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A LANDESMAN-LAZER CONDITION FOR THE BOUNDARY-VALUE PROBLEM $-u'' = au^+ - bu^- + g(u)$ WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. In this article we prove the existence of solutions for the boundary-value problem

$$\begin{gathered} -u'' = au^+ - bu^- + g(u) \\ u(0) = u(2\pi) \\ u'(0) = u'(2\pi), \end{gathered}$$

where $(a, b) \in \mathbb{R}^2$, $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$, and $g : \mathbb{R} \to \mathbb{R}$ is a bounded, continuous function. We consider both the resonance and nonresonance cases relative to the Fučík Spectrum. For the resonance case we assume a generalized Landesman-Lazer condition that depends upon the average values of g at $\pm\infty$. Our theorems generalize the results in [1] by removing certain restrictions on (a, b). Our proofs are also different in that they rely heavily on a variational characterization of the Fučík Spectrum given in [3].

1. Introduction

We are interested in the boundary-value problem

$$\begin{gathered} -u'' = au^+ - bu^- + g(u) \\ u(0) = u(2\pi) \\ u'(0) = u'(2\pi), \end{gathered} \tag{1.1}$$

where $(a, b) \in \mathbb{R}^2$, $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$, and $g : \mathbb{R} \to \mathbb{R}$ is a bounded, continuous function.

It has been known since the 1970s with the work of Fučík [5] that the existence of solutions to (1.1) depends on the parameter values (a, b). Consider the related

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boundary-value problem

$$\begin{aligned} -u'' &= au^+ - bu^- \\ u(0) &= u(2\pi) \\ u'(0) &= u'(2\pi), \end{aligned} \tag{1.2}$$

and define the Fučík Spectrum

$$\Sigma := \{(a,b) \in \mathbb{R}^2 : \text{ There exists a nontrivial solution to (1.2) } \}.$$

The Fučík spectrum represents a nonlinear resonance set for our boundary-value problem, and therefore, in analogy to the Fredholm Alternative for linear operators, we expect that there should exist solutions to (1.1) without further restrictions when $(a,b) \notin \Sigma$; i.e., the *nonresonance* case. However, when $(a,b) \in \Sigma$; i.e., the *resonance* case, we will require the use of a generalized orthogonality condition, referred to in the literature as a Landesman-Lazer condition, in order to establish the existence of solutions. For future reference we let $\{\lambda_k\}_{k=1}^{\infty}$ represent the eigenvalues of (1.2) for the special case $a = b = \lambda$, and $\{\phi_k\}$ represents the associated L^2-normalized eigenfunctions. We will assume throughout that for some fixed k we have $\lambda_k < a < \lambda_{k+1}$ and $b \geq a$.

In this work, we will prove existence theorems for both the resonance and non-resonance cases. Our proofs make use of the variational characterization of Σ due to Castro and Chang in [3], and a Saddle Point Theorem originally due to Rabinowitz in [10]. We refer to a version of the theorem found in [11].

We improve upon the results in [1] by completely removing several restrictions on (a,b). If those restrictions are translated into our setting, then the authors are assuming $\lambda_k < a < \lambda_{k+1}, b \geq a$, $b < \lambda_{k+2}$ and $b < \frac{9}{4}a$. On the other hand our assumptions about g are not as general as those in [1] where, for example, they allow $g(x,u)$ to be Caratheodory. An important goal of our analysis was to demonstrate how the characterization of Σ in [3] can support an improved understanding of related resonance and nonresonance problems. Our arguments rely more on that general characterization and less on careful estimates of the Fučík Spectrum, and associated eigenfunctions, for this ODE case. Finally, we make some improvement on the methods in [8] where the Landesman-Lazer condition also relied on the average values of g at $\pm\infty$ rather than on the actual limits.

2. Preliminary material

Our problems will be set in the Hilbert space $H := W^{1,2}_{2\pi}(\mathbb{R})$; i.e., the 2π periodic functions in $W^{1,2}(\mathbb{R})$. Standard embedding and compactness theorems can be found in [6]. We use the inner products

$$\begin{gathered} \langle u,v\rangle_{L^2} := \int_0^{2\pi} u(x)v(x)dx, \\ \langle u,v\rangle_H := \langle u',v'\rangle_{L^2} + \langle u,v\rangle_{L^2} \end{gathered}$$

The variational characterization of the Fučík spectrum relies on an analysis of the functional

$$J(u) = \frac{1}{2}\int_0^{2\pi} (u')^2 dt - \frac{a}{2}\int_0^{2\pi} (u^+)^2 dt - \frac{b}{2}\int_0^{2\pi} (u^-)^2 dt, \quad u \in H. \tag{2.1}$$

It is straight forward to see that the critical points of this functional are weak solutions to (1.2).

We also consider the functional,

$$E(u) = J(u) - \int_0^{2\pi} G(u)dt, \quad u \in H, \tag{2.2}$$

where $G(t) := \int_0^t g(s)ds$. The critical points of E are weak solutions to (1.1).

The proofs of our existence theorems will apply the following saddle point theorem found in [11].

Theorem 2.1 (Saddle point theorem)**.** *Let H be a Hilbert space, let X be a finite dimensional subspace of H, and let $E : H \to \mathbb{R}$ be a C^1 functional. Let $B_R := \{x \in X : \|x\| \le R\}$, let $\gamma_0 : \partial B_R \to H$ be a continuous function, and let $\Gamma := \{\gamma : B_R \to H : \gamma \in C(B_R, H), \gamma\big|_{\partial B_R} \equiv \gamma_0\}$. If $\sup_{x\in\partial B_R} E(\gamma_0(x)) < \inf_{\gamma\in\Gamma} \sup_{x\in B_R} E(\gamma(x))$ and if E satisfies (PS), then $c := \inf_{\gamma\in\Gamma} \sup_{x\in B_R} E(\gamma(x))$ is a critical value of E.*

A large part of this article is devoted to proving the necessary inequalities to guarantee the appropriate geometry of our functional, E, and then to establishing the Palais-Smale compactness condition (PS). We will see that the geometry of J dominates the geometry of E to a large extent, and that the geometry of J is derived as a consequence of Castro and Chang's characterization of Σ. A first step towards that characterization will be to identify appropriate complementary orthogonal subspaces, X and Y, of H, and then to use the concavity of J on the X subspace to reduce the functional to a $\tilde{J}$ defined on Y. (See details below). Given that background we will use the variational characterization of the Fučík Spectrum:

Theorem 2.2. *Let $\lambda_k < a < \lambda_{k+1}$ and define*

$$b(a) := \sup\{b \ge a : \inf_{\|y\|_{L^2}=1} \tilde{J}(y) > 0\}.$$

Then,

1. *either $(a, b(a)) \in \Sigma$, or $b(a) = \infty$,*
2. *if $a \le b < b(a)$, then $(a, b) \notin \Sigma$, and*
3. *$b(a) \ge \lambda_{k+1}$.*

To prove an existence theorem for the resonance case we will assume the following generalized Landesman-Lazer condition.

Definition 2.3 (LLD)**.** Let $(a, b) \in \Sigma$. If

$$G^{\pm} = \lim_{u\to\pm\infty} \frac{G(u)}{u},$$

and

$$\Big[G^+ \int_{\Psi>0} \Psi\, dt + G^- \int_{\Psi<0} \Psi\, dt\Big] < 0,$$

for every nontrivial eigenfunction Ψ associated with (a, b), then condition (LLD) is satisfied.

Remark 2.4. The classic Landesman-Lazer condition, and associated existence theorems, can be found in [7]. That condition requires g to have limits at infinity. Here, however, we only require that g attain a finite limit on average at infinity. Simple examples to consider are $g(t) = \arctan(u) + c\sin(u)$ for $c \in \mathbb{R}$. The classical Landesman-Lazer condition is applicable for $c = 0$, but the generalized condition above is applicable for arbitrary c. Related generalized conditions are found in [1],[8] and [9] and references therein, where the latter two references are for the case $a = b$. We note that the condition in [1] does not require $\frac{G(u)}{u}$ to have limits at infinity, but rather expresses its condition in terms of lim inf and lim sup.

3. Properties of the functional J

We begin by defining $X := \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_k\}$ and let $Y := X^{\perp}$; i.e, $Y = \operatorname{span}\{\phi_{k+1}, \phi_{k+2}, \dots, \phi_n, \dots\}$, and let the parameter $a \in (\lambda_k, \lambda_{k+1})$. With this splitting in hand, we prove the following lemmas.

Lemma 3.1. *Choose ϵ such that $a = (1+\epsilon)\lambda_k$, and let $s = b - a \geq 0$. Let $\delta = \min\{\frac{\epsilon}{2}\lambda_k, \frac{\epsilon}{2}\}$ and let*
$$D = \langle \nabla J(x_2 + y_2) - \nabla J(x_1 + y_1), x_2 - x_1 \rangle,$$
where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then
$$D \leq -\delta \|x_2 - x_1\|_H^2 + s\big(\|x_2 - x_1\|_{L^2} + \|y_2 - y_1\|_{L^2}\big)\|y_2 - y_1\|_{L^2} \tag{3.1}$$

Proof. Observe that for $u, v \in H$,
$$\begin{aligned} \nabla J(u) \cdot v &= \int_0^{2\pi} u'v' - a\int_0^{2\pi} u^+ v + b\int_0^{2\pi} u^- v \\ &= \int_0^{2\pi} u'v' - a\int_0^{2\pi} uv + s\int_0^{2\pi} u^- v, \end{aligned}$$
so
$$\begin{aligned} D = {}& \langle (x_2+y_2)' - (x_1+y_1)', (x_2-x_1)'\rangle_{L^2} - a\langle (x_2+y_2) - (x_1+y_1), x_2 - x_1\rangle_{L^2} \\ & + s\langle (x_2+y_2)^- - (x_1+y_1)^-, x_2 - x_1\rangle_{L^2} \end{aligned}$$
Using the orthogonality of X and Y to cancel terms, and then recollecting we obain
$$D = \|x_2' - x_1'\|_{L^2}^2 - a\|x_2 - x_1\|_{L^2}^2 + s\langle (x_2+y_2)^- - (x_1+y_1)^-, x_2 - x_1\rangle_{L^2}$$
To estimate the last part of this expression we make the substitution $x_2 - x_1 = (x_2 + y_2) - (x_1 + y_1) - (y_2 - y_1)$, use the monotonicity of the function $f(t) = t^-$, and the fact that $|f(t_2) - f(t_1)| \leq |t_2 - t_1|$, to see that
$$\begin{aligned} \langle (x_2+y_2)^- - (x_1+y_1)^-, x_2 - x_1\rangle_{L^2} &\leq -\langle (x_2+y_2)^- - (x_1+y_1)^-, y_2 - y_1\rangle_{L^2} \\ &\leq \|(x_2+y_2)^- - (x_1+y_1)^-\|_{L^2}\|y_2 - y_1\|_{L^2} \\ &\leq \|(x_2+y_2) - (x_1+y_1)\|_{L^2}\|y_2 - y_1\|_{L^2} \\ &\leq \big(\|x_2 - x_1\|_{L^2} + \|y_2 - y_1\|_{L^2}\big)\|y_2 - y_1\|_{L^2} \end{aligned}$$
To estimate the other part we let $a = (1+\epsilon)\lambda_k$, and write
$$\|x_2' - x_1'\|_{L^2}^2 - a\|x_2 - x_1\|_{L^2}^2 = (1+\frac{\epsilon}{2})\|x_2' - x_1'\|_{L^2}^2 - \frac{\epsilon}{2}\|x_2' - x_1'\|_{L^2}^2 - (1+\epsilon)\lambda_k\|x_2 - x_1\|_{L^2}^2.$$
Standard estimates show that $\|x'\|_{L^2}^2 \leq \lambda_k\|x\|_{L^2}^2$ for all $x \in X$, so
$$\|x_2' - x_1'\|_{L^2}^2 - a\|x_2 - x_1\|_{L^2}^2 \leq -\frac{\epsilon\lambda_k}{2}\|x_2 - x_1\|_{L^2}^2 - \frac{\epsilon}{2}\|x_2' - x_1'\|_{L^2}^2.$$
The desired inequality now follows. □

Remark 3.2. While the necessity of the above lemma may not be immediately obvious, it was determined after repeated estimations of the same type that such an inequality would have wide application to this particular problem.

Theorem 3.3 (Reduction theorem)**.**

(1) *For fixed* $y \in Y$, J *is concave and anticoercive on the set* $y + X := \{y + x : x \in X\}$ *and achieves a unique maximum on this set.*
(2) *Let* $r : Y \to X$ *such that* $J(r(y)+y) = \max_{x\in X} J(x+y)$, *then* r *is Lipschitz continuous as a function of* $L^2(0, 2\pi)$ *into* H, *and* $r(cy) = cr(y)$ *for any* $c \geq 0$ *and* $y \in Y$.
(3) *Let* $\tilde{J} : Y \to \mathbb{R} : \tilde{J}(y) = J(r(y) + y)$, *then*
  (a) $\tilde{J} \in C^1(Y, \mathbb{R})$, *and*
  (b) $\tilde{J}(cy) = c^2\tilde{J}(y)$ *for all* $c \geq 0$ *and* $y \in Y$.

This theorem is well-known from works such as [3] and [2], but we reproduce it here for completeness, because the properties of r and J play a crucial role in subsequent proofs, and because the approach below is somewhat different in certain details from that found in the references.

Proof. Let $u_1, u_2 \in y + X$. By definition of the set $y + X$, write $u_2 = y + x_1$ and $u_1 = y + x_1$. Consider the quantity,

$$\begin{aligned}\langle \nabla J(u_2) - \nabla J(u_1), u_2 - u_1\rangle &= \langle \nabla J(x_2 + y) - \nabla J(x_1 + y), x_2 - x_1\rangle \\ &\leq -\delta\|x_2 - x_1\|_H^2,\end{aligned}$$

by (3.1). Therefore, J is strictly concave on $y+X$. That J is anticoercive on $y+X$ follows from the strict concavity and the Fundamental Theorem of Calculus. The concavity of J on $y+X$ implies that J is weakly upper semicontinuous, and therefore, J must achieve a maximum. That this maximum is unique is a consequence of the strict concavity.

The argument so far shows that r is well defined. To see that $r(y)$ is a continuous function, we let $x_1 = r(y_1), x_2 = r(y_2)$ in (3.1) and note that $D = \langle \nabla J(r(y_2) + y_2) - \nabla J(r(y_1) + y_1), r(y_2) - r(y_1)\rangle = 0$, since $r(y_1), r(y_2)$ are both critical points with respect to X. Making these substitutions, we can solve (3.1) to get

$$\|r(y_2) - r(y_1)\|_H \leq \Big(\frac{s + \sqrt{s^2 + 4\delta s}}{2\delta}\Big)\|y_2 - y_1\|_{L^2}. \tag{3.2}$$

Remark 3.4. Note also that if $\{y_k\}_{k=1}^\infty$ is a bounded sequence in H, then $\{y_k\}_{k=1}^\infty$ has a convergent subsequence in $L^2(0, 2\pi)$, call it $\{y_{k_i}\}_{i=1}^\infty$, which, by (3.2), gives us that $\{r(y_{k_i})\}_{i=1}^\infty$ converges in H. Hence, r is a compact mapping, sending bounded sets in $L^2(0, 2\pi)$ into precompact sets in H.

We now show that $\tilde{J} \in C^1(Y, \mathbb{R})$.

Remark 3.5. While it may seem at first to be a trivial consequence of the Chain Rule, we must be careful to note that r is not necessarily a C^1 function, and therefore, a more technical argument must be made.

Consider the quantity $\tilde{J}(y_2) - \tilde{J}(y_1)$. We note that

$$\begin{aligned}\tilde{J}(y_2) - \tilde{J}(y_1) &= J(r(y_2) + y_2) - J(r(y_1) + y_1) \\ &= (J(r(y_2) + y_2) - J(r(y_2) + y_1)) \\ &\quad + (J(r(y_2) + y_1) - J(r(y_1) + y_1)) \\ &\leq (J(r(y_2) + y_2) - J(r(y_2) + y_1)),\end{aligned} \tag{3.3}$$

since $J(r(y_1) + y_1)$ maximizes $J(x + y_1)$. Then

$$\tilde{J}(y_2) - \tilde{J}(y_1) \leq \nabla J(r(y_2) + y_1) \cdot (y_2 - y_1) + o(\|y_2 - y_1\|_H)$$

since $J \in C^1(H, \mathbb{R})$. Then

$$
\begin{aligned}
\tilde{J}(y_2) - \tilde{J}(y_1) &\le \Big(\nabla J(r(y_1) + y_1) + \nabla J(r(y_2) + y_1) - \nabla J(r(y_1) + y_1)\Big) \cdot (y_2 - y_1) \\
&\quad + o(\|y_2 - y_1\|_H) \\
&= \nabla J(r(y_1) + y_1) \cdot (y_2 - y_1) \\
&\quad + \Big(\nabla J(r(y_2) + y_1) - \nabla J(r(y_1) + y_1)\Big) \cdot (y_2 - y_1) + o(\|y_2 - y_1\|_H) \\
&= \nabla J(r(y_1) + y_1) \cdot (y_2 - y_1) + o(\|y_2 - y_1\|_H),
\end{aligned}
$$

since

$$
\begin{aligned}
&\lim_{\|y_2 - y_1\|_H \to 0} \Big| \frac{(\nabla J(r(y_2) + y_1) - \nabla J(r(y_1) + y_1)) \cdot (y_2 - y_1)}{\|y_2 - y_1\|_H} \Big| \\
&\le \lim_{\|y_2 - y_1\|_H \to 0} \|\nabla J(r(y_2) + y_1) - \nabla J(r(y_1) + y_1)\|_H = 0,
\end{aligned}
$$

by the continuity of both ∇J and r. If, instead of adding and subtracting $J(r(y_2) + y_1)$ in (3.3), we had added and subtracted $J(r(y_1) + y_2)$, we would have concluded that

$$
\tilde{J}(y_2) - \tilde{J}(y_1) \ge \nabla J(r(y_1) + y_1) \cdot (y_2 - y_1) + o(\|y_2 - y_1\|_H).
$$

Combining these two results, we conclude that

$$
\tilde{J}(y_2) - \tilde{J}(y_1) = \nabla J(r(y_1) + y_1) \cdot (y_2 - y_1) + o(\|y_2 - y_1\|_H),
$$

and therefore $\tilde{J} \in C^1(Y, \mathbb{R})$ and $\nabla \tilde{J}(y) = \nabla_Y J(r(y) + y)$.

We finish the proof by showing two homogeneity properties, one of the function r and another of the functional $\tilde{J}$. The case $c = 0$ is trivial, so we assume $c > 0$. For all $u \in H$,

$$
\begin{aligned}
J(cu) &= \frac{1}{2}\Big(\int_0^{2\pi} ((cu)')^2\, dt - a\int_0^{2\pi} \left((cu)^+\right)^2 dt - b\int_0^{2\pi} \left((cu)^-\right)^2 dt\Big) \\
&= \frac{1}{2}\Big(c^2\int_0^{2\pi} ((u)')^2\, dt - ac^2\int_0^{2\pi} \left((u)^+\right)^2 dt - bc^2\int_0^{2\pi} \left((u)^-\right)^2 dt\Big) \\
&= c^2 J(u),
\end{aligned}
\tag{3.4}
$$

since positive constants can be factored out of $(\cdot)^+$ and $(\cdot)^-$.

Now consider

$$
J(x + cy) = J\left(c\left(\frac{x}{c} + y\right)\right) = c^2 J\left(\frac{x}{c} + y\right) \tag{3.5}
$$

Since $J(x + cy)$ is uniquely maximized at $x = r(cy)$ and $c^2 J(\frac{x}{c} + y)$ will be uniquely maximized at $\frac{x}{c} = r(y)$, then $r(cy) = cr(y)$. Finally, combining (3.4) and (3.5) we see that

$$
\tilde{J}(cy) = J(r(cy) + cy) = J(cr(y) + cy) = c^2 J(r(y) + y) = c^2 \tilde{J}(y).
$$

□

Remark 3.6. Note that some of the arguments above can be simplified using the fact that X is finite dimensional, and so, for example, the L^2 and H norms are equivalent on X. We avoid that simplification to preserve some generality in the arguments that is of interest in other situations. For example, for other choices of boundary conditions we often have two primary curves emanating from the point $(\lambda_{k+1}, \lambda_{k+1})$. The lower curve is characterized as we have described here, and the upper curve can be characterized via a similar process where the roles of X and Y are switched in the reduction of J. Thus the arguments for X, above, must be adapted to the infinite dimensional space Y.

The properties of $\tilde{J}$ above lead to the proof of Theorem 2.2. The main idea is to obtain a critical value of $\tilde{J}$ on the set $\{y \in Y : \|y\|_{L^2} = 1\}$ by minimization. One then observes that a critical point of $\tilde{J}$ is only a critical point of J if the associated critical value is 0. We leave the remaining details of this argument for the reader to find in [2]. However, we do state the following lemma, which will be helpful in the next section.

Lemma 3.7. Assume that $a < b \leq b(a)$ as in Theorem 2.2. Then

- (i) If $b < b(a)$, then there exists an $\epsilon > 0$ such that $\tilde{J}(y) \geq \epsilon$ for all $y \in Y$ such that $\|y\|_{L^2} = 1$.
- (ii) If $b = b(a)$, then $\tilde{J}(y) \geq 0$ for all $y \in Y$ such that $\|y\|_{L^2} = 1$, and $\tilde{J}(y) = 0$ if and only if f y is an eigenfunction associated with $(a, b) \in \Sigma$.

Now that we know something about the geometry of J, we can say something about the geometry of E.

4. The geometry of the functional E

As noted previously, the geometry of E is dominated to a large extent by the geometry of J. For the nonresonance case this leads to relatively straight forward arguments to prove a saddle geometry, and later the (PS) condition. For the resonance case we will see that the nonresonance arguments are sufficient to reduce both questions to analyzing what happens to the functional in the direction of an eigenfunction. In that case the (LLD) condition will provide a sufficient tool for finishing the analysis.

Lemma 4.1. *Assume that $b < b(a)$ and let $\mathscr{Y} := \{r(y) + y : y \in Y\}$. There exists some $R >> 0$ sufficiently large such that*

$$\sup_{x \in X, \|x\| = R} E(x) < \inf_{u \in \mathscr{Y}} E(u).$$

Proof. Consider the functional E restricted to the subspace X. If we assume that g is bounded, then, since J is anticoercive on X, and in fact satisfies a quadratic estimate, we can conclude that

$$E(x) \leq -\eta \|x\|_H^2 + M\|x\|_{L^2}, \tag{4.1}$$

for appropriate $\eta > 0$ and $M > 0$, and therefore E is anticoercive on X.

Now consider E restricted to the set $\mathscr{Y}$. Note first that for $y \neq 0$

$$\tilde{J}(y) = \tilde{J}\Big(\|y\|_{L^2} \frac{y}{\|y\|_{L^2}}\Big) = \|y\|_{L^2}^2 \tilde{J}(\frac{y}{\|y\|_{L^2}}), \tag{4.2}$$

and so if $\inf_{\|y\|_{L^2}=1} \tilde{J}(y) \geq \epsilon$, as in Theorem 3.7, we conclude that $\tilde{J}(y) \geq \epsilon \|y\|_{L^2}^2$.

Recall that $r(y)$ is Lipschitz continuous, so $\|r(y)\|_H \leq M'\|y\|_{L^2}$, for some $M' > 0$, and we see that

$$\begin{aligned} E(r(y) + y) &\geq \epsilon \|y\|_{L^2}^2 - M\|r(y) + y\|_{L^2} && (4.3)\\ &\geq \epsilon \|y\|_{L^2}^2 - M\left(\|r(y)\|_{L^2} + \|y\|_{L^2}\right) && (4.4)\\ &\geq \epsilon \|y\|_{L^2}^2 - M(M' + 1)\|y\|_{L^2}. && (4.5) \end{aligned}$$

Thus E is bounded below on $\mathscr{Y}$ (In fact, E is coercive, but that is not necessary here).

It follows that there exists some R sufficiently large such that,

$$\sup_{\|x\|=R} E(x) < \inf_{u\in\mathscr{Y}} E(u).$$

$\square$

A similar estimate is possible in the resonance case, but the proof requires the (LLD) condition.

Lemma 4.2. *Assume that* $b = b(a)$ *and that* (LLD) *is satisfied. There exists some* $R >> 0$ *sufficiently large such that*

$$\sup_{x\in X, \|x\|=R} E(x) < \inf_{u\in\mathscr{Y}} E(u).$$

Proof. The argument that E is anticoercive on X remains the same as above. The analysis of E restricted to $\mathscr{Y}$ requires more care. Once again we will show that E restricted to $\mathscr{Y}$ is bounded below, but this time we use an argument by contradiction.

Recall that for $u = r(y) + y \in \mathscr{Y}$ we have

$$E(u) = \tilde{J}(y) - \int_0^{2\pi} G(u)\,.$$

Let $\{u_n\} \subset \mathscr{Y}$ be a minimizing sequence for E, with $u_n = r(y_n) + y_n$. If $E(u_n)$ is bounded below, then we are done, so we assume, without loss of generality, that $E(u_n) \downarrow -\infty$. We know that $\tilde{J}(y_n) \geq 0$. If $\{y_n\}$ were L^2 bounded, then $\{u_n\}$ would be bounded, and the integral of $G(u_n)$ would be bounded and we would be done, so, without loss of generality, $\|y_n\|_{L^2} \to \infty$. Moreover, if there is an $\epsilon > 0$ such that $\tilde{J}(\frac{y_n}{\|y_n\|_{L^2}}) \geq \epsilon$, then we have $E(u_n) \to \infty$ by the estimates in the previous proof. Thus $\tilde{J}(\frac{y_n}{\|y_n\|_{L^2}}) \downarrow 0$.

Let $v_n = y_n/\|y_n\|_{L^2}$ and let $w_n = u_n/\|y_n\|_{L^2} = r(v_n) + v_n$. It is clear that $\{v_n\}$ is L^2 bounded. It follows that $\{r(v_n)\}$ is bounded in H, and thus that $\{w_n\}$ is L^2 bounded. Since the functional values $J(w_n) = \tilde{J}(v_n)$ are also bounded, it follows that $\{w_n\}$ is H bounded. Thus, without loss of generality, we have $w_n \rightharpoonup w, v_n \rightharpoonup v$ in H, and $w_n \to w, v_n \to v$ in $L^2(0, 2\pi)$. By continuity, $r(v_n) \to r(v)$ in H, so $w = r(v) + v \in \mathscr{Y}$. Moreover, v must be a unit vector in $L^2(0, 2\pi)$ so w is nontrivial.

By weak lower semicontinuity, we have

$$\int_0^{2\pi} (w')^2 \leq \liminf \Big(\int_0^{2\pi} (w_n')^2 \Big),$$

but $J(w_n) = \tilde{J}(v_n) \to 0$, so

$$\lim \Big(\int_0^{2\pi} (w_n')^2 \Big) = \lim \Big(a \int_0^{2\pi} (w_n^+)^2 + b \int_0^{2\pi} (w_n^-)^2 \Big) = a \int_0^{2\pi} (w^+)^2 + b \int_0^{2\pi} (w^-)^2.$$

Hence

$$\int_0^{2\pi} (w')^2 \leq a \int_0^{2\pi} (w^+)^2 + b \int_0^{2\pi} (w^-)^2,$$

i.e. $J(w) = \tilde{J}(v) \leq 0$. But we already know that $\tilde{J}(v) \geq 0$, so it must be that we have equality. Hence w is a nontrivial eigenfunction associated with (a, b). Moreover, it must be that

$$\lim \Big(\int_0^{2\pi} (w_n')^2 \Big) = \int_0^{2\pi} (w')^2,$$

so $w_n \to w$ in H. Applying (LLD) we obtain

$$\frac{1}{\|y_n\|_{L^2}} \int_0^{2\pi} G(u_n) = \int_0^{2\pi} \Big(\frac{G(u_n}{u_n} \Big) w_n \to G^+ \int_0^{2\pi} w^+ - G^- \int_0^{2\pi} w^- < 0.$$

Finally, this leads to

$$E(u_n) = \tilde{J}(y_n) - \int_0^{2\pi} G(u_n) \geq - \int_0^{2\pi} G(u_n) \to \infty,$$

which is a contradiction. The proof is complete. □

We now fix $R \gg 0$ such that

$$\sup_{x \in X, \|x\| = R} E(x) < \inf_{u \in \mathscr{Y}} E(u).$$

The final element in establishing the geometry of E is the following *linking* lemma.

Lemma 4.3. *Suppose that either $a \leq b < b(a)$, or that $b = b(a)$ and* (LLD)*, as above. Let $B_R := \{x \in X : \|x\|_{L^2} \leq R\}$, and let*

$$\Gamma := \{\gamma : B_R \subseteq X \to H : \gamma|_{\partial B_R}(x) = x, \gamma \in C\}.$$

Then

$$\inf_{\gamma \in \Gamma} \sup_{x \in B_R} E(\gamma(x)) > \sup_{x \in \partial B_R} E(x).$$

Proof. Let $\gamma : B_R \subseteq X \to H$ be a continuous function such that $\gamma(\partial B_R) = \{x + y : y = 0, \|x\|_{L^2} = R\}$. Let $\gamma(x) = \gamma_X(x) + \gamma_Y(x)$ where $\gamma_X(x) \in X$ and $\gamma_Y(x) \in Y$. To show that $\gamma(B_R) \cap \mathscr{Y} \neq \emptyset$, we wish to find $x \in B_R$ so that $\gamma_X(x) = r(\gamma_Y(x))$. Let $F(x) = \gamma_X(x) - r(\gamma_Y(x))$. Now, let $h(x, t) = tF(x) + (1 - t)x$. Note first that if $x \in \partial B_R$, then $F(x) = x \neq 0$, so $h(x, t) = tx + (1 - t)x = x$. Applying Brouwer degree, we see that $\deg(F, B_R, 0) = \deg(I, B_R, 0) = 1$, and hence,

$$\inf_{\gamma \in \Gamma} \sup_{x \in B_R} E(\gamma(x)) \geq \inf_{u \in \mathscr{Y}} E(u) > \sup_{x \in \partial B_R} E(x)$$

□

5. The Palais-Smale condition

In this section it is simpler to prove (PS) for both the nonresonance and resonance cases in one theorem.

Theorem 5.1. *If $(a, b) \notin \Sigma$, or if $(a, b) \in \Sigma$ and (LLD) is satisfied, then the functional E satisfies (PS).*

Proof. First, suppose that $\{u_k\}_{k=1}^\infty$ is a sequence such that $\{E(u_k)\}_{k=1}^\infty$ is bounded and $\nabla E(u_k) \to 0$ in H. We must show that $\{u_k\}$ has a converging subsequence in H. The crucial step is to show that some subsequence is L^∞ bounded.

Suppose to the contrary that $\|u_k\|_\infty \to \infty$. Then let $v_k = u_k/\|u_k\|_\infty$. Note that if we divide the energy functional through by $\|u_k\|_\infty^2$, we obtain

$$\frac{E(u_k)}{\|u_k\|_\infty^2} = \frac{1}{2}\int_0^{2\pi} (v_k')^2\,dt - \frac{a}{2}\int_0^{2\pi} (v_k^+)^2\,dt - \frac{b}{2}\int_0^{2\pi} (v_k^-)^2\,dt - \int_0^{2\pi} \frac{G(u_k)}{\|u_k\|_\infty^2}\,dt$$

If we take a limit, the term $E(u_k)/\|u_k\|_\infty^2 \to 0$, since $\{E(u_k)\}_{k=1}^\infty$ is bounded, and

$$\int_0^{2\pi} \frac{G(u_k)}{\|u_k\|_\infty^2}\,dt \to 0,$$

since $G' = g$ is a bounded function, and thus $|G(u_k)| \le C|u_k|$, where $|g(u_k)| \le C$ $\forall u_k$. Also note that $\|v_k^\pm\|_\infty \le 1$, so $\int_0^{2\pi} (v^\pm)^2\,dt$ is likewise bounded. Therefore, we may conclude that

$$\frac{1}{2}\int_0^{2\pi} (v_k')^2\,dt,$$

is bounded and therefore $\|v_k\|_H$ is bounded.

Thus, without loss of generality, there exists $\Psi \in H$ such that $v_k \rightharpoonup \Psi$ in H and $v_k \to \Psi$ in $L^2(0,2\pi)$ and $C[0,2\pi]$, by Alaoglu's theorem and a standard compact embedding theorem. We know that $\|\Psi\|_\infty = 1$ since $\|v_k\|_\infty = 1$ $\forall k$ and convergence is uniform, so Ψ is nontrivial. Using this convergence, we can now show that, for any $w \in H$,

$$\begin{aligned}
0 &= \lim_{k\to\infty} \frac{\nabla E(u_k)}{\|u_k\|_\infty}\cdot w \\
&= \lim_{k\to\infty} \Big[\int_0^{2\pi} v_k' w'\,dt - a\int_0^{2\pi} v_k^+ w\,dt + b\int_0^{2\pi} v_k^- w\,dt - \int_0^{2\pi} \frac{g(u_k)}{\|u_k\|_\infty} w\,dt\Big] \\
&= \int_0^{2\pi} \Psi' w'\,dt - a\int_0^{2\pi} \Psi^+ w\,dt + b\int_0^{2\pi} \Psi^- w\,dt
\end{aligned}$$

Thus Ψ is a weak solution of the Fučík eigenvalue problem, (1.2), and hence Ψ is a nontrivial Fučík eigenfunction.

If $(a,b) \notin \Sigma$, then this is a contradiction and $\|u_k\|_\infty$ is bounded as claimed. If $(a,b) \in \Sigma$, then consider the quantity,

$$\frac{2E(u_k) - \nabla E(u_k)\cdot u_k}{\|u_k\|_\infty} = -2\int_0^{2\pi} \frac{G(u_k)}{\|u_k\|_\infty} + \int_0^{2\pi} g(u_k)\frac{u_k}{\|u_k\|_\infty} \tag{5.1}$$

Note first that, by assumption,

$$\lim_{k\to\infty} \frac{2E(u_k) - \nabla E(u_k)\cdot u_k}{\|u_k\|_\infty} = 0.$$

We can rewrite the first term on the right hand side of (5.1) so that

$$\begin{aligned}
\lim_{k\to\infty}\int_0^{2\pi} \frac{G(u_k)}{\|u_k\|_\infty}\,dt &= \lim_{k\to\infty}\int_0^{2\pi} \frac{G(u_k)}{u_k}\frac{u_k}{\|u_k\|_\infty}\,dt \\
&= \lim_{k\to\infty}\int_{\Psi<0} \frac{G(u_k)}{u_k}\frac{u_k}{\|u_k\|_\infty}\,dt + \int_{\Psi>0} \frac{G(u_k)}{u_k}\frac{u_k}{\|u_k\|_\infty}\,dt \\
&= G^-\int_{\Psi<0} v\,dt + G^+\int_{\Psi>0} v\,dt,
\end{aligned} \tag{5.2}$$

where we have used the fact that Ψ is only 0 on a finite set, and that the integrands converge uniformly to their limits.

Now, we need only to determine what the last integral in (5.1) converges to in order to reach a contradiction, which will show that $\|u_k\|_\infty$ is bounded. The following two lemmas establish the convergence properties of the parts of the integrand.

Lemma 5.2. $\frac{u_k}{\|u_k\|_\infty}$ *has a convergent subsequence in* H.

Proof. Let

$$P(u)\cdot v := \langle u, v\rangle_H$$
$$S(u)\cdot v := -(a+1)\int_0^{2\pi} u^+ v\,dt + (b+1)\int_0^{2\pi} u^- v\,dt$$
$$T(u)\cdot v := -\int_0^{2\pi} g(u)v\,dt$$

so that

$$\nabla E(u)\cdot v = (P(u)+S(u)+T(u))\cdot v.$$

First, let us consider $S(u)$. Since $u_k/\|u_k\|_\infty \xrightarrow{L^2} \Psi$, it follows that $\big(u_k/\|u_k\|_\infty\big)^+ \xrightarrow{L^2} \Psi^+$ and $\big(u_k/\|u_k\|_\infty\big)^- \xrightarrow{L^2} \Psi^-$ by the Lebesgue Dominated Convergence Theorem. Noting that

$$\begin{aligned}
\frac{S(u_k)}{\|u_k\|_\infty}\cdot v &= S\Big(\frac{u_k}{\|u_k\|_\infty}\Big)\cdot v\\
&= -(a+1)\int_0^{2\pi}\Big(\frac{u_k}{\|u_k\|_\infty}\Big)^+ v\,dt + (b+1)\int_0^{2\pi}\Big(\frac{u_k}{\|u_k\|_\infty}\Big)^- v\,dt,
\end{aligned}$$

we conclude that $S\big(u_k/\|u_k\|_\infty\big)\cdot v \to S(\Psi)\cdot v$ for all $v\in H$. Since

$$\begin{aligned}
&\Big|\Big(S\big(\frac{u_k}{\|u_k\|_\infty}\big) - S(\Psi)\Big)\cdot v\Big|\\
&= \Big| -(a+1)\int_0^{2\pi}\Big(\big(\frac{u_k}{\|u_k\|_\infty}\big)^+ - \Psi^+\Big)v\,dt + (b+1)\int_0^{2\pi}\Big(\big(\frac{u_k}{\|u_k\|_\infty}\big)^- - \Psi^-\Big)v\,dt\Big|\\
&\le (a+1)\Big\|\big(\frac{u_k}{\|u_k\|_\infty}\big)^+ - \Psi^+\Big\|_{L^2} + (b+1)\|\big(\frac{u_k}{\|u_k\|_\infty}\big)^- - \Psi^-\|_{L^2},
\end{aligned}$$

for $\|v\|_{L^2}\le 1$, then $S(u_k/\|u_k\|_\infty)\to S(\Psi)$ in H^*.

Now, considering $T(u)$, we see that

$$T(u)\cdot v = -\int_0^{2\pi} g(u)v\,dt,$$

so $\{T(u_k)\}$ is bounded in H^* since

$$\|T(u)\|_{H^*}\le \|g(u)\|_{L^2}\le C.$$

So $\|T(u_k)/\|u_k\|_\infty\|_{H^*}\to 0$ as $k\to\infty$.

Finally we consider P. By the Riesz Representation Theorem, there is an isomorphism, $i: H^*\to H$ such that $i\circ P(u) = u$ for all $u\in H$. So, P is an invertible linear operator with continuous inverse.

Recalling that $\nabla E(u) = P(u)+S(u)+T(u)$ and that, by a hypothesis of the Palais-Smale condition, $\nabla E(u_k)\to 0$ in H^* as $k\to\infty$, we see that

$$\frac{\nabla E(u_k)}{\|u_k\|_\infty} = P\Big(\frac{u_k}{\|u_k\|_\infty}\Big) + S\Big(\frac{u_k}{\|u_k\|_\infty}\Big) + \frac{T(u_k)}{\|u_k\|_\infty}$$

can be rewritten as

$$\frac{u_k}{\|u_k\|_\infty} = P^{-1}\Big(\frac{\nabla E(u_k)}{\|u_k\|_\infty} - S\big(\frac{u_k}{\|u_k\|_\infty}\big) - \frac{T(u_k)}{\|u_k\|_\infty}\Big).$$

Therefore, invoking the continuity of P^{-1} and taking a limit as $k \to \infty$, we conclude that

$$\frac{u_k}{\|u_k\|_\infty} \xrightarrow{H} P^{-1}\big(0 - S(\Psi) - 0\big) = P^{-1}(-S(\Psi)) = \Psi.$$

□

Lemma 5.3.

$$g(u_k) \rightharpoonup G^+\chi_{\Psi>0} + G^-\chi_{\Psi<0}$$

Proof. By Alaoglu's Theorem, we know that $\{g(u_k)\}_{k=1}^\infty$ has a weakly convergent subsequence since $\{g(u_k)\}_{k=1}^\infty$ is bounded in $L^2[0,2\pi]$. Let $g(u_k) \rightharpoonup \mathscr{G}$. Now we need only to show that

$$\mathscr{G} = G^+\chi_{\Psi>0} + G^-\chi_{\Psi<0}$$

It will be helpful to recall some standard properties of Fučík eigenfunctions, Ψ, namely that they are continuously differentiable and have a finite number of critical points. For a proof of such properties and an explicit formulation for such Ψ, see [1].

Let $v = \chi_{[c,d]}$ be the characteristic function of some closed interval where $0 \le c < d \le 2\pi$ and $[c,d] \subset \{x : \Psi(x) > 0, \Psi'(x) > 0\}$. Then we may write

$$\begin{aligned} \int_0^{2\pi} g(u_k)\chi_{[c,d]} &= \int_c^d g(u_k) \\ &= \int_c^d g(u_k)\Big(1 - \frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e)}\Big) + \int_c^d g(u_k)\Big(\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e)}\Big), \end{aligned} \tag{5.3}$$

where $c < e < d$ such that $\Psi'(e) = \frac{\Psi(d)-\Psi(c)}{d-c}$, as guaranteed by the Mean Value Theorem. Analyzing the second term, we find

$$\begin{aligned} \int_c^d g(u_k)\Big(\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e)}\Big) &= \frac{1}{\Psi'(e)\|u_k\|_\infty}\int_c^d g(u_k)u_k' \\ &= \frac{1}{\Psi'(e)\|u_k\|_\infty}\left(G(u_k(d)) - G(u_k(c))\right) \\ &= \frac{1}{\Psi'(e)}\Big(\frac{G(u_k(d))}{u_k(d)}\frac{u_k(d)}{\|u_k\|_\infty} - \frac{G(u_k(c))}{u_k(c)}\frac{u_k(c)}{\|u_k\|_\infty}\Big) \end{aligned}$$

Now, taking a limit of both sides, we see that

$$\begin{aligned} \lim_{k\to\infty}\int_c^d g(u_k)\Big(\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e)}\Big) &= \lim_{k\to\infty}\frac{1}{\Psi'(e)}\Big(\frac{G(u_k(d))}{u_k(d)}\frac{u_k(d)}{\|u_k\|_\infty} - \frac{G(u_k(c))}{u_k(c)}\frac{u_k(c)}{\|u_k\|_\infty}\Big) \\ &= \frac{1}{\Psi'(e)}\left(G^+\Psi(d) - G^+\Psi(c)\right) \\ &= (d-c)G^+ \\ &= \int_0^{2\pi} G^+\chi_{[c,d]} \end{aligned}$$

Focusing now on the first term of (5.3), we note that,

$$\Big(1-\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e)}\Big)\to\Big(1-\frac{\Psi'(x)}{\Psi'(e)}\Big),$$

in $L^2(0,2\pi)$, so

$$\int_c^d g(u_k)\Big(1-\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e)}\Big)\to\int_c^d \mathscr{G}\Big(1-\frac{\Psi'(x)}{\Psi'(e)}\Big).$$

Let $\epsilon>0$ and define $M:=\|\mathscr{G}\|_\infty$. The fact that M exists is a consequence of the boundedness of g. Choose c_i,d_i,e_i such that

$$\cup_{i=i}^n[c_i,d_i]=[c,d],\ |d-c|=\sum_{i=1}^n|d_i-c_i|,\ \text{and } |1-\frac{\Psi'(x)}{\Psi'(e_i)}|<\frac{\epsilon}{M}\quad\forall x\in[c_i,d_i].$$

Then

$$\begin{aligned}\int_0^{2\pi}g(u_k)\chi_{[c,d]}&=\sum_{i=1}^n\int_{c_i}^{d_i}g(u_k)\\&=\sum_{i=1}^n\int_{c_i}^{d_i}g(u_k)\Big(1-\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e_i)}\Big)+\sum_{i=1}^n\int_{c_i}^{d_i}g(u_k)\Big(\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e_i)}\Big).\end{aligned}$$

We see that

$$\sum_{i=1}^n\int_{c_i}^{d_i}g(u_k)\Big(\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e_i)}\Big)\to\sum_{i=1}^n\int_{c_i}^{d_i}G^+\chi_{[c_i,d_i]}=\int_c^d G^+\chi_{[c,d]},$$

while

$$\sum_{i=1}^n\int_{c_i}^{d_i}g(u_k)\Big(1-\frac{\frac{u_k'}{\|u_k\|_\infty}}{\Psi'(e_i)}\Big)\to\sum_{i=1}^n\int_{c_i}^{d_i}\mathscr{G}\Big(1-\frac{\Psi'(x)}{\Psi'(e_i)}\Big)$$

and

$$\Big|\sum_{i=1}^n\int_{c_i}^{d_i}\mathscr{G}\big(1-\frac{\Psi'(x)}{\Psi'(e_i)}\big)\Big|\le\sum_{i=1}^n\int_{c_i}^{d_i}\Big|\mathscr{G}\big(1-\frac{\Psi'(x)}{\Psi'(e_i)}\big)\Big|\le\sum_{i=1}^n\epsilon|d_i-c_i|=\epsilon(d-c)$$

Since ϵ was chosen arbitrarily, we may let $\epsilon\to0$, and hence

$$\lim_{k\to\infty}\int_0^{2\pi}g(u_k)\chi_{[c,d]}=\int_0^{2\pi}G^+\chi_{[c,d]}\ \forall[c,d]\subset\{x:\Psi(x)>0,\Psi'(x)>0\}.\tag{5.4}$$

The exact same calculations will show that, given $[c,d]\subset\{x:\Psi(x)>0,\Psi'(x)<0\}$, we get the same conclusion as in (5.4). For $[c,d]\subset\{x:\Psi(x)<0,\Psi'(x)>0\}$ and $[c,d]\subset\{x:\Psi(x)<0,\Psi'(x)<0\}$, we can complete the same calculations, but will this time find that

$$\lim_{k\to\infty}\int_0^{2\pi}g(u_k)\chi_{[c,d]}=\int_0^{2\pi}G^-\chi_{[c,d]}.$$

Hence, we may recombine the integrals to see that

$$\lim_{k\to\infty}\int_0^{2\pi}g(u_k)\chi_{[c,d]}=\int_0^{2\pi}\big(G^+\chi_{\Psi>0}+G^-\chi_{\Psi<0}\big)\chi_{[c,d]}.\tag{5.5}$$

We proceed, via standard arguments, to replace $\chi_{[c,d]}$ in (5.5) by arbitrary $v\in L^2(0,2\pi)$. So far the closed intervals above avoid critical points of ψ. If an interval does include critical points, however, we may delete an arbitrarily small neighborhood of each of the finitely many critical points so that the total change in the integral is less than some ϵ. Hence (5.5) holds for arbitrary $[c, d]$ up to an arbitrary ϵ. Let ϵ go to zero and we have (5.5) for all closed subintervals of $[0, 2\pi]$. We can immediately generalize to step functions, and then to arbitrary $v \in L^2(0, 2\pi)$ by taking limits of approximating step functions. Thus we have

$$\lim_{k\to\infty} \int_0^{2\pi} g(u_k)v = \int_0^{2\pi} \left(G^+\chi_{\Psi>0} + G^-\chi_{\Psi<0}\right) v,$$

which proves the lemma. □

As a consequence of this lemma we have

$$\int_0^{2\pi} g(u_k)\frac{u_k}{\|u_k\|_\infty} \to G^+ \int_{\Psi>0} \Psi + G^- \int_{\Psi<0} \Psi. \tag{5.6}$$

Combining (5.1), (5.2), and (5.6), we now find that

$$0 = -\Big[G^+ \int_{\Psi>0} \Psi + G^- \int_{\Psi<0} \Psi\Big],$$

a contradiction of (LLD). Hence, $\{u_k\}_{k=1}^\infty$ is a bounded sequence in L^∞.

We note that $\{E(u_k)\}_{k=1}^\infty$ is bounded by hypothesis and all the integral terms, except the one involving u_k', are bounded by virtue of $\{u_k\}_{k=1}^\infty$ being bounded in L^∞. Hence, $\{u_k\}_{k=1}^\infty$ is a bounded sequence in H.

Now, as before, consider $\nabla E(u_k) = P(u_k) + S(u_k) + T(u_k)$. Since $\{u_k\}_{k=1}^\infty$ is bounded in H, then there exists a subsequence such that $u_k \overset{H}{\rightharpoonup} u$ and $u_k \overset{L^2,C}{\to} u$. Now, taking a limit, we see that

$$0 = \lim_{k\to\infty} \nabla E(u_k) = \lim_{k\to\infty} \left(P(u_k) + S(u_k) + T(u_k)\right)$$

and since P is invertible, $S(u_k) \to S(u)$, and $T(u_k) \to T(u)$, we may rearrange the equation to see that

$$u_k \overset{H}{\to} u = P^{-1}(-S(u) - T(u)).$$

Hence $\{u_k\}_{k=1}^\infty$ has a subsequence which converges in H, and therefore we have satisfied (PS). □

6. Main Result

Theorem 6.1. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is bounded and continuous. If $(a, b) \notin \Sigma$ or if $(a, b) \in \Sigma$ and (LLD) is satisfied, then there exists at least one weak solution to* (1.1).

Proof. Recall that the functional E satisfies (PS) due to Theorem 5.1. Also, if

$$\Gamma := \{\gamma : B_R \subseteq X \to H : \gamma\big|_{\partial B_R}(x) = x, \gamma \text{ cont. }\},$$

then

$$\inf_{\gamma\in\Gamma} \sup_{x\in B_R} E(\gamma(x)) > \sup_{x\in\partial B_R} E(x),$$

due to Lemma 4.3. Hence, by Theorem 2.1,

$$c := \inf_{\gamma\in\Gamma} \sup_{x\in X} E(\gamma(x))$$

is a critical value, and so (1.1) has a weak solution. □

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