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# A PRIORI ESTIMATES FOR A CRITICAL SCHRÖDINGER-NEWTON EQUATION

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Abstract. Under natural energy and decay assumptions, we derive a priori estimates for solutions of a Schrödinger-Newton type of equation with critical exponent. On the one hand, such an equation generalizes the traditional Schrödinger-Newton and Choquard equations; while, on the other hand, it is naturally related to problems involving scalar curvature and conformal deformation of metrics.

## 1. Introduction

We shall study the behavior of positive solutions to the equation

$$\Delta u + \Big(\frac{1}{|x|^{\ell}} * u^{\frac{2n}{n-2}}\Big) u^{\frac{n+2}{n-2}} - Vu = 0 \tag{1.1}$$

in $\mathbb{R}^n$, $n \geq 3$, where $\Delta$ is the Euclidean Laplacian, $*$ means convolution, $\ell$ is a real number, and $V$ is a smooth real valued function. Equation (1.1) will be referred to as *critical Schrödinger-Newton equation*. We are concerned with a priori estimates; i.e., bounds on $u$ and its derivatives, which are necessary conditions for any (positive) solution of (1.1). Needless to say, not only are a priori estimates one of the main tools towards an existence theory for a given PDE, but also they reveal important features about the behavior of solutions. The form of the bounds one generally seeks to establish depends, of course, on specific characteristics of the equation, which in the case of (1.1), will be captured by suitable hypotheses on $\ell$, $V$ and asymptotic conditions for $u$. In order to state natural assumptions for the critical Schrödinger-Newton equation as well as to highlight why one would consider (1.1) in the first place, we first turn our attention to some related problems.

Recall that the Schrödinger-Newton equation is

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi - Gm^2\Big(\frac{1}{|x|} * |\Psi|^2\Big)\Psi, \quad x \in \mathbb{R}^3, \tag{1.2}$$

where $\Psi = \Psi(t,x)$ is a function on $\mathbb{R}\times\mathbb{R}^3$, and $m$, $\hbar$ and $G$ are constants. Physically, $\Psi$ is the wave-function of a self-gravitating quantum system of mass $m$ with gravitational interaction given by Newton's law of gravity; $\hbar$ and $G$ are Planck's

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and Newton's constants, respectively. Equation (1.1) is obtained by considering the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + mU\Psi \tag{1.3}$$

with a Newtonian gravitational potential $U$ that is sourced by a distribution of mass given by $\Psi$ itself,

$$\Delta U = 4\pi Gm|\Psi|^2. \tag{1.4}$$

In other words, the mass distribution is given in probabilistic terms, with its probability amplitude evolving according to the Schrödinger equation, as is usual in quantum mechanical systems. Writing $U$ in terms of the right hand side of (1.4) and the fundamental solution of the Laplacian, and using the resulting expression into (1.3), formally produces (1.2).

The Schrödinger-Newton equation was first introduced by Ruffini and Bonazzola in their study of equilibrium of self-gravitating bosons and spin-half fermions [60] and gained notoriety with Penrose's ideas about the role of gravity in the collapse of the wave function [55, 56]. More recently, it was used in discussions of semi-classical quantum gravity [14, 36, 59].

Notice that the power of the convoluted term $\frac{1}{|x|}$ becomes $n-2$ in higher dimensions. Parallel to this situation, the following generalization of (1.2) has been considered,

$$-i\frac{\partial\varphi}{\partial t} = \Delta\varphi + p\Big(\frac{1}{|x|^\ell} * |\varphi|^p\Big)\varphi|\varphi|^{p-2}, \quad x\in\mathbb{R}^n, \tag{1.5}$$

where $p \geq 2$ and $\ell \in (0,n)$. For the remainder of the paper, as in (1.5), dimensional constants such as $\hbar$ and $G$ are set to one. Equation (1.5) is used in certain approximating regimes of the Hartree-Fock theory for a one component plasma; see e.g. [43, 44]. In three spatial dimensions, with $\ell = 1$ and $p = 2$, equation (1.5) has been extensively studied, see [7, 15, 33, 35, 45, 46, 58, 64] and references therein.

As in many situations in Physics, one is particularly interested in wave-front-like solutions of the form $\varphi(t,x) = e^{i\omega t}u(x)$, $\omega\in\mathbb{R}$, which, upon plugging into (1.5), leads to

$$\Delta u + \omega u + p\Big(\frac{1}{|x|^\ell} * |u|^p\Big)u|u|^{p-2} = 0.$$

Considering the more general situation where $\omega \mapsto -V = -V(x)$ and dropping the factor $p$ in front of the convolution, we find

$$\Delta u - Vu + \Big(\frac{1}{|x|^\ell} * |u|^p\Big)u|u|^{p-2} = 0, \tag{1.6}$$

which is referred to as generalized non-linear Choquard equation. A detailed study of (1.6), including existence results, has been recently carried out by Ma and Zhao [49]; Cingolani, Clapp and Secchi [19]; Clapp and Salazar [20]; and Moroz and van Schaftingen [53]. These works deal with the case where the exponent $p$ is sub-critical, i.e., $p < \frac{2n}{n-2}$, while the case $p = \frac{2n}{n-2}$ is called critical. Criticality is here understood in the usual sense of the Sobolev embedding theorems. We recall that, roughly speaking, equations with sub-critical non-linearity are suited for treatment via calculus of variation techniques (see e.g. [5]), provided that the equation can be derived from an action principle — which is the case for (1.6), see the above references.

Equations with critical exponent appear in several situations in Physics and Mathematics (see e.g. [5, 18, 40] and references therein). One important case is the Yamabe equation

$$\Delta_g u - \frac{n-2}{4(n-1)} R_g u + K u^{\frac{n+2}{n-2}} = 0, \ u > 0, \tag{1.7}$$

where $\Delta_g$ and $R_g$ are, respectively, the Laplace-Beltrami operator and the scalar curvature of a given metric $g$, and $K$ is a constant. Equation (1.7) figures in the famous Yamabe problem [66]. We recall that this corresponds to finding a constant scalar curvature metric in the conformal class of a given closed[1] $n$ dimensional ($n \geq 3$) Riemannian manifold. The complete solution of the Yamabe problem through the works of Yamabe [66], Trudinger [65], Aubin [6] and Schoen [62] was probably the first instance of a satisfactory existence theory for equations with critical non-linearity (see [42] for a complete overview). The analogous equation for the Euclidean metric was studied in great detail by Caffarelli, Gidas and Spruck [13].

Interest in the Yamabe problem has not faded with its resolution. On the contrary, the discovery of Pollack, that it is possible to find an arbitrary large number of solutions to the Yamabe equation on manifolds with positive Yamabe invariant, has led to an intensive investigation of the properties of the space $\Phi$ of solutions to (1.7) — see [42] for a definition of the Yamabe invariant and [57] for a precise statement of Pollack's result. A quite satisfactory account of the topology of $\Phi$ was given through the combined works of Khuri, Marques and Schoen [41]; Brendle [9]; and Brendle and Marques [11] (see also [22, 47, 48, 52, 61, 63] for earlier results). These results imply that $\Phi$ is compact in the $C^{2,\alpha}$ topology for $n \leq 24$ and non-compact otherwise[2]. Such results were extended to manifolds with boundary in [21].

From an analytic perspective, the richness surrounding equation (1.7), including the surprising cut-off in dimension $n = 24$, is a direct consequence of the critical exponent. It should be expected, therefore, that allowing $p = \frac{2n}{n-2}$ in (1.5) will lead to many interesting new phenomena, adding to the already sophisticated nature of the generalized non-linear Choquard equation. A contribution in this direction is the goal of the present work. In order not to lose sight of the relation between what has been just described and our objectives in the rest of the manuscript, notice that from the point of view of the theory of partial differential equations, the aforementioned compactness of $\Phi$ corresponds to a priori bounds for solutions of (1.7). The reader should also notice the similarities between (1.1) and (1.7), specially if we are given a metric $g$ in $\mathbb{R}^n$, with $\Delta_g$ replacing $\Delta$ and $V$ being the scalar curvature.

We shall present a priori estimates for positive solutions to (1.1). Such estimates constitute the first step towards an existence theory for this equation. They also provide insight on the structure of the space of solutions to (1.1), at least for those solutions satisfying some additional requirements. We also give an account

[1]The Yamabe problem for manifolds with boundary was studied in [2, 3, 4, 10, 16, 21, 26, 27, 28, 31, 32, 37, 38, 50, 51].

[2]This under the assumption that the Yamabe invariant is positive, and the underlying manifold is not conformally equivalent to the round sphere. The cases of negative and zero Yamabe invariant are trivial. The geometric reasons for singling out the sphere, and the relation between the compactness of $\Phi$ and the geometry of the manifold, are discussed in [12, 54].

of the profile of blowing-up solutions. We point out that since, to the best of our knowledge, equation (1.1) has not been considered before in the literature, we shall not attempt to derive very general results; rather, our focus will be on conditions that allow, on one hand, a good grasp on the behavior of $u$ without, on the other hand, rendering the problem uninteresting. We also stress that our methods may shed new light in the study of equations (1.2) and (1.5), in that we investigate the pointwise behavior of solutions as opposed to the $L^2$ techniques previously employed to deal with these equations. We make some general comments on the pointwise blow-up techniques we employ in section 4.

## 2. Setting and statement of the results

**Notation 2.1.** From now on, $u$ will denote a positive solution of (1.1).

The first thing we investigate is the range of $\ell$ values which will be allowed. For the integral

$$\frac{1}{|x|^\ell} * u^{\frac{2n}{n-2}} = \int_{\mathbb{R}^n} \frac{1}{|y|^\ell} u^{\frac{2n}{n-2}}(x-y)\,dy \tag{2.1}$$

to be finite near the origin without expecting any vanishing of $u$ in the neighborhood of zero, we must have $\ell \in (0,n)$. Next, we ask what kind of asymptotic behavior should be required for $u$. Experience with equations with critical exponent [13, 42] suggests that we should adopt

$$u = O(|x|^{2-n}) \text{ as } |x| \to \infty. \tag{2.2}$$

Then (2.2) and $\ell \in (0,n)$ guarantee that the integral (2.1) is finite.

**Definition 2.2.** Given real numbers $\varrho > 0$ and $L > 0$, we say that $u$ has $(\varrho, L)$-*decay* if it satisfies

$$u(x) \le L|x|^{2-n}, \quad \text{for } |x| \ge \varrho.$$

Denote by $\mathcal{C}_{\varrho,L}$ the set of solutions $u$ with $(\varrho, L)$-decay.

We shall also need some energy conditions. In order to motivate them, multiply (1.1) by $u$, integrate by parts, and assume that all the integrals are finite. Then

$$\int_{\mathbb{R}^n} \left(|\nabla u|^2 + V u^2\right) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1}{|y|^\ell} u^{\frac{2n}{n-2}}(x-y) u^{\frac{2n}{n-2}}(x)\,dy\,dx. \tag{2.3}$$

The left-hand side is just the energy associated with the linear operator $\Delta - V$. If we had a constant rather than $\frac{1}{|x|^\ell} * u^{\frac{2n}{n-2}}$, the above expression would produce the analogue of the Yamabe quotient for our equation. This motivates the following.

**Definition 2.3.** We call the convolution $q_u(x) := \frac{1}{|x|^\ell} * u^{\frac{2n}{n-2}}$ the *quotient of* $u$ (which is always non-negative). For a given real number $K > 0$, denote by $\mathcal{Q}_K$ the set of solutions $u$ whose quotient is less than or equal to $K$. More precisely

$$\mathcal{Q}_K := \Big\{ u : q_u(x) \le K \text{ for all } x \in \mathbb{R}^n \Big\}.$$

**Remark 2.4.** Since $q_u$ is related to the energy on the left-hand side of (2.3), $u \in \mathcal{Q}_K$ can be thought of as an energy-type of condition. This should not be confused, however, with the more physically appealing notion of energy for (1.5) used in [19, 20, 49, 53].

To motivate the extra hypotheses that will be needed, we have to say a few words about the general situation that will be investigated. We shall employ blow-up techniques to study sequences $\{u_i\}$ of $C^{2,\alpha}$ solutions to (1.1), and we are primarily concerned with the constraints that the equation imposes on blow-up up sequences. Understanding how a sequence $\{u_i\}$ can be unbounded in, say, the $C^0$-norm, is important not only because such families of solutions are obstructions to the application of standard compactness arguments, but also because this type of behavior is expected for many critical equations (see [1, 8, 17, 23, 24, 29, 30, 39] and references therein).

Whenever blow-up occurs, i.e., $\|u_i\|_{C^0(\mathbb{R}^n)} \to \infty$ as $i \to \infty$, condition (2.2) restricts the blow-up to within a compact set, in which case, we can assume $u_i$ to diverge along a sequence of points $x_i \to \bar{x}$. An analysis of the sequence $\{u_i\}$ is carried out by rescaling the solutions and the coordinates, leading to an appropriate blow-up model for equation (1.1). In this situation, one expects that the blow up of $u_i$, together with $\{u_i\} \subset \mathcal{Q}_K$, implies that the rescaled $q_{u_i}$'s are very close to a constant in the neighborhood of $\bar{x}$. But in order to avoid substantial extra work that would distract us from the main goals of the paper, we shall simply assume that $q_{u_i}$ has this desired property. Moreover, although our analysis will be local in nature, to avoid the introduction of further cumbersome hypotheses, such an assumption will be taken to hold on a big compact set. The precise behavior of $q_u$ has to be ultimately determined by a more refined analysis of the solutions to equation (1.1), what is beyond the scope of this paper. A technical condition on the *rate* at which $q_{u_i}$ approaches a constant will also be assumed, although probably this can be relaxed. With this in mind we now state our results, whose essence is that *control over the convolution $q_u$ yields uniform control over the solutions.* From elliptic theory, one would expect that the required control on $q_u$ should be in the $C^{0,\alpha}$ topology, and that turns out to be in fact the case.

**Theorem 2.5.** *Fix positive numbers $\varrho$, $L$ and $K$. Let $u_i$ be a sequence of $C^{2,\alpha}$ positive solutions to* (1.1)*, $0 < \alpha < 1$, satisfying*

$$\{u_i\}_{i=1}^{\infty} \subset \mathcal{C}_{\varrho,L} \cap \mathcal{Q}_K,$$

*and suppose that there exists a constant $Q$ such that*

$$q_{u_i} \to Q > 0 \text{ in } C^{0,\alpha}(B_r(0)) \text{ as } i \to \infty,$$

*for some $r > \varrho$. Suppose further that $n \geq 6$.*

*If $\|u_i\|_{C^0(\mathbb{R}^n)} \to \infty$ and $\|q_{u_i} - Q\|_{C^{0,\alpha}(B_r(0))} \|u_i\|^{n-2}_{C^0(\mathbb{R}^n)} \to 0$ as $i \to \infty$, then, up to a subsequence, the following holds. There exist $\bar{x} \in B_\varrho(0)$, a sequence $x_i \to \bar{x}$ and a positive number $\sigma$ such that*

$$\|u_i\|_{C^0(\mathbb{R}^n)} = u_i(x_i),$$
$$\|(u_i(x_i))^{-1}u - (u_i(x_i))^{-1}z_i\|_{C^0(B_\sigma(\bar{x}))} \to 0 \text{ as } i \to \infty,$$

*where*

$$z_i(x) := (u_i(x_i))^{-1}\Big((u_i(x_i))^{-\frac{4}{n-2}} - \frac{Q}{n(n-2)}|x - x_i|^2\Big)^{\frac{2-n}{2}}.$$

*Furthermore, the following estimate holds*

$$\|(u_i(x_i))^{-1}u - (u_i(x_i))^{-1}z_i\|_{C^0(B_\sigma(\bar{x}))} \leq C(u_i(x_i))^{-\frac{4}{n-2}},$$

*where $C = C(L, \varrho, K, Q, r, n, \alpha, \|V\|_{C^{0,\alpha}(B_r(0))})$.*

**Remark 2.6.** The restriction to $n \geq 6$ is used to obtain the most direct proof without considering variations of (2.2). We believe that this can be removed by a more careful application of the techniques here presented.

Theorem 2.5 identifies a blow-up model for equation (1.1). In other words, it states that under suitable energy and decay conditions, and up to a subsequence, any family of solutions that blows up is approximated, after rescaling and near the blow up point $\bar{x}$, by the radially symmetric functions $z_i$. The following corollary says that if we also rescale the coordinates, then the $C^0$ convergence of theorem 2.5 is improved to $C^2$ convergence.

**Theorem 2.7.** *Assume the same hypotheses and notation of theorem 2.5. If $\|u_i\|_{C^0(\mathbb{R}^n)} \to \infty$ and $\|q_{u_i} - Q\|_{C^{0,\alpha}(B_r(0))}\|u_i\|^{n-2}_{C^0(\mathbb{R}^n)} \to 0$ as $i \to \infty$, letting $x_i$ and $\bar{x}$ be as in the conclusion of theorem 2.5, the following holds. Define*

$$\begin{aligned} v_i(y) &:= (u_i(x_i))^{-1} u_i(x_i + (u_i(x_i))^{\frac{2}{2-n}} y), \\ Z_i(y) &:= (u_i(x_i))^{-1} z_i(x_i + (u_i(x_i))^{\frac{2}{2-n}} y). \end{aligned}$$

*Then $Z_i(y) = (1 + \frac{Q}{n(n-2)}|y|^2)^{\frac{2-n}{2}} \equiv Z(y)$ for every $i$, and*

$$\|v_i - Z\|_{C^2(B_{\lambda_i}(0))} \to 0 \quad as \quad i \to \infty,$$

*where $\lambda_i = (u_i(x_i))^{\frac{2}{n-2}}\eta$, with $\eta$ a small positive number. Furthermore, the following estimate holds*

$$\|v_i - Z\|_{C^2(B_{\lambda_i}(0))} \leq C(u_i(x_i))^{-\frac{4}{n-2}},$$

*where $C = C(L, \varrho, K, Q, r, n, \alpha, \|V\|_{C^{0,\alpha}(B_r(0))})$.*

The function $Z$ in theorem 2.7 is well known in conformal geometry. The metric on the sphere written in a coordinate chart via stereographic projection is conformal to the Euclidean metric, with the conformal factor being a multiple of $Z$. $Z$ has also been extensively used in the study of the Yamabe problem.

One naturally wonders about the boundedness of families of solutions to (1.1) as well as the possibility that sequences $u_i > 0$ degenerate in the limit; i.e., points where $\lim_{i\to\infty} u_i(x) = 0$. The next theorem gives some sufficient conditions for bounds from above and below on $u$.

**Theorem 2.8.** *Assume the same hypothesis of theorem 2.5, suppose that $V \neq 0$ and that it does not change sign in $B_r(0)$. A sufficient condition for the existence of a constant $C = C(L, \varrho, K, Q, r, n, \alpha, \|V\|_{C^{0,\alpha}(B_r(0))})$ such that the inequalities*

$$\begin{aligned} \|u\|_{C^0(\mathbb{R}^n)} + \|u\|_{C^{2,\alpha}(B_\varrho(0))} &\leq C, \\ \|u\|_{C^0(B_\varrho(0))} &\geq \frac{1}{C} \end{aligned}$$

*hold for any $u \in \mathcal{C}_{\varrho,L} \cap \mathcal{Q}_K$, is that the inequality of theorem 2.7 be true in the $C^0$-norm with $(u_i(x_i))^{-\frac{4}{n-2}}$ replaced by $(u_i(x_i))^{-\frac{4}{n-2}-\delta}$, for some $\delta > 0$.*

## 3. Proof of theorems

The hypotheses and notation of theorem 2.5 will be assumed throughout this section. Let $\{u_i\}$ be such that $\|u_i\|_{C^0(\mathbb{R}^n)} \to \infty$ as $i \to \infty$. Since $u$ has $(\varrho, L)$-decay, we can assume that for large $i$

$$\|u_i\|_{C^0(\mathbb{R}^n)} = \|u_i\|_{C^0(B_\varrho(0))}.$$

Letting $x_i$ be such that $\|u_i\|_{C^0(\overline{B_\varrho(0)})} = u(x_i)$, and passing to a subsequence we can assume that $x_i \to \bar{x}$ for some $\bar{x} \in \overline{B_\varrho(0)}$, and taking $i$ sufficiently large if necessary we can also suppose that $x_i$ and $\bar{x}$ are interior points. Define a sequence of real numbers $\{\varepsilon_i\}_{i=1}^{\infty}$ by

$$\varepsilon_i^{\frac{2-n}{2}} := u_i(x_i).$$

Notice that $\varepsilon_i \to 0$ when $i \to \infty$. $\varepsilon_i$ measures the rate at which $u_i$ blows-up. For each fixed $i$, consider the change of coordinates

$$y = \varepsilon_i^{-1}(x - x_i),$$

and define the rescaled functions

$$v_i(y) := \varepsilon^{\frac{n-2}{2}} u_i(x_i + \varepsilon_i y).$$

Then 0 is a local maximum for $v_i$ with $v_i(0) = 1$ and

$$0 < v_i \le 1. \tag{3.1}$$

A direct computation shows that $v_i$ satisfies

$$\Delta v_i + \widetilde{q}_i v_i^{\frac{n+2}{n-2}} - \varepsilon_i^2 \widetilde{V} v_i = 0, \tag{3.2}$$

where $\widetilde{q}_i(y) = q_{u_i}(x_i + \varepsilon_i y)$ and $\widetilde{V}(y) = V(x_i + \varepsilon_i y)$. Let $z_i$ and $Z$ be as in theorems 2.5 and 2.7. In the sequel, we shall evoke several standard estimates of elliptic theory. A full account of these results can be found in [34].

**Lemma 3.1.** *With the above definitions, up to a subsequence, it holds that $v_i \to Z$ in $C^2_{\rm loc}(\mathbb{R}^n)$.*

*Proof.* Fix $R > 0$ and $d > 0$, and set $R' = R + d$, $R'' = R + 2d$. For any $p > 1$, we have by $L^p$ estimates that

$$\|v_i\|_{W^{2,p}(B_{R'}(0))} \le C\big(\|v_i\|_{L^p(B_{R''}(0))} + \|\widetilde{q}_i v_i^{\frac{n+2}{n-2}}\|_{L^p(B_{R''}(0))} + \varepsilon_i^2 \|\widetilde{V} v_i\|_{L^p(B_{R''}(0))}\big),$$

where $C = C(n, p, R', R'')$ and $W^{2,p}$ is the usual Sobolev space of functions with 2 weak derivatives in $L^p$. From (3.1) and our hypothesis it follows that

$$\|v_i\|_{W^{2,p}(B_{R'}(0))} \le C(1 + \varepsilon_i^2) \le 2C,$$

for large $i$, where $C = C(n, p, R', R'', K, \|\widetilde{V}\|_{C^0(B_{R''}(0))})$. Choosing $p$ such that $2 > n/p$, we then obtain by the Sobolev embedding theorem that $v_i$ is bounded in $C^{1,\alpha'}(B_{R'}(0))$, for some $0 < \alpha' < 1$, therefore there exists a subsequence, still denoted $v_i$, which converges in $C^{1,\alpha}(B_{R'}(0))$, $0 < \alpha < \alpha'$, to a limit $v_\infty$.

Next, evoke Schauder estimates to obtain

$$\begin{aligned} &\|v_i\|_{C^{2,\alpha}(B_R(0))} \\ &\le C\big(\|v_i\|_{C^0(B_{R'}(0))} + \|\widetilde{q}_i v_i^{\frac{n+2}{n-2}}\|_{C^{0,\alpha}(B_{R'}(0))} + \varepsilon_i^2 \|\widetilde{V} v_i\|_{C^{0,\alpha}(B_{R'}(0))}\big), \end{aligned}$$

where $C = C(n, \alpha, R, R')$. Combining this inequality, the interpolation inequality, the previous bound on the $C^{1,\alpha}$-norm of $v_i$ and the hypotheses of theorem 2.5, we conclude that

$$\|v_i\|_{C^{2,\alpha}(B_R(0))} \le C,$$

with $C = C(n, \alpha, p, R, d, K, Q, \|\widetilde{V}\|_{C^0(B_{R''}(0))})$. As a consequence, up to a subsequence and decreasing $\alpha$ if necessary, the above $C^{1,\alpha}$ convergence is in fact $C^{2,\alpha}$ convergence, and we can pass to the limit in (3.2) to conclude that $v_\infty$ satisfies

$$\Delta v_\infty + Q v_\infty^{\frac{n+2}{n-2}} = 0 \text{ in } B_R(0).$$

Take now a sequence $R_j \to \infty$. Using the above argument on each $B_{R_j}(0)$ along with a standard diagonal subsequence construction produces a subsequence $v_i$ that converges to a limit $v_\infty$ satisfying

$$\Delta v_\infty + Q v_\infty^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbb{R}^n, \tag{3.3}$$

with the convergence being in $C^{2,\alpha}$ on each fixed $B_{R_j}(0)$. Therefore $v \to v_\infty$ in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^n)$. Solutions to (3.3) have been studied by Caffarelli, Gidas and Spruck in [13]. From their results and the fact $v_\infty(0) = 1$, we obtain

$$v_\infty(y) = \Big(1 + \frac{Q}{n(n-2)}|y|\Big)^{\frac{2-n}{2}} \equiv Z(y).$$

□

**Proposition 3.2.** *There exists a constants $C > 0$, independent of $i$, such that*

$$|v_i - Z|(y) \le C\varepsilon_i^2 \text{ for every } |y| \le \varepsilon_i^{-1},$$

*possibly after passing to a subsequence.*

*Proof.* Set

$$A_i = \max_{|y| \le \varepsilon_i^{-1}} |v_i - Z| = |v_i - Z|(y_i),$$

where $y_i$ is defined by this relation as a point where the maximum of $|v_i - Z|$ is achieved for $|y| \le \varepsilon_i^{-1}$. Suppose first that, up to a subsequence, $|y_i| \le \frac{\varepsilon_i^{-1}}{2}$. If the result is not true, then there exists a subsequence such that

$$\frac{\varepsilon_i^2}{A_i} \to 0 \text{ as } i \to \infty. \tag{3.4}$$

Define $w_i(y) = A_i^{-1}(v_i - Z)(y)$. Then it satisfies

$$\Delta w_i + a_i w_i = \frac{\varepsilon_i^2}{A_i}\Big(\widetilde{V} v_i + \frac{Q - \widetilde{q}_i}{\varepsilon_i^2} v_i^{\frac{n+2}{n-2}}\Big), \tag{3.5}$$

where

$$a_i = Q\frac{v_i^{\frac{n+2}{n-2}} - Z^{\frac{n+2}{n-2}}}{v_i - Z}.$$

By Taylor's theorem,

$$v_i^{\frac{n+2}{n-2}} - Z^{\frac{n+2}{n-2}} = \frac{n+2}{n-2} Z^{\frac{4}{n-2}}(v_i - Z) + O(Z^{\frac{6-n}{n-2}}|v_i - Z|^2). \tag{3.6}$$

From our hypotheses, expansion (3.6), the $(\varrho, L)$-decay of $u$, lemma 3.1, (3.4) and equation (3.5), it follows by an argument similar to the proof of lemma 3.1, that $w_i$ is bounded in $C^2_{\text{loc}}(\mathbb{R}^n)$ and

$$|a_i(y)| \le \frac{C}{(1+|y|)^4}, \tag{3.7}$$

for some constant $C > 0$ independent of $i$. Passing to a subsequence, we see that $w_i \to w_\infty$ in $C^2_{\rm loc}(\mathbb{R}^n)$, and that $w_\infty$ satisfies

$$\Delta w_\infty + \frac{n+2}{n-2} Q Z^{\frac{4}{n-2}} w_\infty = 0. \tag{3.8}$$

On the other hand, using the Green's function for the Laplacian with Dirichlet boundary condition, the representation formula and (3.7) show that for $|y| \le \frac{\varepsilon_i^{-1}}{2}$,

$$|w_i(y)| \le \frac{C}{1+|y|} + C\frac{\varepsilon_i^2}{A_i}, \tag{3.9}$$

for some constant $C > 0$ independent of $i$. In particular, $w_\infty$ has the property that

$$\lim_{|y|\to\infty} w_\infty(y) = 0. \tag{3.10}$$

Up to a harmless rescaling, solutions to (3.8) with the property (3.10) have also been studied by Caffarelli, Gidas and Spruck in [13]. They have the property that $w_\infty \equiv 0$ if $w_\infty(0) = 0 = |\nabla w_\infty(0)|$. Recalling that $v_i(0) = 1$ and that 0 is a local maximum of $v_i$, and noticing that $Z(0) = 1$, $\nabla Z(0) = 0$, we see that $w_i(0) = 0 = |\nabla w_i(0)|$. Therefore, $w_\infty(0) = 0 = |\nabla w_\infty(0)|$ holds, and hence $w_\infty$ vanishes identically. Since $w_i(y_i) = 1$ by construction, we must have $|y_i| \to \infty$, but this contradicts (3.9) because of (3.4) and $w_i(y_i) = 1$.

It remains to prove the proposition in the case when $|y_i| > \frac{\varepsilon_i^{-1}}{2}$. Notice that from the $(\varrho, L)$-decay of $u$, we obtain that $v_i(y) \le C|y|^{\frac{2-n}{2}}$ for $|y| \ge \varrho\varepsilon_i^{-1}$, while $Z$ obeys the estimate $Z(y) \le C(1+|y|)^{2-n}$. From these we get $|v_i - Z|(y_i) \le C|y_i|^{\frac{2-n}{2}} \le C\varepsilon_i^2$ if $|y_i| > \frac{\varepsilon_i^{-1}}{2}$. □

*Proof of theorem 2.5.* Notice that

$$z_i(x) = \varepsilon_i^{\frac{2-n}{2}} Z(\varepsilon_i^{-1} x).$$

Writing the estimate of proposition 3.2 in $x$-coordinates and recalling the definition of $\varepsilon_i$, we obtain

$$\|(u_i(x_i))^{-1} u - (u_i(x_i))^{-1} z_i\|_{C^0(B_1(x_i))} \le C\varepsilon_i^2.$$

Since $x_i \to \bar{x}$, choosing $\sigma > 0$ small and $i$ large, we obtain the result. □

*Proof of theorem 2.7.* For the $C^0$-norm this is simply the estimate of proposition 3.2 written in $x$-coordinates. From the $C^0$ bound, we obtain the $C^2$ bound by standard elliptic estimates, after considering a smaller ball via the introduction of $\eta$. □

*Proof of theorem 2.8.* We shall first show that $\|u_i\|_{C^0(\mathbb{R}^n)} \le C$ for a constant independent of $i$. If this is not the case, then we can find a sequence $u_i$ that blows up as in the assumptions of theorem 2.5. In light of elliptic theory and shrinking $\eta$ if necessary, we obtain that the hypothesis on the $C^0$ decay of $v_i - Z$ then yields the estimate

$$\|v_i - Z\|_{C^2(B_{\lambda_i}(0))} \le C\varepsilon_i^{2+\delta},$$

which then gives

$$\varepsilon_i^{\frac{n-2}{2}} \|\nabla^m (u_i - z_i)\|_{C^0(B_\eta(x_i))} \le C\varepsilon_i^{2+\delta-m}, \quad m = 0,1,2. \tag{3.11}$$

Notice that
$$\varepsilon_i^{\frac{n-2}{2}}\Delta(u_i - z_i) + Q\varepsilon_i^{\frac{n-2}{2}}\Big(u_i^{\frac{n+2}{n-2}} - z_i^{\frac{n+2}{n-2}}\Big) + \varepsilon_i^{\frac{n-2}{2}}\Big(q_{u_i} - Q\Big)u_i^{\frac{n+2}{n-2}} = \varepsilon_i^{\frac{n-2}{2}} V u_i. \quad (3.12)$$
From (3.11), and the hypotheses on $q_{u_i}$ we conclude that
$$\varepsilon_i^{\frac{n-2}{2}} Q\Big(u_i^{\frac{n+2}{n-2}} - z_i^{\frac{n+2}{n-2}}\Big)(x_i) + \varepsilon_i^{\frac{n-2}{2}}\Big(q_{u_i} - Q\Big)(x_i)u_i^{\frac{n+2}{n-2}}(x_i) \to 0 \quad \text{as } i \to \infty. \quad (3.13)$$
Inequality (3.11) also gives
$$\varepsilon_i^{\frac{n-2}{2}}\Delta(u_i - z_i)(x_i) \to 0 \text{ as } i \to \infty. \quad (3.14)$$
But (3.13) and (3.14) yield a contradiction with (3.12) since
$$\varepsilon_i^{\frac{n-2}{2}} u_i(x_i) = 1 \text{ for every } i,$$
and $V$ is bounded away from zero in $B_\varrho(0)$.

This establishes a bound on the $C^0$-norm of $u_i$. Restricting the problem to $B_r(0)$ and using standard elliptic estimates, produces a bound on the $C^2$-norm within $B_\varrho(0)$. With the $C^2$ bound at hand, we write
$$u_i^{\frac{n+2}{n-2}} = u_i^{\frac{4}{n-2}} u_i,$$
and consider $u_i^{\frac{4}{n-2}}$ as a given coefficient. In this case, we can treat the equation as a linear equation (for $u_i$) for which the Harnack inequality can be applied, producing the desired bound from below on $u_i$. □

## 4. Discussion

The use of blow-up techniques has a long history in PDE theory (see e.g. [25] for a general theory, and references therein for further applications). Such techniques provide a powerful tool to analyze the singular behavior of solutions of a given PDE. Singular behavior, in this context, generally means the existence of a point $\bar{x}$ such that $u(x) \to \infty$, in some appropriate topology, as $x \to \bar{x}$. More generally, singular behavior can also be understood as the existence of a family of solutions $u_i$ such that $u_i \to \infty$, again in some suitable topology, as $i \to \infty$.

In order to analyze these singularities, the conventional wisdom consists of finding the natural scaling symmetry for the equation. One then carries out a suitable rescale of all quantities — this is the so-called a blow-up of, say, the coordinates — in such a way that, in the limit, a singular model can be extracted from the rescaled equation. In many instances, the singular model is simple, yet retains much of the essential singular behavior of the original equation. When this is the case, a careful analysis of the singular model unveils important features of the singular structure of the problem under consideration.

In our case, we have applied these ideas to understand properties of singular, or blow-up, solutions of (1.1), for the reasons discussed in detail in the introduction. We finish by briefly mentioning how one can try to apply a similar set of ideas to the time-dependent problem (1.5), of which (1.2) is a particular case.

From the point of view of the time dependence, a blow-up behavior $u \to \infty$ is expected when $t \to T$, if the solution is defined on a time interval $[0, T)$, and cannot be extended pass $T$. Hence, the type of blow-up analysis mentioned above should capture some of the essential characteristics of the solution near $T$. Furthermore, this can provide us with explicit breakdown criteria that can be applied to particular examples when investigating the question of long-time existence to equation (1.5). Since the scaling properties of a singular equation are dictated essentially by its non-linear part, we expect that many of the arguments here presented can be adapted to the time-dependent setting.

Finally, our estimates can be also understood as describing the behavior of stationary solutions to (1.5) when these exist. As in many physical models, stationary solutions arise in the limit $t \to \infty$. Understanding the properties that such solutions ought to have (when they exist) can provide insights in the more difficult question of the long-time behavior of equation (1.5).

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## References

[1] A. Ambrosetti, A. Malchiodi; *A multiplicity result for the Yamabe problem on $S^n$*. J. Funct. Anal. 168, no. 2, 529-561 (1999).

[2] M. O. Ahmedou; *A Riemannian mapping type Theorem in higher dimensions, Part I: the conformally flat case with umbilic boundary.* in: Nonlinear equations: methods, models and applications (Bergamo 2001), volume 54 of Progr. Nonlinear Differential Equations Appl, pp 1-18, Birkhäuser, 2003.

[3] S. Almaraz; *A compactness theorem for scalar flat metrics on manifolds with boundary.* Calc. Var. Partial Differential Equations 41, no. 3-4, 341-386 (2011). arXiv: 0906.0927v1 [math.DG].

[4] S. Almaraz; *An existence theorem of conformal scalar-flat metrics on manifolds with boundary.* Pacific J. Math. 248 (2010), no. 1, 1-22. arXiv:0907.0798v1 [math.DG].

[5] T. Aubin; *Some Nonlinear Problems in Riemannian Geometry.* Springer (1998).

[6] T. Aubin; *Équations différentielles non linéaires et probléme de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. 55 269-296 (1976).

[7] R. Benguria, H. Brezis, E. H. Lieb; *The Thomas-Fermi-von Weizsäcker theory of atoms and molecules.* Comm. Math. Phys. 79, 167-180 (1981).

[8] M. Berti, A. Malchiodi; *Non-compactness and multiplicity results for the Yamabe problem on $S^n$.* J. Funct. Anal. 180, no. 1, 210-241 (2001).

[9] S. Brendle; *Blow-up phenomena for the Yamabe equation.* J. Amer. Math. Soc. 21, 951-979 (2008).

[10] S. Brendle, S. S. Chen; *An existence theorem for the Yamabe problem on manifolds with boundary.* arXiv:0908.4327v2 [math.DG].

[11] S. Brendle, F. C. Marques; *Blow-up phenomena for the Yamabe equation II*, J. Differential Geom. Volume 81, Number 2, 225-250 (2009).

[12] S. Brendle, F. C. Marques; *Recent progress on the Yamabe problem.* Surveys in geometric analysis and relativity, 29-47, Adv. Lect. Math. (ALM), 20, Int. Press, Somerville, MA, (2011). arXiv:1010.4960v1 [math.DG].

[13] L. A. Caffarelli, B. Gidas, J. Spruck; *Asymptotic Symmetry and Local Behavior of Semi-linear Elliptic Equations with Critical Sobolev Growth.* Communications on Pure and Applied Mathematics, Vol. XLII 271-297 (1989).

[14] S. Carlip; *Is quantum gravity necessary?* Class. Quant. Grav. 25:(15) 107-144 (2008). arXiv: 0803.3456v1 [gr-qc].

[15] T. Cazenave, P. L. Lions; *Orbital stability of standing waves for some nonlinear Schrödinger equations.* Comm. Math. Phys. 85, 549-561 (1982).

[16] S. S. Chen; *Conformal deformation to scalar flat metrics with constant mean curvature on the boundary in higher dimensions.* arXiv: 0912.1302v2 [math.DG].

[17] W.-Y. Chen, J.-C. Wei, S.-S. Yan; *Innitely many solutions for the Schrödinger equations in $\mathbb{R}^n$ with critical growth.* J. Differential Equations 252, no. 3, 2425-2447 (2012).

[18] Y. Choquet-Bruhat; *General Relativity and the Einstein Equations.* Oxford University Press, USA (2009).
[19] S. Cingolani, M. Clapp, and S. Secchi; *Multiple solutions to a magnetic nonlinear Choquard equation*. Z. angew. Math. Phys. 63 233-248 (2012).
[20] M. Clapp, D. Salazar; *Positive and sign changing solutions to a nonlinear Choquard equation*. arXiv:1205.6286 [math.AP] (2012).
[21] M. M. Disconzi, M. A. Khuri; *Compactness and non-compactness for the Yamabe problem on manifolds with boundary.* arXiv:1201.4559v2 [math.DG], 47 pages (2012).
[22] O. Druet; *Compactness for Yamabe metrics in low dimensions*. Int. Math. Res. Not 23 1143-1191 (2004).
[23] O. Druet; *From one bubble to several bubbles: the low-dimensional case*. J. Differential Geom. 63,no. 3, 399-473 (2003).
[24] O. Druet, E. Hebey; *Blow-up examples for second order elliptic PDEs of critical Sobolev growth.* Trans. Amer. Math. Soc. 357, no. 5, 1915-1929 (electronic) (2005).
[25] Druet, E. Hebey, F. Robert; *Blow-up Theory for Elliptic PDEs in Riemannian Geometry.* Princeton University Press (2004).
[26] J. F. Escobar; *The Yamabe problem on manifolds with boundary.* Journal of Differential Geometry, Vol. 35, Number 1 , 21-84 (1992).
[27] J. F. Escobar; *Conformal deformations of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary.* Annals of Mathematics, Second Series, Vol. 136, Number 1 , 1-50 (Jul. 1992).
[28] J. F. Escobar; *Conformal metrics with prescribed mean curvature on the boundary.* Calc. var. 4, 559-592 (1996).
[29] P. Esposito, A. Pistoia, J. Vétois; *Blow-up solutions for linear perturbations of the Yamabe equation.* arXiv:1210.6165 [math.AP] (2012).
[30] P. Esposito, A. Pistoia, J. Vétois; *The effect of linear perturbations on the Yamabe problem.* arXiv:1210.7979 [math.AP] (2012).
[31] V. Felli, M. O. Ahmedou; *Compactness results in conformal deformations of Riemannian metrics on manifolds with boundary.* Mathematische Zeitschrift, Vol. 244, 175-210 (2003).
[32] V. Felli, M. O. Ahmedou; *On a geometric equation with critical nonlinearity on the boundary.* Pacific Journal of Mathematics 218, 75-99 (2005).
[33] B. Gidas, W. M. Ni, L. Nirenberg; *Symmetry of Positive Solutions of Nonlinear Elliptic Equations in* $\mathbb{R}^n$. Mathematical Analysis and Appl. Academic Press, New York, (1981).
[34] D. Gilbarg, N. T. Trudinger; *Elliptic partial differential equations of second order.* Springer (2000).
[35] J. Ginibre, G. Velo; *On a class of nonlinear Schrödinger equations with nonlocal interaction.* Math. Z. 170, 109-136 (1980).
[36] D. Giulini, A. Grossardt; *Gravitationally induced inhibitions of dispersion according to a modified Schrödinger-Newton equation for a homogeneous-sphere potential.* arXiv:1212.5146 [gr-qc] (2012).
[37] Z. Han, Y. Y. Li; *The Yamabe problem on manifolds with boundary: existence and compactness results.* Duke Mathematical Journal, Vol. 99, Number 3 (1999), 489-541.
[38] Z. C. Han, Y. Y. Li; *The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature.* Comm. Anal. and Geom. Vol. 8, No. 4, 809-869 (2000).
[39] E. Hebey, M. Vaugon; *From best constants to critical functions.* Math. Z. 237, no. 4, 737-767 (2001).
[40] J. Jost; *Riemannian Geometry and Geometric Analysis.* Springer; 6th ed. (2011).
[41] M. A. Khuri, F. C. Marques, R. M. Schoen; *A compactness theorem for the Yamabe problem.* Journal of Differential Geometry, Vol. 81, Number 1 , 143-196 (2009).
[42] J. M. Lee, T. H. Parker; *The Yamabe problem.* Bulletin (New Series) of the American Mathematical Society, Vol. 17, Number 1 (July 1987).
[43] E. H. Lieb; *Existence and uniqueness of the minimizing solution of Choquardas nonlinear equation.* Stud. Appl. Math. 57, 93-105 (1977).
[44] E. H. Lieb, B. Simon; *The HartreeFock theory for Coulomb systems.* Comm. Math. Phys. 53, 185-194 (1977).
[45] P. L. Lions; *The Choquard equation and related questions.* Nonlinear Anal. T.M.A. 4, 1063-1073 (1980).

[46] P. L. Lions; *Compactness and topological methods for some nonlinear variational problems of mathematical physics*. Nonlinear Problems: Present and Future, pp. 17-34, (1982).

[47] Y. Y. Li, L. Zhang; *Compactness of solutions to the Yamabe problem II*. Calculus of Variations 24, 185-237 (2005).

[48] Y. Y. Li, L. Zhang; *Compactness of solutions to the Yamabe problem III*. Journal of Functional Analysis 245 438-474 (2007).

[49] L. Ma, L. Zhao; *Classification of Positive Solitary Solutions of the Nonlinear Choquard Equation*. Archive for Rational Mechanics and Analysis, Volume 195, Issue 2, pp 455-467 (2010).

[50] F. C. Marques; *Existence results for the Yamabe problem on manifolds with boundary*. Indiana University Mathematics Journal, Vol. 54, No. 6 (2005).

[51] F. C. Marques; *Conformal deformations to scalar-flat metrics with constant mean curvature on the boundary*. Comm. Anal. and Geom. Vol. 15, No. 2, 381-405 (2007).

[52] F. C. Marques; *A priori estimates for the Yamabe problem in the non-locally conformally flat case*. Journal of Differential Geometry, Vol. 71, Number 2, 315-346 (2005).

[53] V. Moroz, J. van Schaftingen; *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*. arXiv:1205.6286v1 [math.AP] (2012).

[54] M. Obata; *The conjectures on conformal transformations of Riemannian manifolds*. J. Diff. Geom. 6 247-258 (1972).

[55] R. Penrose, *On Gravity's Role in Quantum State Reduction*. General Relativity and Gravitation 28 5, 581-600 (1996).

[56] R. Penrose; *Quantum computation, entanglement and state reduction*. Phil. Trans. R. Soc. Lond. A 356 no. 1743, 1927-1939 (1998).

[57] D. Pollack; *Nonuniqueness and high energy solutions for a conformally invariant scalar curvature equation*. Comm. Anal. and Geom. 1 347-414 (1993).

[58] P. Rabinowitz; *On a class of nonlinear Schrödinger equations*. Z. Angew. Math. Phys. 43, 270-291 (1992).

[59] P. J. Salzman, S. Carlip; *A possible experimental test of quantized gravity*. Based on the PhD thesis of P. Salzman "Investigation of the Time Dependent Schrödinger-Newton Equation", U. C. at Davis, 2005. arXiv:0606120 [gr-qc] (2005).

[60] R. Runi, S. Bonazzola; *Systems of Self-Gravitating Particles in General Relativity and the concept of an Equation of State*, Phys. Rev. 187 5, 1767-1783 (1969).

[61] R. Schoen; *On the number of constant scalar curvature metrics in a conformal class*. in: 'Differential Geometry: A symposium in honor of Manfredo do Carmo' (H.B. Lawson and K. Teneblat, eds.), Wiley, 311-320 (1991).

[62] R. Schoen; *Conformal deformation of a Riemannian metric to constant scalar curvature*. J. Differential Geometry 20 479-495 (1984).

[63] R. Schoen, D. Zhang; *Prescribed scalar curvature on the $n$-sphere*. Calc. Var. and PDEs 4 (1996).

[64] J. Shatah, W. Strauss; *Instability of nonlinear bound states*. Comm. Math. Phys. 100, 173-190 (1985).

[65] N. Trudinger; *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 22 (3) 165-274 (1968).

[66] H. Yamabe; *On a deformation of Riemannian structures on compact manifolds*. Osaka Math. J. 12 21-37 (1960).



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