implicit character which can be naturally managed in explicit representation form. Note that the local solvability of those implicit algebraic equations exhibited by methods (5.2) needs to be discussed and it would lead to additional computational errors which could impact significantly the accuracy of approximations in the course of numerical integration.

Theorem 5.1 (Explicit Representation + Stability of Methods (LIM)). *Suppose that*

$$a_2 \geq 0, \quad \forall n \in \mathbb{N} : (a_1 - \sigma^2\pi^2/L^2)h_n < 1.$$

Then the method (LIM) governed by (5.3) *has the nonexploding explicit representation*

$$c_k(n+1) = \frac{c_k(n) + b_k \Delta W_n^k}{1 + h_n\Big(\sigma^2\frac{k^2\pi^2}{L^2} - a_1 + a_2\sum_{l=1}^{N}[c_l(n)]^2\Big)} \tag{5.4}$$

where $n \in \mathbb{N}$, $b_k = b\alpha_k$ and $\Delta W_n^k \in \mathcal{N}(0, h_n)$. Moreover, if $\sigma^2\pi^2 \geq a_1L^2$, their second moments are linearly bounded in time t; i.e.,

$$\mathbb{E}[c_k(n+1)]^2 \leq \mathbb{E}[c_k(n)]^2 + (b_k)^2h_n \leq \mathbb{E}[c_k(0)]^2 + (b_k)^2\ t_{n+1} \tag{5.5}$$

for all $k = 1, 2, \dots, N$, where $n \in \mathbb{N}$. Hence, we have in the limit (as both $N \to +\infty$ and $h_n \to 0+$)

$$\mathbb{E}[\|u(t_n, \cdot)\|_H^2] = \sum_{k=1}^{\infty}\mathbb{E}[c_k(n)]^2 \leq \sum_{k=1}^{\infty}\mathbb{E}[c_k(0)]^2 + b^2\sum_{k=1}^{\infty}\alpha_k^2\ t_n \tag{5.6}$$

which replicates the consistent estimate of second moments of underlying exact solution u in the course of integration, provided that

$$\frac{\sigma^2\pi^2}{L^2} \geq a_1, \quad \mathbb{E}[\|u(0, \cdot)\|_H^2] = \sum_{k=1}^{\infty}\mathbb{E}[c_k(0)]^2 < +\infty, \quad \sum_{k=1}^{\infty}\alpha_k^2 < +\infty.$$

Proof. Suppose that $1 + h_n(\sigma^2\pi^2/L^2 - a_1) > 0$. The explicit representation (5.4) is finite and a rather obvious result due to the linear-implicit character of method

(5.3). It remains to consider the second moments

$$\begin{aligned}
\mathbb{E}[c_k(n+1)]^2 &= \mathbb{E}\Big[\frac{c_k(n)+b_k\Delta W_n^k}{1+h_n\Big(\sigma^2\frac{k^2\pi^2}{L^2}-a_1+a_2\sum_{l=1}^N[c_l(n)]^2\Big)}\Big]^2 \\
&= \mathbb{E}\Big[\frac{[c_k(n)]^2+2c_k(n)\Delta W_n^k+b_k^2(\Delta W_n^k)^2}{\Big[1+h_n\Big(\sigma^2\frac{k^2\pi^2}{L^2}-a_1+a_2\sum_{l=1}^N[c_l(n)]^2\Big)\Big]^2}\Big] \\
&= \mathbb{E}\Big[\frac{[c_k(n)]^2+b_k^2h_n}{\Big[1+h_n\Big(\sigma^2\frac{k^2\pi^2}{L^2}-a_1+a_2\sum_{l=1}^N[c_l(n)]^2\Big)\Big]^2}\Big]
\end{aligned}$$

since the increments $\Delta W_n^k = W_k(t_{n+1})-W_k(t_n) \in \mathcal{N}(0,h_n)$ are independent (Here, note that we exploited a tower property of conditional expectations). Now, suppose that $\sigma^2\pi^2 \geq L^2a_1$. In this case one can estimate these second moments as stated by (5.5). Finally, the relation (5.5) is summed over k to verify the claim (5.6) of Theorem 5.1. □

Recall the following definition (e.g. see [32, 34]). Let c_k^h denote the numerical approximation of the k-th Fourier coefficients c_k along partitions of fixed time-intervals $[0,T]$ of the form

$$0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots < t_{n_T} = T.$$

Then the numerical approximation $c^h = (c_k^h)_{k=1,2,\dots,N}$ is said to be **mean consistent** with **rate** r_0 iff there are a constant $C_0 = C_0(T)$ and a positive continuous function V (or functional) such that

$$\forall n = 0,1,2,\dots,n_T-1 : \|\mathbb{E}[c(n+1)] - \mathbb{E}[c^h(n+1)]\|_N \leq C_0V(c(n))h_n^{r_0}$$

along any (nonrandom) partitions with sufficiently small step sizes $h_n \leq \delta \leq 1$, where $\|\cdot\|_N$ is the Euclidean vector norm in $\mathbb{R}^N$, provided that one has nonrandom data $c(n) = c^h(n)$. Moreover, the numerical approximation $(c_k^h)_{k=1,2,\dots,N}$ is said to be **p-th mean consistent** with **rate** r_p if and only if there are a constant $C_p = C_p(T)$ and a positive continuous function V (or functional) such that

$$\forall n = 0,1,2,\dots,n_T-1 : \Big(\mathbb{E}\Big[\|c(t_{n+1}) - c^h(n+1)\|_N^p\Big]\Big)^{1/p} \leq C_pV(c(t_n))h_n^{r_p}$$

along any (nonrandom) partitions with sufficiently small step sizes $h_n \leq \delta \leq 1$, where $\|\cdot\|_N$ is the Euclidean vector norm in $\mathbb{R}^N$, provided that one has nonrandom data $c(t_n) = c^h(n)$. Note that the choice of vector norm $\|\cdot\|_N$ in $\mathbb{R}^N$ is not so essential for the qualitative property of consistency due to the equivalence of all vector norms in $\mathbb{R}^N$ (only the constants C_p and functional V could differ for different norms).

Theorem 5.2. *The method (LIM) governed by* (5.3) *is mean consistent with rate* $r_0 = 1.5$ *and p-th mean consistent with rate* $r_p = 1.0$*, where* $p \geq 1$.

Proof. Let c^h be governed by the method (5.3). Suppose that we have nonrandom local initial data satisfying

$$c(t_n) = c^h(n)$$

along partitions $(t_n)_{n\in\mathbb{N}}$ of time-intervals $[0,T]$ with current step sizes $h_n = t_{n+1} - t_n \leq 1$. Let $\alpha = diag(\alpha_1,\alpha_2,\dots,\alpha_N)$ be the diagonal matrix in $\mathbb{R}^{N\times N}$ with diagonal entries α_k and W the N-dimensional vector of the Wiener processes W_k. Furthermore, define

$$f_h(c^h(n)) = \operatorname{diag}\Big(\frac{-\sigma^2\frac{k^2\pi^2}{L^2} + a_1 - a_2\sum_{l=1}^N [c_l^h(n)]^2}{1 + h_n\Big(\sigma^2\frac{k^2\pi^2}{L^2} - a_1 + a_2\sum_{l=1}^N [c_l^h(n)]^2\Big)}\Big)(c^h(n) + b\Delta W_n),$$
$$g_h(c(n)) = b\alpha$$

where $c(n)$ is the vector of Fourier coefficients $c_k(n)$ for all $n \in \mathbb{N}$. Besides, note that the method (5.3) poessesses the explicit one-step representation

$$c^h(n+1) = c^h(n) + f_h(c^h(n))h_n + g_h(c^h(n))\Delta W_n.$$

Consider the property of mean consistency by estimating

$$\begin{aligned}
&\|\mathbb{E}[c(t_{n+1}) - c^h(n+1)]\|_N \\
&= \Big\|\mathbb{E}[c(t_n) + \int_{t_n}^{t_{n+1}} f(c(s))ds \\
&\quad + b\alpha\int_{t_n}^{t_{n+1}} dW(s) - c^h(n) - f_h(c^h(n))h_n - g_h(c^h(n))\Delta W_n]\Big\|_N \\
&= \Big\|\mathbb{E}[\int_{t_n}^{t_{n+1}} f(c(s))ds - f_h(c(t_n))h_n]\Big\|_N \quad (\text{since } c^h(n) = c(t_n)) \\
&= \|\mathbb{E}\Big[\int_{t_n}^{t_{n+1}} [f(c(s)) - f_h(c(n))]ds\Big]\|_N \\
&= \|\int_{t_n}^{t_{n+1}} \mathbb{E}[f(c(s)) - f_h(c(t_n))]ds\|_N \quad (\text{for nonrandom partitions } (t_n)_{n\in\mathbb{N}}) \\
&\le \mathbb{E}\Big[\int_{t_n}^{t_{n+1}} \|f(c(s)) - \bar{f}_h(c(t_n))\|_N ds\Big] \quad (\text{due to } \Delta \text{ -inequality}) \\
&\le \mathbb{E}\Big[\int_{t_n}^{t_{n+1}} \|f(c(s)) - f(c(t_n))\|_N ds\Big] + \mathbb{E}\Big[\int_{t_n}^{t_{n+1}} \|f(c(t_n)) - \bar{f}_h(c(t_n))\|_N ds\Big] \\
&\le C_0(1 + [V(c(t_n))]^2)h_n^{3/2}
\end{aligned}$$

where V is the Lyapunov functional (4.2) with appropriate constant C_0 and

$$\bar{f}_h(c(t_n)) = \operatorname{diag}\Big(\frac{-\sigma^2\frac{k^2\pi^2}{L^2} + a_1 - a_2\sum_{l=1}^N [c_l(t_n)]^2}{1 + h_n\Big(\sigma^2\frac{k^2\pi^2}{L^2} - a_1 + a_2\sum_{l=1}^N [c_l(t_n)]^2\Big)}\Big)c(t_n)$$

and

$$f(c(s)) = \operatorname{diag}\Big(-\sigma^2\frac{k^2\pi^2}{L^2} + a_1 - a_2\sum_{l=1}^N [c_l(s)]^2\Big)c(s).$$

Thus, the method (5.3) has at least a mean consistency rate $r_0 \ge 1.5$. Similarly, one may establish an estimation of the rate $r_p = 1.0$ of p-th mean consistency for $p \ge 1$. Consequently, the proof of Theorem 5.2 can be completed. □

Anyway, a detailed simulation study using those methods and comparing them to others with respect to their performance should follow. An overview of standard numerical methods for SDEs can be found in [1, 6, 10, 17, 24, 32, 36, 42] among others. For SPDEs with Lipschitzian coefficients, direct standard difference methods and finite element techniques have also been investigated, e.g. see

[11, 13, 26, 40, 45, 46]. It can be shown that some nonstandard methods such as the linear-implicit method possess an expected total energy which is linearly bounded in time (a fact which shows its dynamical consistency with the estimates from Section 4). However, this requires much more explanations and space, and hence it is beyond of the scope of this paper.

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