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D'ALEMBERT'S FORMULA AND PERIODIC MILD SOLUTIONS TO ITERATED HIGHER-ORDER DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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Abstract. We give necessary and sufficient conditions for the periodicity of solutions of mild solutions to the iterated higher-order differential equation

$$\prod_{j=1}^{n}(\frac{d}{dt} - A_j)u(t) = f(t), \quad 0 \le t \le T,$$

in a Hilbert space. Our results are illustrated with examples and applications.

1. Introduction

In this article we study the periodicity of solutions of the iterated higher-order differential equation

$$\prod_{j=1}^{n}\big(\frac{d}{dt} - A_j\big)u(t) = f(t), \quad 0 \le t \le T, \tag{1.1}$$

where A_j are linear, closed and mutually commuting operators on a Hilbert space E, and f is a function from $[0,T]$ to E.

The asymptotic behavior and, in particular, the periodicity of solutions of the higher-order differential equation

$$u^{(n)}(t) = Au(t) + f(t), \quad 0 \le t \le T, \tag{1.2}$$

has been a subject of intensive study for recent decades. When $n = 1$, it is well known [7] that, if A is an $n \times n$ matrix on $\mathbb{C}^n$, then (1.2) admits a unique T-periodic solution for each continuous T-periodic forcing term f if and only if $\lambda_k = 2k\pi/T$, $k \in \mathbb{Z}$, are not eigen-values of A. That result was extended by Krein and Dalecki [4] to the Cauchy problem in an abstract Banach space. It was shown [4, Theorem II 4.3] that, if A is a linear, bounded operator on E, then (1.2) admits a unique T-periodic solution for each $f \in C[0,T]$ if and only if $2k\pi i/T \in \varrho(A)$, $k \in \mathbb{Z}$. Here $\varrho(A)$ denotes the resolvent set of A. For an unbounded operator A, the situation changes dramatically and the above statement generally fails. When A generates a strongly continuous semigroup, periodicity of solutions of (1.4) has intensively been

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studied recently (see e.g. [9, 10, 14, 18]). Corresponding results on the periodic solutions of the second order differential equation were obtained in [3, 20], when A is the generator of a cosine family. Related results on the periodicity of solutions of (1.2), when A is a closed operator, can be found in [5, 8, 12, 13, 19] and the references therein.

Unfortunately, for the complete higher-order differential equations, we have little consideration about the regularity of their solutions, mainly because of the complexity of the structure of the equation. In [15] and [16], the authors studied the iterated higher-order Cauchy problem of the type

$$\begin{gathered}\prod_{j=1}^{n}\big(\frac{d}{dt}-A_j\big)u(t)=0, \quad t>0,\\ u^{(j)}(0)=x_j\in E \quad (j=0,1,\dots,n-1)\end{gathered} \tag{1.3}$$

and stated that, under some certain conditions, (1.3) is well posed if and only if A_i are generators of C_0-semigroups. Moreover, they found the formula of solutions in the form $u(t)=\sum_1^n u_i(t)$, where $(d/dt-A_j)u_i=0$. That result suggests that (1.3) is in some sense the correct way to consider higher-order Cauchy problems. Later, in [17], the nonautonomous version of iterated evolution equation (1.3) was studied, where a nice structure of the solutions was found.

In this paper we investigate the periodicity of mild solutions of the iterated higher-order differential equation (1.1) when A_j, $j=0,1,\dots,n-1$, are linear and closed operators on a Hilbert space E. The main tool we use here is the Fourier series method. For an integrable function $f(t)$ from $[0,T]$ to E, the Fourier coefficient of $f(t)$ is defined by

$$f_k=\frac{1}{T}\int_0^T f(s)e^{-2k\pi is/T}ds, \quad k\in Z.$$

Then $f(t)$ can be represented by Fourier series

$$f(t)\approx\sum_{k=-\infty}^{\infty}e^{2k\pi it/T}f_k.$$

We first give the definition of mild solution to (1.1), when $n=1$.

Definition 1.1. (i) A continuous function $u(\cdot)$ is a mild solution of the differential equation

$$u'(t)=Au(t)+f(t), \quad t\in[0,T], \tag{1.4}$$

if $\int_0^t u(s)ds\in D(A)$ and

$$u(t)=u(0)+A\int_0^t u(s)ds+\int_0^t f(s)ds$$

for all $t\in[0,T]$.

(ii) Suppose f is a continuous function. A function $u(\cdot)$ is a classical solution of (1.4) if $u(t)$ is continuously differentiable, $u(t)\in D(A)$, and (1.4) holds for all $t\in[0,T]$.

It is not hard to see that, if a mild solution of (1.4) is continuously differentiable, then it is a classical solution. Furthermore, if $u(t)$ is a mild solution on $[0,T]$ with $u(0)=u(T)$, then $u(t)$ can be continuously extended to a T-periodic mild solution of (1.4) on $\mathbb{R}$, provided $f(t)$ has been extended T-periodically, too. Therefore, we call a mild solution of (1.4) T-periodic if $u(0) = u(T)$.

We now consider the iterated differential equation (1.1) and employ the substitution (see also [15]) by defining $U(\cdot) := (u_1(\cdot), u_2(\cdot), \dots, u_n(\cdot))^T$ with

$$\begin{gathered}
u_1(\cdot) = u(\cdot) \\
u_2(\cdot) = u_1(\cdot)' - A_1 u_1(\cdot) \\
\dots \\
u_n(\cdot) = u_{n-1}(\cdot)' - A_{n-1} u_{n-1}(\cdot).
\end{gathered}$$

Then we have

$$\begin{aligned}
u_1(\cdot)' &= A_1 u_1(\cdot) + u_2(\cdot); \\
u_2(\cdot)' &= A_2 u^1(\cdot) + u^3(\cdot); \\
&\dots \\
u_{n-1}(\cdot)' &= A_{n-1} u^{n-1}(\cdot) + u_n(\cdot); \\
u_n(\cdot)' &= A_n u_n(\cdot) + f(\cdot).
\end{aligned}$$

That can be written in matrix form as

$$U'(t) = \mathcal{C}U(t) + F(t), \quad t \in \mathbb{R}, \tag{1.5}$$

on the product space E^n, where $F(t) = (0, 0, \dots, 0, f(t))^T$ and

$$\mathcal{C} := \begin{pmatrix} A_1 & I & 0 & \cdots & \cdots & 0 \\ 0 & A_2 & I & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & A_{n-1} & I \\ 0 & \cdots & & \cdots & 0 & A_n \end{pmatrix} \tag{1.6}$$

with $D(\mathcal{C}) := D(A_1) \times D(A_2) \times \dots \times D(A_n)$. Note that the product space E^n is again a Hilbert space with the norm $\|(x_1, x_2, \dots, x_n)^T\| := \sqrt{\sum_1^n \|x_i\|^2}$. In [15], is was stated that $\mathcal{C}$ is generator of a C_0-semigroup in E^n if and only if A_i $(i = 1, 2, \dots, n)$ are generators of C_0 semigroups in E. That suggests the following definition of mild (classical) solutions for iterated higher-order differential equation.

Definition 1.2. A continuous function $u(\cdot)$ is a mild (classical) solution of the higher-order differential equation (1.1) if u is the first component of a mild (classical) solution of the first-order differential equation (1.5).

We next establish the relationship between the Fourier coefficients of the periodic solutions of (1.1) and those of the inhomogeneity f. Then, as the main result, we give an equivalent condition so that (1.1) admits a unique periodic solution for each inhomogeneity f in a certain function space. Our result generalizes some well-known ones, as in Section 3 we present several particular cases, among which, A generates a C_0 semigroup and a cosine family.

Throughout this article, if not otherwise indicated, we assume that E is a complex Hilbert space and A_i, $i = 1, \dots, n$, are linear, closed and mutually commuting operators on E with $D = D(A_j)$, $j = 1, 2, \dots, n$, dense in E. The spectrum and resolvent set of A are denoted by $\sigma(A)$ and $\varrho(A)$, respectively and $(\lambda - A)^{-1}$ is denoted by $R(\lambda, A)$. Two unbounded operators A and B are said to commute if for each $\lambda_1 \in \varrho(A)$ and $\lambda_2 \in \varrho(B)$ we have $(\lambda_1 - A)^{-1}(\lambda_2 - B)^{-1} = (\lambda_2 - B)^{-1}(\lambda_1 - A)^{-1}$. That definition is equivalent to the fact that $AB = BA$ as the following simple lemma shows.

Lemma 1.3. *Suppose A and B are two commuting operators. Then for each $x \in D$ with $Bx \in D$ we have $Ax \in D$ and $BAx = ABx$.*

Proof. Let $\alpha \in \varrho(A)$ and $\beta \in \varrho(B)$ and put $y = ABx$. Then

$$(\alpha - A)(\beta - B)x = \alpha\beta x - \beta Ax - \alpha Bx + y$$

or

$$\begin{aligned} x &= (\beta - B)^{-1}(\alpha - A)^{-1}(\alpha\beta x - \beta Ax - \alpha Bx + y) \\ &= (\alpha - A)^{-1}(\beta - B)^{-1}(\alpha\beta x - \beta Ax - \alpha Bx + y), \end{aligned}$$

which implies

$$(\beta - B)(\alpha - A)x = \alpha\beta x - \beta Ax - \alpha Bx + y$$

or $BAx = y = ABx$. □

Let $J = [0, T]$. For the sake of simplicity (and without loss of generality) we assume $T = 1$. For $p \geq 1$, $L_p(J)$ denotes the space of E-valued p-integrable functions on J with $\|f\|_{L_p(J)} = (\int_0^1 \|f(t)\|^p dt)^{1/p} < \infty$ and $C(J)$ the space of continuous functions on J with $\|f\|_{C(J)} = \max_J \|f(t)\|$. Moreover, if $m \geq 1$, we define the function space

$$W_2^m(J) := \{f \in L_2(J) : f', f'', \dots, f^{(m)} \in L_2(J)\}$$

which is a Hilbert space with the norm

$$\|f\|_{W_2^m} := \Big(\sum_{j=0}^{m} \|f^{(j)}\|^2_{L_2(J)}\Big)^{1/2}.$$

We will use the following simple lemma.

Lemma 1.4. *If F is an absolutely continuous function on J such that $f = F' \in L_p(J)$, then for $k \neq 0$ we have*

$$F_k = \frac{1}{2k\pi i} f_k + \frac{F(0) - F(1)}{2k\pi i},$$

where f_k and F_k are the Fourier coefficients of f and F, respectively.

Finally, a continuous function $u(\cdot)$ is said to be a 1-periodic solution of (1.1) (or to be a solution in $W_2^m(J)$) if the corresponding mild solution $\mathcal{U}(\cdot)$ of (1.5) is 1-periodic (or in $W_2^m(J, E^n)$) respectively.

2. Periodic mild solutions of higher-order differential equations

Proposition 2.1. *Suppose $f \in L_p(J)$ and u is a mild solution of the first-order differential equation*
$$u'(t) = Au(t) + f(t), \quad t \in J. \tag{2.1}$$
Then
$$(2k\pi i - A)u_k - f_k = u(0) - u(1) \tag{2.2}$$
for $k \in \mathbb{Z}$.

Proof. Let u be a mild solution of (1.4), i.e.,
$$u(t) = u(0) + A\int_0^t u(s)ds + \int_0^t f(s)ds. \tag{2.3}$$
First, if $k = 0$, then using (2.3) for $t = 1$ we have $u(1) = u(0) + Au_0 + f_0$, from which (2.2) holds for $k = 0$.

Next, if $k \neq 0$, taking the k^{th} Fourier coefficient on both sides of (2.3), we obtain
$$\begin{aligned} u_k &= A\int_0^1 e^{-2k\pi is}\int_0^s u(\tau)d\tau ds + \int_0^1 e^{-2k\pi is}\int_0^s f(\tau)d\tau ds \\ &= AU_k + F_k, \end{aligned}$$
where U_k is the k^{th} Fourier coefficient of $U(t) = \int_0^t u(\tau)d\tau$ and F_k is the k^{th} Fourier coefficient of $F(t) = \int_0^t f(\tau)d\tau$. Using now Lemma 1.4 for $U(t) = \int_0^t u(\tau)d\tau$ and $F(t) = \int_0^t f(\tau)d\tau$ we obtain
$$u_k = \frac{A(u_k - U(1))}{2k\pi i} + \frac{f_k - F(1)}{2k\pi i},$$
from which we have
$$\begin{aligned} (2k\pi i - A)u_k &= f_k - (AU(1) + F(1)) \\ &= f_k - (A\int_0^1 u(s)ds + \int_0^1 f(s)ds) \\ &= f_k + (u(0) - u(1)). \end{aligned}$$
Hence, (2.2) holds. Here we used the fact that u is a mild solution of (1.4), implying $u(1) = u(0) + A\int_0^1 u(s)ds + \int_0^1 f(s)ds$. $\square$

If u is a 1-periodic solution of (1.4), then we have a nice relationship between Fourier coefficients of u and those of f, as the following result shows.

Corollary 2.2. *Suppose $f \in L_p(J)$ and u is a continuous mild solution of* (1.4)*. Then u is 1-periodic if and only if*
$$(2k\pi i - A)u_k = f_k \tag{2.4}$$
for every $k \in \mathbb{Z}$.

Next we give a sufficient condition for the existence of 1-periodic mild solutions of (1.1).

Proposition 2.3. *Suppose $f \in L_p(J)$. Then the iterated differential equation* (1.1) *admits a continuous, 1-periodic mild solution if and only if there is a sequence $(u_k)_{k=-\infty}^{\infty} \subset E$, such that*

(i) *For each m, $0 \le m \le n-1$, the function*
$$v_m(t) := \sum_{k=-\infty}^{\infty} e^{-2k\pi it}[\prod_{j=1}^{m}(2k\pi i - A_j)]u_k \tag{2.5}$$
is continuous on $[0,1]$ and

(ii) *The equality*
$$\prod_{j=1}^{n}(2k\pi i - A_j)u_k = f_k \tag{2.6}$$
holds for every $k \in \mathbb{Z}$.

Proof. Suppose (1.1) admits a 1-periodic mild solution u. By the definition of solution u, there is a 1-periodic mild solution $U(t) = (u_1(t), u_2(t), \dots, u_n(t))^T$ of (1.5) with $u = u_1$. By Corollary 2.2, we have
$$(2k\pi i - \mathcal{C})U_k = (0, 0, \dots, f_k)^T$$
or
$$\begin{pmatrix} (2k\pi i - A_1) & -I & 0 & \cdots & \cdots & 0 \\ 0 & (2k\pi i - A_2) & -I & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & -I \\ 0 & \cdots & & \cdots & 0 & (2k\pi i - A_n) \end{pmatrix} \begin{pmatrix} (u_1)_k \\ (u_2)_k \\ \vdots \\ (u_n)_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f_k \end{pmatrix},$$
which implies
$$\begin{aligned} (u_2)_k &= (2k\pi i - A_1)(u_1)_k = (2k\pi i - A_1)u_k; \\ (u_3)_k &= (2k\pi i - A_2)(u_2)_k = (2k\pi i - A_2)(2k\pi i - A_1)u_k; \\ &\dots \\ (u_n)_k &= (2k\pi i - A_{n-1})(u_{n-1})_k = \prod_{j=1}^{n-1}(2k\pi i - A_j)u_k; \\ f_k &= (2k\pi i - A_n)(u_n)_k = \prod_{j=1}^{n}(2k\pi i - A_j)u_k. \end{aligned} \tag{2.7}$$
Hence, for each j, $0 \le j \le n-1$, the function
$$v_j(t) := \sum_{k=-\infty}^{\infty} e^{2k\pi it}[\prod_{z=1}^{j}(2k\pi i - A_z)]u_k$$
is the same as $u_j(t)$, which is continuous on $[0,1]$. Moreover, (2.6) follows from (2.7).

Conversely, suppose for each j, $0 \le j \le n-1$, the function (2.5) is continuous on $[0,1]$ and(2.6) holds. We show that there exists a mild solution U of (1.5), which is 1-periodic. To this end, for each $k \in \mathbb{Z}$ we define
$$u_1(t) := \sum_{k=-\infty}^{\infty} e^{2k\pi it}u_k;$$

$$u_2(t) := \sum_{k=-\infty}^{\infty} e^{2k\pi it}(2k\pi i - A_1)u_k;$$
$$\dots$$
$$u_n(t) := \sum_{k=-\infty}^{\infty} e^{2k\pi it}\prod_{j=1}^{n-1}(2k\pi i - A_j)u_k.$$

Then, by the assumption, $U(t) := (u_1(t), u_2(t), \dots, u_n(t))^T$ is a continuous function with the following Fourier coefficients:

$$\begin{aligned}
(u_2)_k &= (2k\pi i - A_1)u_k;\\
(u_3)_k &= (2k\pi i - A_2)(u_2)_k = (2k\pi i - A_2)(2k\pi i - A_1)u_k\\
&\dots\\
(u_n)_k &= (2k\pi i - A_{n-1})(u_{n-1})_k = \prod_{j=1}^{n-1}(2k\pi i - A_j)u_k
\end{aligned}$$

and by (2.6),

$$(2k\pi i - A_n)(u_n)_k = \prod_{j=1}^{n}(2k\pi i - A_j)u_k = f_k$$

Hence,

$$\begin{pmatrix}
(2k\pi i - A_1) & -I & 0 & \cdots & \cdots & 0\\
0 & (2k\pi i - A_2) & -I & \ddots & & \vdots\\
\vdots & & & \ddots & \ddots & \vdots\\
\vdots & & & \ddots & \ddots & \vdots\\
\vdots & & & \ddots & \ddots & -I\\
0 & \cdots & & \cdots & 0 & (2k\pi i - A_n)
\end{pmatrix}
\begin{pmatrix}(u_1)_k\\(u_2)_k\\ \vdots\\(u_n)_k\end{pmatrix}
=\begin{pmatrix}0\\0\\ \vdots\\ f_k\end{pmatrix}$$

or $(2k\pi - \mathcal{C})U_k = (0, 0, \dots, f_k)^T$. By Corollary 2.2, $U(t) = (u_1(t), u_2(t), \dots, u_n(t))^T$ is a 1-periodic mild solution of (1.5) and hence, $u(t)$ is a 1-periodic mild solution of (1.1). $\square$

Note that Proposition 2.1, Corollary 2.2 and Proposition 2.3 also hold if E is a Banach space. We now can state the main result of the paper.

Theorem 2.4. *Suppose E is a Hilbert space. Then the following are equivalent*

(i) *For each function $f \in W_2^1(J)$, Equation (1.1) admits a unique 1-periodic mild solution in $W_2^1(J)$;*

(ii) *For each $k \in \mathbb{Z}$ and $1 \le j \le n$, $2k\pi i \in \varrho(A_j)$ and there is a number $M > 0$ such that*

$$\sup_{k\in\mathbb{Z}} \|(2k\pi i - A_j)^{-1}(2k\pi i - A_{j+1})^{-1}\cdots(2k\pi i - A_n)^{-1}\| = M < \infty. \tag{2.8}$$

Proof. (i) $\Rightarrow$ (ii): Suppose for each function $f \in W_2^1(J)$, Equation (1.1) admits a unique 1-periodic mild solution $u \in W_2^1(J)$. By the definition of solution u in $W_2^1(J)$, the corresponding solution U of (1.5) belongs to $W_2^1(J, E^n)$. We prove that U is the only mild solution of (1.5) corresponding to f by showing that $U \equiv 0$ is the only mild solution of (1.5)corresponding to $f \equiv 0$. Indeed, if $f \equiv 0$, then $u \equiv 0$.

Hence, its Fourier coefficients $u_k = 0$ for all $k \in \mathbb{Z}$. In the proof of Theorem 2.3 we have $(u_2)_k = (2k\pi i - A_1)u_k = 0$ for all $k \in \mathbb{Z}$. Hence, $u_2(t) \equiv 0$. Similarly, we have $u_j(t) \equiv 0$ for all $1 \le j \le n$ and thus, $U(t) \equiv 0$.

Define the operator:

$$G : f \in W_2^1(J) \mapsto Gf \in W_2^1(J, E^n)$$

as follows: $(Gf)(t)$ is the unique solution of (1.5) corresponding to f. Then G is a linear, everywhere defined operator. We will prove its boundedness by showing G is a closed operator.

To this end, suppose $\{f_m\}_{m=1}^{\infty}$ is a sequence of functions in $F_1 = W_2^1(J)$ such that $f_m \to f$ in F_1 and Gf_m approaches some function $V = (V_1, V_1, \dots, V_n)^T$ in $F_2 = W_2^1(J, E^n)$ as $m \to \infty$. We show that $f \in D(G)$ and $Gf = V$.

Since Gf_m is a mild solution of (1.5) corresponding to f_m, we have

$$Gf_m(t) = Gf_m(0) + \mathcal{C}\int_0^t Gf_m(s)ds + \int_0^t F_m(s)ds.$$

Hence,

$$\begin{aligned}
(Gf_m)_1(t) &= (Gf_m)_1(0) + A_1\int_0^t (Gf_m)_1(s)ds + \int_0^t (Gf_m)_2(s)ds \\
(Gf_m)_2(t) &= (Gf_m)_2(0) + A_2\int_0^t (Gf_m)_2(s)ds + \int_0^t (Gf_m)_3(s)ds \\
&\dots \\
(Gf_m)_{n-1}(t) &= (Gf_m)_{n-1}(0) + A_{n-1}\int_0^t (Gf_m)_{n-1}(s)ds + \int_0^t (Gf_m)_n(s)ds \\
(Gf_m)_n(t) &= (Gf_m)_n(0) + A_n\int_0^t (Gf_m)_n(s)ds + \int_0^t f_m(s)ds.
\end{aligned} \tag{2.9}$$

Consider now the sequence $\{x_m\}_{m\ge1}$ in E, where $x_m = \int_0^t (Gf_m)_1(s)ds$. We have

$$x_m = \int_0^t (Gf_m)_1(s)ds \to \int_0^t V_1(s)ds$$

as $m \to \infty$, and from (2.9),

$$\begin{aligned}
A_1 x_m = A_1\int_0^t (Gf_m)_1(s)ds &= (Gf_m)_1(t) - (Gf_m)_1(0) - \int_0^t (Gf_m)_2(s)ds \\
&\to V_1(t) - V_1(0) - \int_0^t V_2(s)ds
\end{aligned}$$

as $m \to \infty$. Since A_1 is a closed operator, we have $\int_0^t V_1(s)ds \in D(A_1)$ and

$$A_1\int_0^t V_1(s)ds = V_1(t) - V_1(0) - \int_0^t V_2(s)ds,$$

which implies

$$V_1(t) = V_1(0) + A_1\int_0^t V_1(s)ds + \int_0^t V_2(s)ds.$$

In the same manner, we can show that

$$V_2(t) = V_2(0) + A_2\int_0^t V_2(s)ds + \int_0^t V_3(s)ds,$$

$$\dots$$

$$V_{n-1}(t) = V_{n-1}(0) + A_{n-1}\int_0^t V_{n-1}(s)ds + \int_0^t V_n(s)ds,$$

$$V_n(t) = V_n(0) + A_n\int_0^t V_n(s)ds + \int_0^t f(s)ds,$$

i.e., V is a mild solution of (1.5) corresponding to f and consequently, $Gf = V$. So, G is a bounded operator from F_1 to F_2.

Next we show that $2k\pi i \in \varrho(A_j)$ for each $k \in \mathbb{Z}$ and each $1 \le j \le n$. Let x be any vector in E, $k \in \mathbb{Z}$ and take $f(t) = e^{2k\pi it}x$ and $V = (V_1, V_2, \dots, V_n)^T$ be the unique mild solution of (1.5) corresponding to f. From Fourier coefficient Identity (2.7) we have

$$\prod_{j=1}^n (2k\pi i - A_j)(V_n)_k = f_k = x,$$

which shows $\prod_{j=1}^n (2k\pi i - A_j)$ and hence, $(2k\pi i - A_n)$, is surjective. Using Lemma 1.3 we have $(2k\pi i - A_j)$ is surjective for each $1 \le j \le n$.

Assume now that for some $1 \le j \le n$, $(2k\pi i - A_j)$ contrarily is not injective. Without loss of generality we can assume that A_j is the first operator with non-injective $(2k\pi i - A_j)$, i.e., $(2k\pi i - A_l)$ are injective for $1 \le l < j$. Then there is a vector $y_0 \ne 0$ in E with $(2k\pi i - A_j)y_0 = 0$. Put $y(t) := e^{2k\pi it}y_0$, then it is not hard to see that

$$y(t) = y(0) + A_j\int_0^t y(s)ds$$

holds for $t \in J$. Hence, we can see that the equation (1.5) with $f \equiv 0$ has two different mild solutions in $W_2^1(J, E^n)$ $U(t) \equiv 0$ and

$$V(t) = e^{2k\pi it}\begin{pmatrix} R(2k\pi i, A_1)\dots R(2k\pi i, A_{j-1})y_0 \\ R(2k\pi i, A_2)\dots R(2k\pi i, A_{j-1})y_0 \\ \vdots \\ R(2k\pi i, A_{j-1})y_0 \\ y_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

which contradicts the uniqueness of mild solutions. Hence, $(2k\pi i - A_j)$ is injective and thus, $2k\pi i \in \varrho(A_j)$ for all $j = 1, 2, \dots, n$.

Finally, we show that (2.8) holds. To this end, for any $x \in E$, let $f(t) := e^{2k\pi it}x$. Then, by (2.7) we see that

$$U(t) = e^{2k\pi it}\begin{pmatrix} R(2k\pi i, A_1)\dots R(2k\pi i, A_n)x \\ R(2k\pi i, A_2)\dots R(2k\pi i, A_n)x \\ \vdots \\ R(2k\pi i, A_n)x \end{pmatrix}$$

is the unique mild solution of (1.5) corresponding to $f = e^{2k\pi it}x$. It is not difficult to compute that

$$\|f\|^2_{W_2^1(J)} = (1 + 4k^2\pi^2)\|x\|^2,$$

$$\|U\|^2_{W^1_2(J,E^n)} = (1+4k^2\pi^2)\sum_{j=1}^{n} \|R(2k\pi i, A_j)\cdot R(2k\pi, A_{j+1})\dots R(2k\pi i, A_n)x\|^2.$$

Using the inequality $\|U\|^2_{W^1_2(J,E^n)} \le \|G\|^2\|f\|^2_{W^1_2(J)}$ we have

$$\sum_{j=1}^{n} \|R(2k\pi i, A_j)R(2k\pi, A_{j+1})\cdots R(2k\pi i, A_n)x\|^2 \le \|G\|^2\|x\|^2$$

for all $x\in E$, from which we obtain

$$\|R(2k\pi i, A_j)R(2k\pi, A_{j+1})\cdots R(2k\pi i, A_n)\| \le \|G\|,$$

and hence, (2.8) holds.

(ii) $\Rightarrow$ (i): Suppose for each $k\in\mathbb{Z}$ and $1\le j\le n$, $2k\pi i\in\varrho(A_j)$ and (2.8) holds. If $f(t)=e^{2k\pi it}x$ for some $k\in\mathbb{Z}$ and $x\in E$, then, from the previous part of the proof, we see that

$$U(t) = e^{2k\pi it}\begin{pmatrix} R(2k\pi i, A_1)\dots R(2k\pi i, A_n)x \\ R(2k\pi i, A_2)\dots R(2k\pi i, A_n)x \\ \vdots \\ R(2k\pi i, A_n)x \end{pmatrix}$$

is the unique mild solution of (1.5), which is in $W^1_2(J,E^n)$.

Next, if $f(t)=\sum_k e^{2k\pi it}x_k$ for any finite sequence $\{x_k\}_k\subset E$. Using the linearity of mild solutions, we see that

$$U(t) = \sum_k e^{2k\pi it}\begin{pmatrix} R(2k\pi i, A_1)\dots R(2k\pi i, A_n)x_k \\ R(2k\pi i, A_2)\dots R(2k\pi i, A_n)x_k \\ \vdots \\ R(2k\pi i, A_n)x_k \end{pmatrix}$$

is the unique mild solution of (1.5) corresponding to f. Moreover, by using the standard calculation we have

$$\|f\|^2_{W^1_2(J)} = \sum_k (1+4k^2\pi^2)\|x_k\|^2$$

and

$$\begin{aligned}
&\|U\|^2_{W^1_2(J,E^n)} \\
&= \sum_k (1+4k^2\pi^2)\sum_{j=1}^{n} \|R(2k\pi i, A_j)R(2k\pi, A_{j+1})\cdots R(2k\pi i, A_n)x_k\|^2 \\
&\le \sum_k (1+4k^2\pi^2)\sum_{j=1}^{n} \|R(2k\pi i, A_j)R(2k\pi, A_{j+1})\cdots R(2k\pi i, A_n)\|^2\|x_k\|^2 \\
&\le \sum_k (1+4k^2\pi^2)\sum_{j=1}^{n} M^2\|x_k\|^2 \\
&= nM^2\sum_k (1+4k^2\pi^2)\|x_k\|^2 \\
&= nM^2\|f\|^2,
\end{aligned}$$

which implies

$$\|U\|_{W_2^1(J,E^n)} \le \sqrt{n}M\|f\|_{W_2^1(J)}. \tag{2.10}$$

Put

$$\mathcal{L}(J) := \{f(t) = \sum_k e^{2k\pi it}x_k : \{x_k\} \text{ is a finite sequence in } E\}.$$

Inequality (2.10) holds for all $f \in \mathcal{L}(J)$. Observe that $\mathcal{L}(J)$ is dense in $W_2^1(J)$. Suppose now that f is any function in $W_2^1(J)$. Then there is a sequence $\{f_m\} \subset \mathcal{L}(J)$ such that $\lim_{m\to\infty} f_m = f$ in $W_2^1(J)$. Let U_m be the unique mild solution of (1.5) corresponding to f_m. Since $(f_m - f_q) \in \mathcal{L}(J)$ for all $m, q \in \mathbb{N}$ we have $\|U_m - U_q\|_{W_2^1(J,E^n)} \le \sqrt{n}M\|f_m - f_q\|_{W_2^1(J)} \to 0$ for $m, q \to \infty$. Hence, there exists a function $U \in W_2^1(J,E^n)$ such that $\lim_{m\to\infty} U_m = U$ in $W_2^1(J,E^n)$. Using the same arguments as in the (i) $\Rightarrow$ (ii) part, where we proved that G is a bounded operator, we can show that U is a mild solution of (1.5) corresponding to f. The uniqueness of U is obvious, and the proof is complete. ☐

Example. Suppose A_i $(i = 1, 2, \dots, n)$ are mutually commuting infinitesimal generators of C_0 semigroups on E. Then $\mathcal{C}$ generates a C_0-semigroup $\mathcal{T}(t)$ in E^n (see [15]) and the mild solution of (1.5) can be expressed by

$$U(t) = \mathcal{T}(t)U(0) + \int_0^t \mathcal{T}(t-\tau)F(\tau)d\tau, \tag{2.11}$$

where $F(t) := (0, 0, \dots, 0, f(t))^T$. In this case each 1-periodic mild solution of (1.1) in $W_2^1(J)$ is a classical solution, as the following theorem states.

Theorem 2.5. *If A_i generates C_0 semigroup in E, then the following statements are equivalent.*

(i) *For each $f \in L_2(J)$ Equation* (1.1) *admits a unique 1-periodic mild solution.*
(ii) *For each $f \in W_2^1(J)$, Equation* (1.1) *admits a unique 1-periodic classical solution.*
(iii) *For each $f \in W_2^1(J)$, Equation* (1.1) *admits a unique 1-periodic mild solution in $W_2^1(J)$.*
(iv) *For each $k \in \mathbb{Z}$ and $0 \le j \le n$, $2k\pi i \in \varrho(A_j)$ and*
$$\sup_{k\in\mathbb{Z}} \|(2k\pi i - A_j)^{-1}(2k\pi i - A_{j+1})^{-1} \cdots (2k\pi i - A_n)^{-1}\| < \infty. \tag{2.12}$$

Proof. The equivalence between (i) and (ii) can be shown by standard argument and between (iii) and (iv) is from Theorem 2.4 and the implication (ii) $\to$(iii) is obvious. It remains to show (iii) $\to$(ii). To this end, let $U(\cdot)$ be the unique 1-periodic mild solution of (1.5), which belong to $W_2^1(J,E^n)$. Since $F(t) \in W_2^1(J,E^n)$, we have $\int_0^t \mathcal{T}(t-\tau)F(\tau)d\tau \in D(\mathcal{C})$ and $t \to \int_0^t \mathcal{T}(t-\tau)F(\tau)d\tau$ is continuously differentiable (see e.g. [11]). From (2.11) we obtain $\mathcal{T}(\cdot)U(0) \in W_p^1(J,E^n)$. It follows that $\mathcal{T}(t)U(0) \in D(\mathcal{C})$ for $t > 0$ (since $t \mapsto \mathcal{T}(t)x$ is differentiable at t_0 if and only if $\mathcal{T}(t_0)x \in D(\mathcal{C})$). Hence, $U(1)$, and thus, $U(0)$ (the same as $U(1)$) belongs to $D(\mathcal{C})$. So U is a classical solution. The uniqueness of the 1-periodic classical solution is obvious. ☐

If $n = 1$, then Theorem 2.5 becomes Gearhart theorem in [6] (See also [14]). We see clearly that statement (iv) in Theorem 2.5 holds if for each j, $0 \le j \le n$, we have $2k\pi i \in \varrho(A_j)$ and

$$\sup_{k\in\mathbb{Z}} \|(2k\pi i - A_j)^{-1}\| < \infty. \tag{2.13}$$

But in general, condition (2.13) is stronger than (2.12) (they are equivalent if $n = 1$). Hence, unless $n = 1$, the existence and uniqueness of 1-periodic mild solution of (1.1) does not imply (2.13). The next example shows that in some special cases the two conditions are equivalent.

Example. Suppose $B = A^2$, where A generates a C_0 group on E. Consider the second-order differential equation

$$u''(t) = Bu(t) + f(t), \quad 0 \le t \le 1. \tag{2.14}$$

We can rewrite (2.14) as

$$(\frac{d}{dt} - A)(\frac{d}{dt} + A)u(t) = f(t).$$

Hence, from Theorem 2.5 we have the following result.

Theorem 2.6. *The following statements are equivalent:*

(i) *For each function $f \in W_2^1(J)$, Equation* (2.14) *admits a unique 1-periodic mild solution in $W_2^1(J)$;*
(ii) *For each function $f \in W_2^1(J)$, Equation* (2.14) *admits a unique 1-periodic classical solution;*
(iii) *For each $k \in \mathbb{Z}$, $2k\pi i \in \varrho(B)$ and*
$$\sup_{k\in\mathbb{Z}} \|(2k\pi i - A)^{-1}\| < \infty. \tag{2.15}$$
(iii) *For each $k \in \mathbb{Z}$, $-4k^2\pi^2 \in \varrho(B)$ and*
$$\sup_{k\in\mathbb{Z}} \|(4k^2\pi^2 i + B)^{-1}\| < \infty. \tag{2.16}$$

Proof. Let $A_1 = -A$ and $A_2 = A$. Then it is easy to see that $\sup_{k\in\mathbb{Z}} \|(2k\pi i - A_1)^{-1}\| < \infty$ is equivalent to $\sup_{k\in\mathbb{Z}} \|(2k\pi i - A_2)^{-1}\| < \infty$, and that completes the proof. □

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