Hilbert Space Methods for Partial Differential Equations

R. E. Showalter

Electronic Journal of Differential Equations
Monograph 01, 1994.

Preface

This book is an outgrowth of a course which we have given almost periodically over the last eight years. It is addressed to beginning graduate students of mathematics, engineering, and the physical sciences. Thus, we have attempted to present it while presupposing a minimal background: the reader is assumed to have some prior acquaintance with the concepts of "linear" and "continuous" and also to believe L^2 is complete. An undergraduate mathematics training through Lebesgue integration is an ideal background but we dare not assume it without turning away many of our best students. The formal prerequisite consists of a good advanced calculus course and a motivation to study partial differential equations.

A problem is called *well-posed* if for each set of data there exists exactly one solution and this dependence of the solution on the data is continuous. To make this precise we must indicate the space from which the solution is obtained, the space from which the data may come, and the corresponding notion of continuity. Our goal in this book is to show that various types of problems are well-posed. These include boundary value problems for (stationary) elliptic partial differential equations and initial-boundary value problems for (time-dependent) equations of parabolic, hyperbolic, and pseudo-parabolic types. Also, we consider some nonlinear elliptic boundary value problems, variational or uni-lateral problems, and some methods of numerical approximation of solutions.

We briefly describe the contents of the various chapters. Chapter I presents all the elementary Hilbert space theory that is needed for the book. The first half of Chapter I is presented in a rather brief fashion and is intended both as a review for some readers and as a study guide for others. Non-standard items to note here are the spaces $C^m(\bar{G})$, V^*, and V'. The first consists of restrictions to the closure of G of functions on $\mathbb{R}^n$ and the last two consist of conjugate-linear functionals.

Chapter II is an introduction to distributions and Sobolev spaces. The latter are the Hilbert spaces in which we shall show various problems are well-posed. We use a primitive (and non-standard) notion of distribution which is adequate for our purposes. Our distributions are conjugate-linear and have the pedagogical advantage of being independent of any discussion of topological vector space theory.

Chapter III is an exposition of the theory of linear elliptic boundary value problems in variational form. (The meaning of "variational form" is explained in Chapter VII.) We present an abstract Green's theorem which permits the separation of the abstract problem into a partial differential equation on the region and a condition on the boundary. This approach has the pedagogical advantage of making optional the discussion of regularity theorems. (We construct an operator ∂ which is an extension of the normal derivative on the boundary, whereas the normal derivative makes sense only for appropriately regular functions.)

Chapter IV is an exposition of the generation theory of linear semigroups of contractions and its applications to solve initial-boundary value problems for partial differential equations. Chapters V and VI provide the immediate extensions to cover evolution equations of second order and of implicit type. In addition to the classical heat and wave equations with standard boundary conditions, the applications in these chapters include a multitude of non-standard problems such as equations of pseudo-parabolic, Sobolev, viscoelasticity, degenerate or mixed type; boundary conditions of periodic or non-local type or with time-derivatives; and certain interface or even global constraints on solutions. We hope this variety of applications may arouse the interests even of experts.

Chapter VII begins with some reflections on Chapter III and develops into an elementary alternative treatment of certain elliptic boundary value problems by the classical Dirichlet principle. Then we briefly discuss certain unilateral boundary value problems, optimal control problems, and numerical approximation methods. This chapter can be read immediately after Chapter III and it serves as a natural place to begin work on nonlinear problems.

There are a variety of ways this book can be used as a text. In a year course for a well-prepared class, one may complete the entire book and supplement it with some related topics from nonlinear functional analysis. In a semester course for a class with varied backgrounds, one may cover Chapters I, II, III, and VII. Similarly, with that same class one could cover in one semester the first four chapters. In any abbreviated treatment one could omit I.6, II.4, II.5, III.6, the last three sections of IV, V, and VI, and VII.4. We have included over 40 examples in the exposition and there are about 200 exercises. The exercises are placed at the ends of the chapters and each is numbered so as to indicate the section for which it is appropriate.

Some suggestions for further study are arranged by chapter and precede the Bibliography. If the reader develops the interest to pursue some topic in one of these references, then this book will have served its purpose.

R. E. Showalter; Austin, Texas, January, 1977.

Contents

Chapter I

Elements of Hilbert Space

1 Linear Algebra

We begin with some notation. A function F with domain $\mathrm{dom}(F) = A$ and range $\mathrm{Rg}(F)$ a subset of B is denoted by $F : A \to B$. That a point $x \in A$ is mapped by F to a point $F(x) \in B$ is indicated by $x \mapsto F(x)$. If S is a subset of A then the *image* of S by F is $F(S) = \{F(x) : x \in S\}$. Thus $\mathrm{Rg}(F) = F(A)$. The pre-image or inverse image of a set $T \subset B$ is $F^{-1}(T) = \{x \in A : F(x) \in T\}$. A function is called *injective* if it is one-to-one, *surjective* if it is onto, and *bijective* if it is both injective and surjective. Then it is called, respectively, an *injection, surjection*, or *bijection*.

$\mathbb{K}$ will denote the field of scalars for our vector spaces and is always one of $\mathbb{R}$ (real number system) or $\mathbb{C}$ (complex numbers). The choice in most situations will be clear from the context or immaterial, so we usually avoid mention of it.

The "strong inclusion" $K \subset\subset G$ between subsets of Euclidean space $\mathbb{R}^n$ means K is compact, G is open, and $K \subset G$. If A and B are sets, their Cartesian product is given by $A \times B = \{[a, b] : a \in A,\ b \in B\}$. If A and B are subsets of $\mathbb{K}^n$ (or any other vector space) their set sum is $A + B = \{a + b : a \in A,\ b \in B\}$.

1.1

A *linear space* over the field $\mathbb{K}$ is a non-empty set V of vectors with a binary operation *addition* $+ : V \times V \to V$ and a *scalar multiplication* $\cdot : \mathbb{K} \times V \to V$

such that $(V,+)$ is an Abelian group, i.e.,

$$\begin{array}{ll}
(x+y)+z = x+(y+z)\ , & x,y,z \in V\ , \\
\text{there is a zero } \theta \in V : x+\theta = x\ , & x \in V\ , \\
\text{if } x \in V, \text{ there is } -x \in V : x+(-x) = \theta\ , & \text{and} \\
x+y = y+x\ , & x,y \in V\ ,
\end{array}$$

and we have

$$\begin{array}{lll}
(\alpha+\beta)\cdot x = \alpha\cdot x+\beta\cdot x\ , & \alpha\cdot(x+y) = \alpha\cdot x+\alpha\cdot y\ , & \\
\alpha\cdot(\beta\cdot x) = (\alpha\beta)\cdot x\ , & 1\cdot x = x\ , & x,y \in V\ ,\ \alpha,\beta \in \mathbb{K}\ .
\end{array}$$

We shall suppress the symbol for scalar multiplication since there is no need for it.

Examples. (a) The set $\mathbb{K}^n$ of n-tuples of scalars is a linear space over $\mathbb{K}$. Addition and scalar multiplication are defined coordinatewise:

$$\begin{array}{l}
(x_1,x_2,\ldots,x_n)+(y_1,y_2,\ldots,y_n) = (x_1+y_1,x_2+y_2,\ldots,x_n+y_n) \\
\alpha(x_1,x_2,\ldots,x_n) = (\alpha x_1,\alpha x_2,\ldots,\alpha x_n)\ .
\end{array}$$

(b) The set $\mathbb{K}^X$ of functions $f : X \to \mathbb{K}$ is a linear space, where X is a non-empty set, and we define $(f_1+f_2)(x) = f_1(x)+f_2(x)$, $(\alpha f)(x) = \alpha f(x)$, $x \in X$.

(c) Let $G \subset \mathbb{R}^n$ be open. The above pointwise definitions of linear operations give a linear space structure on the set $C(G,\mathbb{K})$ of continuous $f : G \to \mathbb{K}$. We normally shorten this to $C(G)$.

(d) For each n-tuple $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ of non-negative integers, we denote by D^α the *partial derivative*

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$$

of order $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n$. The sets $C^m(G) = \{f \in C(G) : D^\alpha f \in C(G) \text{ for all } \alpha,\ |\alpha| \le m\}$, $m \ge 0$, and $C^\infty G = \bigcap_{m\ge1} C^m(G)$ are linear spaces with the operations defined above. We let D^θ be the identity, where $\theta = (0,0,\ldots,0)$, so $C^0(G) = C(G)$.

(e) For $f \in C(G)$, the *support* of f is the closure in G of the set $\{x \in G : f(x) \ne 0\}$ and we denote it by $\mathrm{supp}(f)$. $C_0(G)$ is the subset of those functions in $C(G)$ with compact support. Similarly, we define $C_0^m(G) = C^m(G) \cap C_0(G)$, $m \ge 1$ and $C_0^\infty(G) = C^\infty(G) \cap C_0(G)$.

(f) If $f : A \to B$ and $C \subset A$, we denote $f|_C$ the *restriction* of f to C. We obtain useful linear spaces of functions on the closure $\bar{G}$ as follows:

$$C^m(\bar{G}) = \{f|_{\bar{G}} : f \in C_0^m(\mathbb{R}^n)\} \quad , \quad C^\infty(\bar{G}) = \{f|_{\bar{G}} : f \in C_0^\infty(\mathbb{R}^n)\} \ .$$

These spaces play a central role in our work below.

1.2

A subset M of the linear space V is a *subspace* of V if it is closed under the linear operations. That is, $x + y \in M$ whenever $x, y \in M$ and $\alpha x \in M$ for each $\alpha \in \mathbb{K}$ and $x \in M$. We denote that M is a subspace of V by $M \leq V$. It follows that M is then (and only then) a linear space with addition and scalar multiplication inherited from V.

Examples. We have three chains of subspaces given by

$$\begin{aligned} C^j(G) \leq C^k(G) \ &\leq \ \mathbb{K}^G \ , \\ C^j(\bar{G}) \leq C^k(\bar{G}) \ &, \quad \text{and} \\ \{\theta\} \ \leq \ C_0^j(G) \leq C_0^k(G) \ &, \quad 0 \leq k \leq j \leq \infty \ . \end{aligned}$$

Moreover, for each k as above, we can identify $\varphi \in C_0^k(G)$ with that $\Phi \in C^k(\bar{G})$ obtained by defining Φ to be equal to φ on G and zero on ∂G, the boundary of G. Likewise we can identify each $\Phi \in C^k(\bar{G})$ with $\Phi|_G \in C^K(G)$. These identifications are "compatible" and we have $C_0^k(G) \leq C^k(\bar{G}) \leq C^k(G)$.

1.3

We let M be a subspace of V and construct a corresponding *quotient space*. For each $x \in V$, define a *coset* $\hat{x} = \{y \in V : y - x \in M\} = \{x + m : m \in M\}$. The set $V/M = \{\hat{x} : x \in V\}$ is the *quotient set*. Any $y \in \hat{x}$ is a *representative* of the coset $\hat{x}$ and we clearly have $y \in \hat{x}$ if and only if $x \in \hat{y}$ if and only if $\hat{x} = \hat{y}$. We shall define addition of cosets by adding a corresponding pair of representatives and similarly define scalar multiplication. It is necessary to first verify that this definition is unambiguous.

Lemma *If $x_1, x_2 \in \hat{x}$, $y_1, y_2 \in \hat{y}$, and $\alpha \in \mathbb{K}$, then $(\widehat{x_1 + y_1}) = (\widehat{x_2 + y_2})$ and $(\widehat{\alpha x_1}) = (\widehat{\alpha x_2})$.*

The proof follows easily, since M is closed under addition and scalar multiplication, and we can define $\hat{x} + \hat{y} = \widehat{(x + y)}$ and $\alpha\hat{x} = \widehat{(\alpha x)}$. These operations make V/M a linear space.

Examples. (a) Let $V = \mathbb{R}^2$ and $M = \{(0, x_2) : x_2 \in \mathbb{R}\}$. Then V/M is the set of parallel translates of the x_2-axis, M, and addition of two cosets is easily obtained by adding their (unique) representatives on the x_1-axis.

(b) Take $V = C(G)$. Let $x_0 \in G$ and $M = \{\varphi \in C(G) : \varphi(x_0) = 0\}$. Write each $\varphi \in V$ in the form $\varphi(x) = (\varphi(x) - \varphi(x_0)) + \varphi(x_0)$. This representation can be used to show that V/M is essentially equivalent (isomorphic) to $\mathbb{K}$.

(c) Let $V = C(\bar{G})$ and $M = C_0(G)$. We can describe V/M as a space of "boundary values." To do this, begin by noting that for each $K \subset\subset G$ there is a $\psi \in C_0(G)$ with $\psi = 1$ on K. (Cf. Section II.1.1.) Then write a given $\varphi \in C(\bar{G})$ in the form

$$\varphi = (\varphi\psi) + \varphi(1 - \psi) ,$$

where the first term belongs to M and the second equals φ in a neighborhood of ∂G.

1.4

Let V and W be linear spaces over $\mathbb{K}$. A function $T : V \to W$ is *linear* if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) , \qquad \alpha, \beta \in \mathbb{K} ,\ x, y \in V .$$

That is, linear functions are those which preserve the linear operations. An *isomorphism* is a linear bijection. The set $\{x \in V : Tx = 0\}$ is called the *kernel* of the (not necessarily linear) function $T : V \to W$ and we denote it by $K(T)$.

Lemma *If $T : V \to W$ is linear, then $K(T)$ is a subspace of V, $\mathrm{Rg}(T)$ is a subspace of W, and $K(T) = \{\theta\}$ if and only if T is an injection.*

Examples. (a) Let M be a subspace of V. The identity $i_M : M \to V$ is a linear injection $x \mapsto x$ and its range is M.

(b) The *quotient* map $q_M : V \to V/M$, $x \mapsto \hat{x}$, is a linear surjection with kernel $K(q_M) = M$.

(c) Let G be the open interval (a, b) in $\mathbb{R}$ and consider $D \equiv d/dx$: $V \to C(\bar{G})$, where V is a subspace of $C^1(\bar{G})$. If $V = C^1(\bar{G})$, then D is a linear surjection with $K(D)$ consisting of constant functions on $\bar{G}$. If $V = \{\varphi \in C^1(\bar{G}) :$

$\varphi(a) = 0\}$, then D is an isomorphism. Finally, if $V = \{\varphi \in C^1(\bar{G}) : \varphi(a) = \varphi(b) = 0\}$, then $\mathrm{Rg}(D) = \{\varphi \in C(\bar{G}) : \int_a^b \varphi = 0\}$.

Our next result shows how each linear function can be factored into the product of a linear injection and an appropriate quotient map.

Theorem 1.1 *Let $T : V \to W$ be linear and M be a subspace of $K(T)$. Then there is exactly one function $\widehat{T} : V/M \to W$ for which $\widehat{T} \circ q_M = T$, and $\widehat{T}$ is linear with* $\mathrm{Rg}(\widehat{T}) = \mathrm{Rg}(T)$. *Finally, $\widehat{T}$ is injective if and only if $M = K(T)$.*

Proof: If $x_1, x_2 \in \hat{x}$, then $x_1 - x_2 \in M \subset K(T)$, so $T(x_1) = T(x_2)$. Thus we can define a function as desired by the formula $\widehat{T}(\hat{x}) = T(x)$. The uniqueness and linearity of $\widehat{T}$ follow since q_M is surjective and linear. The equality of the ranges follows, since q_M is surjective, and the last statement follows from the observation that $K(T) \subset M$ if and only if $v \in V$ and $\widehat{T}(\hat{x}) = 0$ imply $\hat{x} = \hat{0}$.

An immediate corollary is that each linear function $T : V \to W$ can be factored into a product of a surjection, an isomorphism, and an injection: $T = i_{\mathrm{Rg}(T)} \circ \widehat{T} \circ q_{K(T)}$.

A function $T : V \to W$ is called *conjugate linear* if

$$T(\alpha x + \beta y) = \bar{\alpha} T(x) + \bar{\beta} T(y) \ , \qquad \alpha, \beta \in \mathbb{K} \ , \ x, y \in V \ .$$

Results similar to those above hold for such functions.

1.5

Let V and W be linear spaces over $\mathbb{K}$ and consider the set $L(V, W)$ of linear functions from V to W. The set W^V of all functions from V to W is a linear space under the pointwise definitions of addition and scalar multiplication (cf. Example 1.1(b)), and $L(V, W)$ is a subspace.

We define V^* to be the linear space of all conjugate linear functionals from $V \to \mathbb{K}$. V^* is called the *algebraic dual* of V. Note that there is a bijection $f \mapsto \bar{f}$ of $\mathcal{L}(V, \mathbb{K})$ onto V^*, where $\bar{f}$ is the functional defined by $\bar{f}(x) = \overline{f(x)}$ for $x \in V$ and is called the *conjugate* of the functional $f : V \to \mathbb{K}$. Such spaces provide a useful means of constructing large linear spaces containing a given class of functions. We illustrate this technique in a simple situation.

Example. Let G be open in $\mathbb{R}^n$ and $x_0 \in G$. We shall imbed the space $C(G)$ in the algebraic dual of $C_0(G)$. For each $f \in C(G)$, define $T_f \in C_0(G)^*$ by

$$T_f(\varphi) = \int_G f\bar{\varphi}\ , \qquad \varphi \in C_0(G)\ .$$

Since $f\bar{\varphi} \in C_0(G)$, the Riemann integral is adequate here. An easy exercise shows that the function $f \mapsto T_f : C(G) \to C_0(G)^*$ is a linear injection, so we may thus identify $C(G)$ with a subspace of $C_0(G)^*$. This linear injection is not surjective; we can exhibit functionals on $C_0(G)$ which are not identified with functions in $C(G)$. In particular, the *Dirac functional* δ_{x_0} defined by

$$\delta_{x_0}(\varphi) = \overline{\varphi(x_0)}\ , \qquad \varphi \in C_0(G)\ ,$$

cannot be obtained as T_f for any $f \in C(G)$. That is, $T_f = \delta_{x_0}$ implies that $f(x) = 0$ for all $x \in G$, $x \neq x_0$, and thus $f = 0$, a contradiction.

2 Convergence and Continuity

The absolute value function on $\mathbb{R}$ and modulus function on $\mathbb{C}$ are denoted by $|\cdot|$, and each gives a notion of length or distance in the corresponding space and permits the discussion of convergence of sequences in that space or continuity of functions on that space. We shall extend these concepts to a general linear space.

2.1

A *seminorm* on the linear space V is a function $p : V \to \mathbb{R}$ for which $p(\alpha x) = |\alpha| p(x)$ and $p(x + y) \le p(x) + p(y)$ for all $\alpha \in \mathbb{K}$ and $x, y \in V$. The pair V, p is called a *seminormed space*.

Lemma 2.1 *If V, p is a seminormed space, then*

(a) $|p(x) - p(y)| \le p(x - y)\ , \qquad x, y \in V\ ,$

(b) $p(x) \ge 0\ , \qquad x \in V$ *, and*

(c) *the kernel $K(p)$ is a subspace of V.*

(d) *If $T \in L(W, V)$, then $p \circ T : W \to \mathbb{R}$ is a seminorm on W.*

(e) *If* p_j *is a seminorm on* V *and* $\alpha_j \geq 0$, $1 \leq j \leq n$, *then* $\sum_{j=1}^{n} \alpha_j p_j$ *is a seminorm on* V.

Proof: We have $p(x) = p(x-y+y) \leq p(x-y)+p(y)$ so $p(x)-p(y) \leq p(x-y)$. Similarly, $p(y) - p(x) \leq p(y-x) = p(x-y)$, so the result follows. Setting $y = 0$ in (a) and noting $p(0) = 0$, we obtain (b). The result (c) follows directly from the definitions, and (d) and (e) are straightforward exercises.

If p is a seminorm with the property that $p(x) > 0$ for each $x \neq \theta$, we call it a *norm*.

Examples. (a) For $1 \leq k \leq n$ we define seminorms on $\mathbb{K}^n$ by $p_k(x) = \sum_{j=1}^{k} |x_j|$, $q_k(x) = (\sum_{j=1}^{k} |x_j|^2)^{1/2}$, and $r_k(x) = \max\{|x_j| : 1 \leq j \leq k\}$. Each of p_n, q_n and r_n is a norm.

(b) If $J \subset X$ and $f \in \mathbb{K}^X$, we define $p_J(f) = \sup\{|f(x)| : x \in J\}$. Then for each finite $J \subset X$, p_J is a seminorm on $\mathbb{K}^X$.

(c) For each $K \subset\subset G$, p_K is a seminorm on $C(G)$. Also, $p_{\bar{G}} = p_G$ is a norm on $C(\bar{G})$.

(d) For each j, $0 \leq j \leq k$, and $K \subset\subset G$ we can define a seminorm on $C^k(G)$ by $p_{j,K}(f) = \sup\{|D^\alpha f(x)| : x \in K,\ |\alpha| \leq j\}$. Each such $p_{j,G}$ is a norm on $C^k(\bar{G})$.

2.2

Seminorms permit a discussion of convergence. We say the sequence $\{x_n\}$ in V *converges* to $x \in V$ if $\lim_{n\to\infty} p(x_n - x) = 0$; that is, if $\{p(x_n - x)\}$ is a sequence in $\mathbb{R}$ converging to 0. Formally, this means that for every $\varepsilon > 0$ there is an integer $N \geq 0$ such that $p(x_n - x) < \varepsilon$ for all $n \geq N$. We denote this by $x_n \to x$ in V, p and suppress the mention of p when it is clear what is meant.

Let $S \subset V$. The *closure* of S in V, p is the set $\bar{S} = \{x \in V : x_n \to x$ in V, p for some sequence $\{x_n\}$ in $S\}$, and S is called *closed* if $S = \bar{S}$. The closure $\bar{S}$ of S is the smallest closed set containing $S : S \subset \bar{S}$, $\bar{S} = \bar{\bar{S}}$, and if $S \subset K = \bar{K}$ then $\bar{S} \subset K$.

Lemma *Let* V, p *be a seminormed space and* M *be a subspace of* V. *Then* $\bar{M}$ *is a subspace of* V.

Proof: Let $x, y \in \bar{M}$. Then there are sequences $x_n, y_n \in M$ such that $x_n \to x$ and $y_n \to y$ in V, p. But $p((x+y)-(x_n+y_n)) \leq p(x-x_n)+p(y-y_n) \to$

0 which shows that $(x_n + y_n) \to x + y$. Since $x_n + y_n \in M$, all n, this implies that $x+y \in \bar{M}$. Similarly, for $\alpha \in \mathbb{K}$ we have $p(\alpha x - \alpha x_n) = |\alpha| p(x - x_n) \to 0$, so $\alpha x \in \bar{M}$.

2.3

Let V, p and W, q be seminormed spaces and $T : V \to W$ (not necessarily linear). Then T is called *continuous at* $x \in V$ if for every $\varepsilon > 0$ there is a $\delta > 0$ for which $y \in V$ and $p(x - y) < \delta$ implies $q(T(x) - T(y)) < \varepsilon$. T is *continuous* if it is continuous at every $x \in V$.

Theorem 2.2 *T is continuous at x if and only if $x_n \to x$ in V, p implies $Tx_n \to Tx$ in W, q.*

Proof: Let T be continuous at x and $\varepsilon > 0$. Choose $\delta > 0$ as in the definition above and then N such that $n \geq N$ implies $p(x_n - x) < \delta$, where $x_n \to x$ in V, p is given. Then $n \geq N$ implies $q(Tx_n - Tx) < \varepsilon$, so $Tx_n \to Tx$ in W, q.

Conversely, if T is not continuous at x, then there is an $\varepsilon > 0$ such that for every $n \geq 1$ there is an $x_n \in V$ with $p(x_n - x) < 1/n$ and $q(Tx_n - Tx) \geq \varepsilon$. That is, $x_n \to x$ in V, p but $\{Tx_n\}$ does not converge to Tx in W, q.

We record the facts that our algebraic operations and seminorm are always continuous.

Lemma *If V, p is a seminormed space, the functions $(\alpha, x) \mapsto \alpha x : \mathbb{K} \times V \to V$, $(x, y) \mapsto x + y : V \times V \to V$, and $p : V \to \mathbb{R}$ are all continuous.*

Proof: The estimate $p(\alpha x - \alpha_n x_n) \leq |\alpha - \alpha_n| p(x) + |\alpha_n| p(x - x_n)$ implies the continuity of scalar multiplication. Continuity of addition follows from an estimate in the preceding Lemma, and continuity of p follows from the Lemma of 2.1.

Suppose p and q are seminorms on the linear space V. We say p is *stronger* than q (or q is *weaker* than p) if for any sequence $\{x_n\}$ in V, $p(x_n) \to 0$ implies $q(x_n) \to 0$.

Theorem 2.3 *The following are equivalent:*

(a) *p is stronger than q,*

(b) *the identity* $I : V, p \to V, q$ *is continuous, and*

(c) *there is a constant* $K \geq 0$ *such that*

$$q(x) \leq Kp(x) \ , \qquad x \in V \ .$$

Proof: By Theorem 2.2, (a) is equivalent to having the identity $I : V, p \to V, q$ continuous at 0, so (b) implies (a). If (c) holds, then $q(x - y) \leq Kp(x - y)$, $x, y \in V$, so (b) is true.

We claim now that (a) implies (c). If (c) is false, then for every integer $n \geq 1$ there is an $x_n \in V$ for which $q(x_n) > np(x_n)$. Setting $y_n = (1/q(x_n))x_n$, $n \geq 1$, we have obtained a sequence for which $q(y_n) = 1$ and $p(y_n) \to 0$, thereby contradicting (a).

Theorem 2.4 *Let* V, p *and* W, q *be seminormed spaces and* $T \in L(V, W)$. *The following are equivalent:*

(a) T *is continuous at* $\theta \in V$,

(b) T *is continuous, and*

(c) *there is a constant* $K \geq 0$ *such that*

$$q(T(x)) \leq Kp(x) \ , \qquad x \in V \ .$$

Proof: By Theorem 2.3, each of these is equivalent to requiring that the seminorm p be stronger than the seminorm $q \circ T$ on V.

2.4

If V, p and W, q are seminormed spaces, we denote by $\mathcal{L}(V, W)$ the set of continuous linear functions from V to W. This is a subspace of $L(V, W)$ whose elements are frequently called the *bounded* operators from V to W (because of Theorem 2.4).

Let $T \in \mathcal{L}(V, W)$ and consider

$$\lambda \equiv \sup\{q(T(x)) : x \in V \ , \quad p(x) \leq 1\} \ ,$$
$$\mu \equiv \inf\{K > 0 : q(T(x)) \leq Kp(x) \ \text{ for all } \ x \in V\} \ .$$

If K belongs to the set defining μ, then for every $x \in V : p(x) \leq 1$ we have $q(T(x)) \leq K$, hence $\lambda \leq K$. This holds for all such K, so $\lambda \leq \mu$. If $x \in V$ with $p(x) > 0$, then $y \equiv (1/p(x))x$ satisfies $p(y) = 1$, so $q(T(y)) \leq \lambda$. That is $q(T(x)) \leq \lambda p(x)$ whenever $p(x) > 0$. But by Theorem 2.4(c) this last inequality is trivially satisfied when $p(x) = 0$, so we have $\mu \leq \lambda$. These remarks prove the first part of the following result; the remaining parts are straightforward.

Theorem 2.5 *Let V, p and W, q be seminormed spaces. For each $T \in \mathcal{L}(V, W)$ we define a real number by $|T|_{p,q} \equiv \sup\{q(T(x)) : x \in V,\ p(x) \leq 1\}$. Then we have $|T|_{p,q} = \sup\{q(T(x)) : x \in V,\ p(x) = 1\} = \inf\{K > 0 : q(T(x)) \leq Kp(x)$ for all $x \in V\}$ and $|\cdot|_{p,q}$ is a seminorm on $\mathcal{L}(V, W)$. Furthermore, $q(T(x)) \leq |T|_{p,q} \cdot p(x)$, $x \in V$, and $|\cdot|_{p,q}$ is a norm whenever q is a norm.*

Definitions. The *dual* of the seminormed space V, p is the linear space $V' = \{f \in V^* : f$ is continuous$\}$ with the norm

$$\|f\|_{V'} = \sup\{|f(x)| : x \in V\ ,\quad p(x) \leq 1\}\ .$$

If V, p and W, q are seminormed spaces, then $T \in \mathcal{L}(V, W)$ is called a *contraction* if $|T|_{p,q} \leq 1$, and T is called an *isometry* if $|T|_{p,q} = 1$.

3 Completeness

3.1

A sequence $\{x_n\}$ in a seminormed space V, p is called *Cauchy* if $\lim_{m,n\to\infty} p(x_m - x_n) = 0$, that is, if for every $\varepsilon > 0$ there is an integer N such that $p(x_m - x_n) < \varepsilon$ for all $m, n \geq N$. Every convergent sequence is Cauchy. We call V, p *complete* if every Cauchy sequence is convergent. A complete normed linear space is a *Banach space*.

Examples. Each of the seminormed spaces of Examples 2.1(a-d) is complete.

(e) Let $G = (0, 1) \subset \mathbb{R}^1$ and consider $C(\bar{G})$ with the norm $p(x) = \int_0^1 |x(t)|\, dt$. Let $0 < c < 1$ and for each n with $0 < c - 1/n$ define $x_n \in C(\bar{G})$ by

$$x_n(t) = \begin{cases} 1\ , & c \leq t \leq 1 \\ n(t - c) + 1\ , & c - 1/n < t < c \\ 0\ , & 0 \leq t \leq c - 1/n \end{cases}$$

For $m \geq n$ we have $p(x_m - x_n) \leq 1/n$, so $\{x_m\}$ is Cauchy. If $x \in C(\bar{G})$, then

$$p(x_n - x) \geq \int_0^{c-1/n} |x(t)|\, dt + \int_c^1 |1 - x(t)|\, d(t)\ .$$

This shows that if $\{x_n\}$ converges to x then $x(t) = 0$ for $0 \leq t < c$ and $x(t) = 1$ for $c \leq t \leq 1$, a contradiction. Hence $C(\bar{G})$, p is not complete.

3.2

We consider the problem of extending a given function to a larger domain.

Lemma *Let $T : D \to W$ be given, where D is a subset of the seminormed space V, p and W, q is a normed linear space. There is at most one continuous $\bar{T} : \bar{D} \to W$ for which $\bar{T}|_D = T$.*

Proof: Suppose T_1 and T_2 are continuous functions from $\bar{D}$ to W which agree with T on D. Let $x \in \bar{D}$. Then there are $x_n \in D$ with $x_n \to x$ in V, p. Continuity of T_1 and T_2 shows $T_1 x_n \to T_1 x$ and $T_2 x_n \to T_2 x$. But $T_1 x_n = T_2 x_n$ for all n, so $T_1 x = T_2 x$ by the uniqueness of limits in the normed space W, q.

Theorem 3.1 *Let $T \in \mathcal{L}(D, W)$, where D is a subspace of the seminormed space V, p and W, q is a Banach space. Then there exists a unique $\bar{T} \in \mathcal{L}(\bar{D}, W)$ such that $\bar{T}|_D = T$, and $|\bar{T}|_{p,q} = |T|_{p,q}$.*

Proof: Uniqueness follows from the preceding lemma. Let $x \in \bar{D}$. If $x_n \in D$ and $x_n \to x$ in V, p, then $\{x_n\}$ is Cauchy and the estimate

$$q(T(x_m) - T(x_n)) \leq K p(x_m - x_n)$$

shows $\{T(x_n)\}$ is Cauchy in W, q, hence, convergent to some $y \in W$. If $x'_n \in D$ and $x'_n \to x$ in V, p, then $Tx'_n \to y$, so we can define $\bar{T} : \bar{D} \to W$ by $T(x) = y$. The linearity of T on D and the continuity of addition and scalar multiplication imply that $\bar{T}$ is linear. Finally, the continuity of seminorms and the estimates

$$q(T(x_n)) \leq |T|_{p,q}\, p(x_n)$$

show $\bar{T}$ is continuous on $|\bar{T}|_{p,q} = |T|_{p,q}$.

3.3

A *completion* of the seminormed space V, p is a complete seminormed space W, q and a linear injection $T : V \to W$ for which $\mathrm{Rg}(T)$ is dense in W and T preserves seminorms: $q(T(x)) = p(x)$ for all $x \in V$. By identifying V, p with $\mathrm{Rg}(T), q$, we may visualize V as being dense and contained in a corresponding space that is complete. The completion of a normed space is a Banach space and linear injection as above. If two Banach spaces are completions of a given normed space, then we can use Theorem 3.1 to construct a linear norm-preserving bijection between them, so the completion of a normed space is essentially unique.

We first construct a completion of a given seminormed space V, p. Let W be the set of all Cauchy sequences in V, p. From the estimate $|p(x_n) - p(x_m)| \leq p(x_n - x_m)$ it follows that $\bar{p}(\{x_n\}) = \lim_{n\to\infty} p(x_n)$ defines a function $\bar{p} : W \to \mathbb{R}$ and it easily follows that $\bar{p}$ is a seminorm on W. For each $x \in V$, let $Tx = \{x, x, x, \ldots\}$, the indicated constant sequence. Then $T : V, p \to W, \bar{p}$ is a linear seminorm-preserving injection. If $\{x_n\} \in W$, then for any $\varepsilon > 0$ there is an integer N such that $p(x_n - x_N) < \varepsilon/2$ for $n \geq N$, and we have $\bar{p}(\{x_n\} - T(x_N)) \leq \varepsilon/2 < \varepsilon$. Thus, $\mathrm{Rg}(T)$ is dense in W. Finally, we verify that $W, \bar{p}$ is complete. Let $\{\bar{x}_n\}$ be a Cauchy sequence in $W, \bar{p}$ and for each $n \geq 1$ pick $x_n \in V$ with $\bar{p}(\bar{x}_n - T(x_n)) < 1/n$. Define $\bar{x}_0 = \{x_1, x_2, x_2, \ldots\}$. From the estimate

$$p(x_m - x_n) = \bar{p}(Tx_m - Tx_n) \leq 1/m + \bar{p}(\bar{x}_m - \bar{x}_n) + 1/n$$

it follows that $\bar{x}_0 \in W$, and from

$$\bar{p}(\bar{x}_n - \bar{x}_0) \leq \bar{p}(\bar{x}_n - Tx_n) + \bar{p}(Tx_n - \bar{x}_0) < 1/n + \lim_{m\to\infty} p(x_n - x_m)$$

we deduce that $\bar{x}_n \to \bar{x}_0$ in $W, \bar{p}$. Thus, we have proved the following.

Theorem 3.2 *Every seminormed space has a completion.*

3.4

In order to obtain from a normed space a corresponding normed completion, we shall identify those elements of W which have the same limit by factoring W by the kernel of $\bar{p}$. Before describing this quotient space, we consider quotients in a seminormed space.

Theorem 3.3 *Let V, p be a seminormed space, M a subspace of V and define*

$$\hat{p}(\hat{x}) = \inf\{p(y) : y \in \hat{x}\} \ , \qquad \hat{x} \in V/M \ .$$

(a) *$V/M, \hat{p}$ is a seminormed space and the quotient map $q : V \to V/M$ has $(p, \hat{p})$-seminorm $= 1$.*

(b) *If D is dense in V, then $\hat{D} = \{\hat{x} : x \in D\}$ is dense in V/M.*

(c) *$\hat{p}$ is a norm if and only if M is closed.*

(d) *If V, p is complete, then $V/M, \hat{p}$ is complete.*

Proof: We leave (a) and (b) as exercises. Part (c) follows from the observation that $\hat{p}(\hat{x}) = 0$ if and only if $x \in \bar{M}$.

To prove (d), we recall that a Cauchy sequence converges if it has a convergent subsequence so we need only consider a sequence $\{\hat{x}_n\}$ in V/M for which $\hat{p}(\hat{x}_{n+1} - \hat{x}_n) < 1/2^n$, $n \geq 1$. For each $n \geq 1$ we pick $y_n \in \hat{x}_n$ with $p(y_{n+1} - y_n) < 1/2^n$. For $m \geq n$ we obtain

$$p(y_m - y_n) \leq \sum_{k=0}^{m-1-n} p(y_{n+1+k} - y_{n+k}) < \sum_{k=0}^{\infty} 2^{-(n+k)} = 2^{1-n} \ .$$

Thus $\{y_n\}$ is Cauchy in V, p and part (a) shows $\hat{x}_n \to \hat{x}$ in V/M, where x is the limit of $\{y_n\}$ in V, p.

Given V, p and the completion $W, \bar{p}$ constructed for Theorem 3.2, we consider the quotient space W/K and its corresponding seminorm $\hat{p}$, where K is the kernel of $\bar{p}$. The continuity of $\bar{p} : W \to \mathbb{R}$ implies that K is closed, so $\hat{p}$ is a norm on W/K. $W, \bar{p}$ is complete, so W/K, $\hat{p}$ is a Banach space. The quotient map $q : W \to W/K$ satisfies $\hat{p}(q(x)) = \hat{p}(\hat{x}) = \bar{p}(y)$ for all $y \in q(x)$, so q preserves the seminorms. Since $\mathrm{Rg}(T)$ is dense in W it follows that the linear map $q \circ T : V \to W/K$ has a dense range in W/K. We have $\hat{p}((q \circ T)x) = \hat{p}(\widehat{Tx}) = p(x)$ for $x \in V$, hence $K(q \circ T) \leq K(p)$. If p is a norm this shows that $q \circ T$ is injective and proves the following.

Theorem 3.4 *Every normed space has a completion.*

3.5

We briefly consider the vector space $\mathcal{L}(V, W)$.

Theorem 3.5 *If V, p is a seminormed space and W, q is a Banach space, then $\mathcal{L}(V, W)$ is a Banach space. In particular, the dual V' of a seminormed space is complete.*

Proof: Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(V, W)$. For each $x \in V$, the estimate

$$q(T_m x - T_n x) \leq |T_m - T_n| p(x)$$

shows that $\{T_n x\}$ is Cauchy, hence convergent to a unique $T(x) \in W$. This defines $T : V \to W$ and the continuity of addition and scalar multiplication in W will imply that $T \in L(V, W)$. We have

$$q(T_n(x)) \leq |T_n| p(x) \ , \qquad x \in V \ ,$$

and $\{|T_n|\}$ is Cauchy, hence, bounded in $\mathbb{R}$, so the continuity of q shows that $T \in \mathcal{L}(V, W)$ with $|T| \leq K \equiv \sup\{|T_n| : n \geq 1\}$.

To show $T_n \to T$ in $\mathcal{L}(V, W)$, let $\varepsilon > 0$ and choose N so large that $m, n \geq N$ implies $|T_m - T_n| < \varepsilon$. Hence, for $m, n \geq N$, we have

$$q(T_m(x) - T_n(x)) < \varepsilon p(x) \ , \qquad x \in V \ .$$

Letting $m \to \infty$ shows that for $n \geq N$ we have

$$q(T(x) - T_n(x)) \leq \varepsilon p(x) \ , \qquad x \in V \ ,$$

so $|T - T_n| \leq \varepsilon$.

4 Hilbert Space

4.1

A *scalar product* on the vector space V is a function $V \times V \to \mathbb{K}$ whose value at x, y is denoted by (x, y) and which satisfies (a) $x \mapsto (x, y) : V \to \mathbb{K}$ is linear for every $y \in V$, (b) $(x, y) = \overline{(y, x)}$, $x, v \in V$, and (c) $(x, x) > 0$ for each $x \neq 0$. From (a) and (b) it follows that for each $x \in V$, the function $y \mapsto (x, y)$ is conjugate-linear, i.e., $(x, \alpha y) = \bar{\alpha}(x, y)$. The pair $V, (\cdot, \cdot)$ is called a *scalar product space*.

Theorem 4.1 *If $V, (\cdot, \cdot)$ is a scalar product space, then*

(a) $|(x, y)|^2 \le (x, x) \cdot (y, y)$, $\qquad x, y \in V$,

(b) $\|x\| \equiv (x, x)^{1/2}$ *defines a norm* $\|\cdot\|$ *on* V *for which*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \ , \qquad x, y \in V \ , \quad \text{and}$$

(c) *the scalar product is continuous from* $V \times V$ *to* K.

Proof: Part (a) follows from the computation

$$0 \le (\alpha x + \beta y, \alpha x + \beta y) = \beta(\beta(y, y) - |\alpha|^2)$$

for the scalars $\alpha = -\overline{(x, y)}$ and $\beta = (x, x)$. To prove (b), we use (a) to verify

$$\|x + y\|^2 \le \|x\|^2 + 2|(x, y)| + \|y\|^2 \le (\|x\| + \|y\|)^2 \ .$$

The remaining norm axioms are easy and the indicated identity is easily verified. Part (c) follows from the estimate

$$|(x, y) - (x_n, y_n)| \le \|x\| \, \|y - y_n\| + \|y_n\| \, \|x - x_n\|$$

applied to a pair of sequences, $x_n \to x$ and $y_n \to y$ in $V, \|\cdot\|$.

A *Hilbert space* is a scalar product space for which the corresponding normed space is complete.

Examples. (a) Let $V = \mathbb{K}^N$ with vectors $x = (x_1, x_2, \ldots, x_N)$ and define $(x, y) = \sum_{j=1}^N x_j \bar{y}_j$. Then $V, (\cdot, \cdot)$ is a Hilbert space (with the norm $\|x\| = (\sum_{j=1}^N |x_j|^2)^{1/2}$) which we refer to as Euclidean space.

(b) We define $C_0(G)$ a scalar product by

$$(\varphi, \psi) = \int_G \varphi \bar{\psi}$$

where G is open in $\mathbb{R}^n$ and the Riemann integral is used. This scalar product space is not complete.

(c) On the space $L^2(G)$ of (equivalence classes of) Lebesgue square-summable $\mathbb{K}$-valued functions we define the scalar product as in (b) but with the Lebesgue integral. This gives a Hilbert space in which $C_0(G)$ is a dense subspace.

Suppose $V, (\cdot,\cdot)$ is a scalar product space and let $B, \|\cdot\|$ denote the completion of $V, \|\cdot\|$. For each $y \in V$, the function $x \mapsto (x,y)$ is linear, hence has a unique extension to B, thereby extending the definition of (x,y) to $B \times V$. It is easy to verify that for each $x \in B$, the function $y \mapsto (x,y)$ is in V' and we can similarly extend it to define (x,y) on $B \times B$. By checking that (the extended) function $(\cdot,\cdot)$ is a scalar product on B, we have proved the following result.

Theorem 4.2 *Every scalar product space has a (unique) completion which is a Hilbert space and whose scalar product is the extension by continuity of the given scalar product.*

Example. $L^2(G)$ is the completion of $C_0(G)$ with the scalar product given above.

4.2

The scalar product gives us a notion of angles between vectors. (In particular, recall the formula $(x,y) = \|x\|\,\|y\|\cos(\theta)$ in Example (a) above.) We call the vectors x, y *orthogonal* if $(x,y) = 0$. For a given subset M of the scalar product space V, we define the *orthogonal complement* of M to be the set

$$M^\perp = \{x \in V : (x,y) = 0 \text{ for all } y \in M\} .$$

Lemma *$M^\perp$ is a closed subspace of V and $M \cap M^\perp = \{0\}$.*

Proof: For each $y \in M$, the set $\{x \in V : (x,y) = 0\}$ is a closed subspace and so then is the intersection of all these for $y \in M$. The only vector orthogonal to itself is the zero vector, so the second statement follows.

A set K in the vector space V is *convex* if for $x, y \in K$ and $0 \le \alpha \le 1$, we have $\alpha x + (1-\alpha)y \in K$. That is, if a pair of vectors is in K, then so also is the line segment joining them.

Theorem 4.3 *A non-empty closed convex subset K of the Hilbert space H has an element of minimal norm.*

Proof: Setting $d \equiv \inf\{\|x\| : x \in K\}$, we can find a sequence $x_n \in K$ for which $\|x_n\| \to d$. Since K is convex we have $(1/2)(x_n + x_m) \in K$ for

$m, n \geq 1$, hence $\|x_n + x_m\|^2 \geq 4d^2$. From Theorem 4.1(b) we obtain the estimate $\|x_n - x_m\|^2 \leq 2(\|x_n\|^2 + \|x_m\|^2) - 4d^2$. The right side of this inequality converges to 0, so $\{x_n\}$ is Cauchy, hence, convergent to some $x \in H$. K is closed, so $x \in K$, and the continuity of the norm shows that $\|x\| = \lim_n \|x_n\| = d$.

We note that the element with minimal norm is unique, for if $y \in K$ with $\|y\| = d$, then $(1/2)(x+y) \in K$ and Theorem 4.1(b) give us, respectively, $4d^2 \leq \|x+y\|^2 = 4d^2 - \|x-y\|^2$. That is, $\|x - y\| = 0$.

Theorem 4.4 *Let M be a closed subspace of the Hilbert space H. Then for every $x \in H$ we have $x = m + n$, where $m \in M$ and $n \in M^\perp$ are uniquely determined by x.*

Proof: The uniqueness follows easily, since if $x = m_1 + n_1$ with $m_1 \in M$, $n_1 \in M^\perp$, then $m_1 - m = n - n_1 \in M \cap M^\perp = \{\theta\}$. To establish the existence of such a pair, define $K = \{x + y : y \in M\}$ and use Theorem 4.3 to find $n \in K$ with $\|n\| = \inf\{\|x+y\| : y \in M\}$. Then set $m = x - n$. It is clear that $m \in M$ and we need only to verify that $n \in M^\perp$. Let $y \in M$. For each $\alpha \in \mathbb{K}$, we have $n - \alpha y \in K$, hence $\|n - \alpha y\|^2 \geq \|n\|^2$. Setting $\alpha = \beta(n, y)$, $\beta > 0$, gives us $|(n,y)|^2(\beta\|y\|^2 - 2) \geq 0$, and this can hold for all $\beta > 0$ only if $(n, y) = 0$.

4.3

From Theorem 4.4 it follows that for each closed subspace M of a Hilbert space H we can define a function $P_M : H \to M$ by $P_M : x = m + n \mapsto m$, where $m \in M$ and $n \in M^\perp$ as above. The linearity of P_M is immediate and the computation

$$\|P_M x\|^2 \leq \|P_M x\|^2 + \|n\|^2 = \|P_M x + n\|^2 = \|x\|^2$$

shows $P_M \in \mathcal{L}(H, H)$ with $\|P_M\| \leq 1$. Also, $P_M x = x$ exactly when $x \in M$, so $P_M \circ P_M = P_M$. The operator P_M is called the *projection on M*.

If $P \in \mathcal{L}(B, B)$ satisfies $P \circ P = P$, then P is called a *projection* on the Banach space B. The result of Theorem 4.4 is a guarantee of a rich supply of projections in a Hilbert space.

4.4

We recall that the (continuous) dual of a seminormed space is a Banach space. We shall show there is a natural correspondence between a Hilbert space H and its dual H'. Consider for each fixed $x \in H$ the function f_x defined by the scalar product: $f_x(y) = (x, y)$, $y \in H$. It is easy to check that $f_x \in H'$ and $\|f_x\|_{H'} = \|x\|$. Furthermore, the map $x \mapsto f_x : H \to H'$ is linear:

$$\begin{aligned} f_{x+z} &= f_x + f_z \ , \qquad x, z \in H \ , \\ f_{\alpha x} &= \alpha f_x \ , \qquad \alpha \in \mathbb{K} \ , \ x \in H \ . \end{aligned}$$

Finally, the function $x \mapsto f_x : H \to H'$ is a norm preserving and linear injection. The above also holds in any scalar product space, but for Hilbert spaces this function is also surjective. This follows from the next result.

Theorem 4.5 *Let H be a Hilbert space and $f \in H'$. Then there is an element $x \in H$ (and only one) for which*

$$f(y) = (x, y) \ , \qquad y \in H \ .$$

Proof: We need only verify the existence of $x \in H$. If $f = \theta$ we take $x = \theta$, so assume $f \neq \theta$ in H'. Then the kernel of f, $K = \{x \in H : f(x) = 0\}$ is a closed subspace of H with $K^\perp \neq \{\theta\}$. Let $n \in K^\perp$ be chosen with $\|n\| = 1$. For each $z \in K^\perp$ it follows that $\overline{f(n)}z - \overline{f(z)}n \in K \cap K^\perp = \{\theta\}$, so z is a scalar multiple of n. (That is, $K^\perp$ is one-dimensional.) Thus, each $y \in H$ is of the form $y = P_K(y) + \lambda n$ where $(y, n) = \lambda(n, n) = \lambda$. But we also have $f(y) = \bar{\lambda} f(n)$, since $P_K(y) \in K$, and thus $f(y) = (\overline{f(n)}n, y)$ for all $y \in H$.

The function $x \mapsto f_x$ from H to H' will occur frequently in our later discussions and it is called the *Riesz map* and is denoted by R_H. Note that it depends on the scalar product as well as the space. In particular, R_H is an isometry of H onto H' defined by

$$R_H(x)(y) = (x, y)_H \ , \qquad x, y \in H \ .$$

5 Dual Operators; Identifications

5.1

Suppose V and W are linear spaces and $T \in L(V,W)$. Then we define the *dual operator* $T' \in L(W^*,V^*)$ by

$$T'(f) = f \circ T \ , \qquad f \in W^* \ .$$

Theorem 5.1 *If V is a linear space, W,q is a seminorm space, and $T \in L(V,W)$ has dense range, then T' is injective on W'. If V,p and W,q are seminorm spaces and $T \in \mathcal{L}(V,W)$, then the restriction of the dual T' to W' belongs to $\mathcal{L}(W',V')$ and it satisfies*

$$\|T'\|_{\mathcal{L}(W',V')} \le |T|_{p,q} \ .$$

Proof: The first part follows from Section 3.2. The second is obtained from the estimate

$$|T'f(x)| \le \|f\|_{W'}|T|_{p,q}\, p(x) \ , \qquad f \in W' \ , \ x \in V \ .$$

We give two basic examples. Let V be a subspace of the seminorm space W,q and let $i : V \to W$ be the identity. Then $i'(f) = f \circ i$ is the restriction of f to the subspace V; i' is injective on W' if (and only if) V is dense in W. In such cases we may actually identify $i'(W')$ with W', and we denote this identification by $W' \le V^*$.

Consider the quotient map $q : W \to W/V$ where V and W,q are given as above. It is clear that if $g \in (W/V)^*$ and $f = q'(g)$, i.e., $f = g \circ q$, then $f \in W^*$ and $V \le K(f)$. Conversely, if $f \in W^*$ and $V \le K(f)$, then Theorem 1.1 shows there is a $g \in (W/V)^*$ for which $q'(g) = f$. These remarks show that $\mathrm{Rg}(q') = \{f \in W^* : V \le K(f)\}$. Finally, we note by Theorem 3.3 that $|q|_{q,\hat{q}} = 1$, so it follows that $g \in (W,V)'$ if and only if $q'(g) \in W'$.

5.2

Let V and W be Hilbert spaces and $T \in \mathcal{L}(V,W)$. We define the *adjoint* of T as follows: if $u \in W$, then the functional $v \mapsto (u,Tv)_W$ belongs to V', so Theorem 4.5 shows that there is a unique $T^*u \in V$ such that

$$(T^*u,v)_V = (u,Tv)_W \ , \qquad u \in W \ , \ v \in V \ .$$

Theorem 5.2 *If V and W are Hilbert spaces and $T \in \mathcal{L}(V,W)$, then $T^* \in \mathcal{L}(W,V)$, $\mathrm{Rg}(T)^\perp = K(T^*)$ and $\mathrm{Rg}(T^*)^\perp = K(T)$. If T is an isomorphism with $T^{-1} \in \mathcal{L}(W,V)$, then T^* is an isomorphism and $(T^*)^{-1} = (T^{-1})^*$.*

We leave the proof as an exercise and proceed to show that dual operators are essentially equivalent to the corresponding adjoint. Let V and W be Hilbert spaces and denote by R_V and R_W the corresponding Riesz maps (Section 4.4) onto their respective dual spaces. Let $T \in \mathcal{L}(V,W)$ and consider its dual $T' \in \mathcal{L}(W',V')$ and its adjoint $T^* \in \mathcal{L}(W,V)$. For $u \in W$ and $v \in V$ we have $R_V \circ T^*(u)(v) = (T^*u, v)_V = (u, Tv)_W = R_W(u)(Tv) = (T' \circ R_W u)(v)$. This shows that $R_V \circ T^* = T' \circ R_W$, so the Riesz maps permit us to study either the dual or the adjoint and deduce information on both. As an example of this we have the following.

Corollary 5.3 *If V and W are Hilbert spaces, and $T \in \mathcal{L}(V,W)$, then $\mathrm{Rg}(T)$ is dense in W if and only if T' is injective, and T is injective if and only if $\mathrm{Rg}(T')$ is dense in V'. If T is an isomorphism with $T^{-1} \in \mathcal{L}(W,V)$, then $T' \in \mathcal{L}(W',V')$ is an isomorphism with continuous inverse.*

5.3

It is extremely useful to make certain identifications between various linear spaces and we shall discuss a number of examples which will appear frequently in the following.

First, consider the linear space $C_0(G)$ and the Hilbert space $L^2(G)$. Elements of $C_0(G)$ are functions while elements of $L^2(G)$ are *equivalence classes* of functions. Since each $f \in C_0(G)$ is square-summable on G, it belongs to exactly one such equivalence class, say $i(f) \in L^2(G)$. This defines a linear injection $i : C_0(G) \to L^2(G)$ whose range is dense in $L^2(G)$. The dual $i' : L^2(G)' \to C_0(G)^*$ is then a linear injection which is just restriction to $C_0(G)$.

The Riesz map R of $L^2(G)$ (with the usual scalar product) onto $L^2(G)'$ is defined as in Section 4.4. Finally, we have a linear injection $T : C_0(G) \to C_0(G)^*$ given in Section 1.5 by

$$(Tf)(\varphi) = \int_G f(x)\bar{\varphi}(x)\,dx\ , \qquad f, \varphi \in C_0(G)\ .$$

Both R and T are possible identifications of (equivalence classes of) functions with conjugate-linear functionals. Moreover we have the important identity

$$T = i' \circ R \circ i \ .$$

This shows that all four injections may be used simultaneously to identify the various pairs as subspaces. That is, we identify

$$C_0(G) \leq L^2(G) = L^2(G)' \leq C_0(G)^* \ ,$$

and thereby reduce each of i, R, i' and T to the identity function from a subspace to the whole space. Moreover, once we identify $C_0(G) \leq L^2(G)$, $L^2(G)' \leq C_0(G)^*$, and $C_0(G) \leq C_0(G)^*$, by means of i, i', and T, respectively, then it follows that the identification of $L^2(G)$ with $L^2(G)'$ through the Riesz map R is possible (i.e., compatible with the three preceding) *only if* the R corresponds to the standard scalar product on $L^2(G)$. For example, suppose R is defined through the (equivalent) scalar-product

$$(Rf)(g) = \int_G a(x) f(x) \overline{g(x)} \, dx \ , \qquad f, g \in L^2(G) \ ,$$

where $a(\cdot) \in L^\infty(G)$ and $a(x) \geq c > 0$, $x \in G$. Then, with the three identifications above, R corresponds to multiplication by the function $a(\cdot)$. Other examples will be given later.

5.4

We shall find the concept of a sesquilinear form is as important to us as that of a linear operator. The theory of sesquilinear forms is analogous to that of linear operators and we discuss it briefly.

Let V be a linear space over the field $\mathbb{K}$. A *sesquilinear form* on V is a $\mathbb{K}$-valued function $a(\cdot, \cdot)$ on the product $V \times V$ such that $x \mapsto a(x, y)$ is linear for every $y \in V$ and $y \mapsto a(x, y)$ is conjugate linear for every $x \in V$. Thus, each sesquilinear form $a(\cdot, \cdot)$ on V corresponds to a unique $\mathcal{A} \in L(V, V^*)$ given by

$$a(x, y) = \mathcal{A}x(y) \ , \qquad x, y \in V \ . \tag{5.1}$$

Conversely, if $\mathcal{A} \in \mathcal{L}(V, V^*)$ is given, then Equation (5.1) defines a sesquilinear form on V.

Theorem 5.4 *Let V, p be a normed linear space and $a(\cdot,\cdot)$ a sesquilinear form on V. The following are equivalent:*

(a) *$a(\cdot,\cdot)$ is continuous at (θ,θ),*

(b) *$a(\cdot,\cdot)$ is continuous on $V \times V$,*

(c) *there is a constant $K \geq 0$ such that*

$$|a(x,y)| \leq Kp(x)p(y) \ , \qquad x, y \in V \ , \tag{5.2}$$

(d) *$\mathcal{A} \in \mathcal{L}(V, V')$.*

Proof: It is clear that (c) and (d) are equivalent, (c) implies (b), and (b) implies (a). We shall show that (a) implies (c). The continuity of $a(\cdot,\cdot)$ at (θ,θ) implies that there is a $\delta > 0$ such that $p(x) \leq \delta$ and $p(y) \leq \delta$ imply $|a(x,y)| \leq 1$. Thus, if $x \neq 0$ and $y \neq 0$ we obtain Equation (5.2) with $K = 1/\delta^2$.

When we consider real spaces (i.e., $\mathbb{K} = \mathbb{R}$) there is no distinction between linear and conjugate-linear functions. Then a sesquilinear form is linear in both variables and we call it *bilinear*.

6 Uniform Boundedness; Weak Compactness

A sequence $\{x_n\}$ in the Hilbert space H is called *weakly convergent* to $x \in H$ if $\lim_{n\to\infty}(x_n, v)_H = (x, v)_H$ for every $v \in H$. The weak limit x is clearly unique. Similarly, $\{x_n\}$ is *weakly bounded* if $|(x_n, v)_H|$ is bounded for every $v \in H$.

Our first result is a simple form of the *principle of uniform boundedness*.

Theorem 6.1 *A sequence $\{x_n\}$ is weakly bounded if and only if it is bounded.*

Proof: Let $\{x_n\}$ be weakly bounded. We first show that on some sphere, $s(x,r) = \{y \in H : \|y-x\| < r\}$, $\{x_n\}$ is uniformly bounded: there is a $K \geq 0$ with $|(x_n, y)_H| \leq K$ for all $y \in s(x,r)$. Suppose not. Then there is an integer n_1 and $y_1 \in s(0,1)$: $|(x_{n_1}, y_1)_H| > 1$. Since $y \mapsto (x_{n_1}, y)_H$ is continuous, there is an $r_1 < 1$ such that $|(x_{n_1}, y)_H| > 1$ for $y \in s(y_1, r_1)$. Similarly, there is an integer $n_2 > n_1$ and $\overline{s(y_2, r_2)} \subset s(y_1, r_1)$ such that $r_2 < 1/2$

and $|(x_{n_2}, y)_H| > 2$ for $y \in s(y_2, r_2)$. We inductively define $\overline{s(y_j, r_j)} \subset s(y_{j-1}, r_{j-1})$ with $r_j < 1/j$ and $|(x_{n_j}, y)_H| > j$ for $y \in s(y_j, r_j)$. Since $\|y_m - y_n\| < 1/n$ if $m > n$ and H is complete, $\{y_n\}$ converges to some $y \in H$. But then $y \in s(y_j, r_j)$, hence $|(x_{n_j}, y)_H| > j$ for all $j \geq 1$, a contradiction.

Thus $\{x_n\}$ is uniformly bounded on some sphere $s(y, r) : |(x_n, y+rz)_H| \leq K$ for all z with $\|z\| \leq 1$. If $\|z\| \leq 1$, then

$$|(x_n, z)_H| = (1/r)|x_n, y + rz)_H - (x_n, y)_H| \leq 2K/r \ ,$$

so $\|x_n\| \leq 2K/r$ for all n.

We next show that bounded sequences have weakly convergent subsequences.

Lemma *If $\{x_n\}$ is bounded in H and D is a dense subset of H, then* $\lim_{n\to\infty}(x_n, v)_H = (x, v)_H$ *for all $v \in D$ (if and) only if $\{x_n\}$ converges weakly to x.*

Proof: Let $\varepsilon > 0$ and $v \in H$. There is a $z \in D$ with $\|v - z\| < \varepsilon$ and we obtain

$$\begin{aligned} |(x_n - x, v)_H| &\leq |(x_n, v - z)_H| + |(z, x_n - x)_H| + |(x, v - z)_H| \\ &< \varepsilon\|x_n\| + |(z, x_n - x)_H| + \varepsilon\|x\| \ . \end{aligned}$$

Hence, for all n sufficiently large (depending on z), we have $|(x_n - x, v)_H| < 2\varepsilon \sup\{\|x_m\| : m \geq 1\}$. Since $\varepsilon > 0$ is arbitrary, the result follows.

Theorem 6.2 *Let the Hilbert space H have a countable dense subset $D = \{y_n\}$. If $\{x_n\}$ is a bounded sequence in H, then it has a weakly convergent subsequence.*

Proof: Since $\{(x_n, y_1)_H\}$ is bounded in $\mathbb{K}$, there is a subsequence $\{x_{1,n}\}$ of $\{x_n\}$ such that $\{(x_{1,n}, y_1)_H\}$ converges. Similarly, for each $j \geq 2$ there is a subsequence $\{x_{j,n}\}$ of $\{x_{j-1,n}\}$ such that $\{(x_{j,n}, y_k)_H\}$ converges in $\mathbb{K}$ for $1 \leq k \leq j$. It follows that $\{x_{n,n}\}$ is a subsequence of $\{x_n\}$ for which $\{(x_{n,n}, y_k)_H\}$ converges for every $k \geq 1$.

From the preceding remarks, it suffices to show that if $\{(x_n, y)_H\}$ converges in $\mathbb{K}$ for every $y \in D$, then $\{x_k\}$ has a weak limit. So, we define $f(y) = \lim_{n\to\infty}(x_n, y)_H$, $y \in \langle D \rangle$, where $\langle D \rangle$ is the subspace of all linear combinations of elements of D. Clearly f is linear; f is continuous, since $\{x_n\}$ is bounded, and has by Theorem 3.1 a unique extension $f \in H'$. But then there is by Theorem 4.5 an $x \in H$ such that $f(y) = (x, y)_H$, $y \in H$. The Lemma above shows that x is the weak limit of $\{x_n\}$.

Any seminormed space which has a countable and dense subset is called *separable*. Theorem 6.2 states that any bounded set in a separable Hilbert space is *relatively sequentially weakly compact*. This result holds in any reflexive Banach space, but all the function spaces which we shall consider are separable Hilbert spaces, so Theorem 6.2 will suffice for our needs.

7 Expansion in Eigenfunctions

7.1

We consider the Fourier series of a vector in the scalar product space H with respect to a given set of orthogonal vectors. The sequence $\{v_j\}$ of vectors in H is called *orthogonal* if $(v_i, v_j)_H = 0$ for each pair i, j with $i \neq j$. Let $\{v_j\}$ be such a sequence of non-zero vectors and let $u \in H$. For each j we define the *Fourier coefficient* of u with respect to v_j by $c_j = (u, v_j)_H/(v_j, v_j)_H$. For each $n \geq 1$ it follows that $\sum_{j=1}^n c_j v_j$ is the projection of u on the subspace M_n spanned by $\{v_1, v_2, \ldots, v_n\}$. This follows from Theorem 4.4 by noting that $u - \sum_{j=1}^n c_j v_j$ is orthogonal to each v_i, $1 \leq j \leq n$, hence belongs to $M_n^\perp$. We call the sequence of vectors *orthonormal* if they are orthogonal and if $(v_j, v_j)_H = 1$ for each $j \geq 1$.

Theorem 7.1 *Let $\{v_j\}$ be an orthonormal sequence in the scalar product space H and let $u \in H$. The Fourier coefficients of u are given by $c_j = (u, v_j)_H$ and satisfy*

$$\sum_{j=1}^{\infty} |c_j|^2 \leq \|u\|^2 \ . \tag{7.1}$$

Also we have $u = \sum_{j=1}^\infty c_j v_j$ if and only if equality holds in (7.1).

Proof: Let $u_n \equiv \sum_{j=1}^n c_j v_j$, $n \geq 1$. Then $u - u_n \perp u_n$ so we obtain

$$\|u\|^2 = \|u - u_n\|^2 + \|u_n\|^2 \ , \qquad n \geq 1 \ . \tag{7.2}$$

But $\|u_n\|^2 = \sum_{j=1}^n |c_j|^2$ follows since the set $\{v_i, \ldots, v_n\}$ is orthonormal, so we obtain $\sum_{j=1}^n |c_j|^2 \leq \|u\|^2$ for all n, hence (7.1) holds. It follows from (7.2) that $\lim_{n\to\infty} \|u - u_n\| - 0$ if and only if equality holds in (7.1).

The inequality (7.1) is *Bessel's inequality* and the corresponding equality is called *Parseval's equation*. The series $\sum_{j=1}^{\infty} c_j v_j$ above is the *Fourier series* of u with respect to the orthonormal sequence $\{v_j\}$.

Theorem 7.2 *Let $\{v_j\}$ be an orthonormal sequence in the scalar product space H. Then every element of H equals the sum of its Fourier series if and only if $\{v_j\}$ is a* basis *for H, that is, its linear span is dense in H.*

Proof: Suppose $\{v_j\}$ is a basis and let $u \in H$ be given. For any $\varepsilon > 0$, there is an $n \geq 1$ for which the linear span M of the set $\{v_1, v_2, \ldots, v_n\}$ contains an element which approximates u within ε. That is, $\inf\{\|u-w\| : w \in M\} < \varepsilon$. If u_n is given as in the proof of Theorem 7.1, then we have $u - u_n \in M^{\perp}$. Hence, for any $w \in M$ we have

$$\|u - u_n\|^2 = (u - u_n, u - w)_H \leq \|u - u_n\| \, \|u - w\| \ ,$$

since $u_n - w \in M$. Taking the infimum over all $w \in M$ then gives

$$\|u - u_n\| \leq \inf\{\|u - w\| : w \in M\} < \varepsilon \ . \tag{7.3}$$

Thus, $\lim_{n\to\infty} u_n = u$. The converse is clear.

7.2

Let $T \in \mathcal{L}(H)$. A non-zero vector $v \in H$ is called an *eigenvector* of T if $T(v) = \lambda v$ for some $\lambda \in \mathbb{K}$. The number λ is the *eigenvalue* of T corresponding to v. We shall show that certain operators possess a rich supply of eigenvectors. These eigenvectors form an orthonormal sequence to which we can apply the preceding Fourier series expansion techniques.

An operator $T \in \mathcal{L}(H)$ is called *self-adjoint* if $(Tu, v)_H = (u, Tv)_H$ for all $u, v \in H$. A self-adjoint T is called *non-negative* if $(Tu, u)_H \geq 0$ for all $u \in H$.

Lemma 7.3 *If $T \in \mathcal{L}(H)$ is non-negative self-adjoint, then $\|Tu\| \leq \|T\|^{1/2}(Tu, u)_H^{1/2}$, $u \in H$.*

Proof: The sesquilinear form $[u, v] \equiv (Tu, v)_H$ satisfies the first two scalar-product axioms and this is sufficient to obtain

$$|[u, v]|^2 \leq [u, u][v, v] \ , \qquad u, v \in H \ . \tag{7.4}$$

(If either factor on the right side is strictly positive, this follows from the proof of Theorem 4.1. Otherwise, $0 \le [u+tv, u+tv] = 2t[u,v]$ for all $t \in \mathbb{R}$, hence, both sides of (7.4) are zero.) The desired result follows by setting $v = T(u)$ in (7.4).

The operators we shall consider are the compact operators. If V, W are seminormed spaces, then $T \in \mathcal{L}(V,W)$ is called *compact* if for any bounded sequence $\{u_n\}$ in V its image $\{Tu_n\}$ has a subsequence which converges in W. The essential fact we need is the following.

Lemma 7.4 *If $T \in \mathcal{L}(H)$ is self-adjoint and compact, then there exists a vector v with $\|v\| = 1$ and $T(v) = \mu v$, where $|\mu| = \|T\|_{\mathcal{L}(H)} > 0$.*

Proof: If λ is defined to be $\|T\|_{\mathcal{L}(H)}$, it follows from Theorem 2.5 that there is a sequence u_n in H with $\|u_n\| = 1$ and $\lim_{n\to\infty} \|Tu_n\| = \lambda$. Then $((\lambda^2 - T^2)u_n, u_n)_H = \lambda^2 - \|Tu_n\|^2$ converges to zero. The operator $\lambda^2 - T^2$ is non-negative self-adjoint so Lemma 7.3 implies $\{(\lambda^2 - T^2)u_n\}$ converges to zero. Since T is compact we may replace $\{u_n\}$ by an appropriate subsequence for which $\{Tu_n\}$ converges to some vector $w \in H$. Since T is continuous there follows $\lim_{n\to\infty}(\lambda^2 u_n) = \lim_{n\to\infty} T^2 u_n = Tw$, so $w = \lim_{n\to\infty} Tu_n = \lambda^{-2}T^2(w)$. Note that $\|w\| = \lambda$ and $T^2(w) = \lambda^2 w$. Thus, either $(\lambda + T)w \ne 0$ and we can choose $v = (\lambda + T)w/\|(\lambda + T)w\|$, or $(\lambda + T)w = 0$, and we can then choose $v = w/\|w\|$. Either way, the desired result follows.

Theorem 7.5 *Let H be a scalar product space and let $T \in \mathcal{L}(H)$ be self-adjoint and compact. Then there is an orthonormal sequence $\{v_j\}$ of eigenvectors of T for which the corresponding sequence of eigenvalues $\{\lambda_j\}$ converges to zero and the eigenvectors are a basis for* $\mathrm{Rg}(T)$.

Proof: By Lemma 7.4 it follows that there is a vector v_1 with $\|v_1\| = 1$ and $T(v_1) = \lambda_1 v_1$ with $|\lambda_1| = \|T\|_{\mathcal{L}(H)}$. Set $H_1 = \{v_1\}^\perp$ and note $T\{H_1\} \subset H_1$. Thus, the restriction $T|_{H_1}$ is self-adjoint and compact so Lemma 7.4 implies the existence of an eigenvector v_2 of T of unit length in H_1 with eigenvalue λ_2 satisfying $|\lambda_2| = \|T\|_{\mathcal{L}(H_1)} \le |\lambda_1|$. Set $H_2 = \{v_1, v_2\}^\perp$ and continue this procedure to obtain an orthonormal sequence $\{v_j\}$ in H and sequence $\{\lambda_j\}$ in $\mathbb{R}$ such that $T(v_j) = \lambda_j v_j$ and $|\lambda_{j+1}| \le |\lambda_j|$ for $j \ge 1$.

Suppose the sequence $\{\lambda_j\}$ is eventually zero; let n be the first integer for which $\lambda_n = 0$. Then $H_{n-1} \subset K(T)$, since $T(v_j) = 0$ for $j \ge n$. Also we see $v_j \in \mathrm{Rg}(T)$ for $j < n$, so $\mathrm{Rg}(T)^\perp \subset \{v_1, v_2, \ldots, v_{n-1}\}^\perp = H_{n-1}$ and from

Theorem 5.2 follows $K(T) = \mathrm{Rg}(T)^\perp \subset H_{n-1}$. Therefore $K(T) = H_{n-1}$ and $\mathrm{Rg}(T)$ equals the linear span of $\{v_1, v_2, \ldots, v_{n-1}\}$.

Consider hereafter the case where each λ_j is different from zero. We claim that $\lim_{j\to\infty}(\lambda_j) = 0$. Otherwise, since $|\lambda_j|$ is decreasing we would have all $|\lambda_j| \geq \varepsilon$ for some $\varepsilon > 0$. But then

$$\|T(v_i) - T(v_j)\|^2 = \|\lambda_i v_i - \lambda_j v_j\|^2 = \|\lambda_i v_i\|^2 + \|\lambda_j v_j\|^2 \geq 2\varepsilon^2$$

for all $i \neq j$, so $\{T(v_j)\}$ has no convergent subsequence, a contradiction. We shall show $\{v_j\}$ is a basis for $\mathrm{Rg}(T)$. Let $w \in \mathrm{Rg}(T)$ and $\sum b_j v_j$ the Fourier series of w. Then there is a $u \in H$ with $T(u) = w$ and we let $\sum c_j v_j$ be the Fourier series of u. The coefficients are related by

$$b_j = (w, v_j)_H = (Tu, v_j)_H = (u, Tv_j)_H = \lambda_j c_j \ ,$$

so there follows $T(c_j v_j) = b_j v_j$, hence,

$$w - \sum_{j=1}^{n} b_j v_j = T\left(u - \sum_{j=1}^{n} c_j v_j\right) \ , \qquad n \geq 1 \ . \tag{7.5}$$

Since T is bounded by $|\lambda_{n+1}|$ on H_n, and since $\|u - \sum_{j=1}^{n} c_j v_j\| \leq \|u\|$ by (7.2), we obtain from (7.5) the estimate

$$\left\| w - \sum_{j=1}^{n} b_j u_j \right\| \leq |\lambda_{n+1}| \cdot \|u\| \ , \qquad n \geq 1 \ . \tag{7.6}$$

Since $\lim_{j\to\infty} \lambda_j = 0$, we have $w = \sum_{j=1}^{\infty} b_j v_j$ as desired.

Exercises

1.1. Explain what “compatible” means in the Examples of Section 1.2.

1.2. Prove the Lemmas of Sections 1.3 and 1.4.

1.3. In Example (1.3.b), show V/M is isomorphic to $\mathbb{K}$.

1.4. Let $V = C(\bar{G})$ and $M = \{\varphi \in C(\bar{G}) : \varphi|_{\partial G} = 0\}$. Show V/M is isomorphic to $\{\varphi|_{\partial G} : \varphi \in C(\bar{G})\}$, the space of “boundary values” of functions in V.

1.5. In Example (1.3.c), show $\hat{\varphi}_1 = \hat{\varphi}_2$ if and only if φ_1 equals φ_2 on a neighborhood of ∂G. Find a space of functions isomorphic to V/M.

1.6. In Example (1.4.c), find $K(D)$ and $\mathrm{Rg}(D)$ when $V = \{\varphi \in C^1(\bar{G}) : \varphi(a) = \varphi(b)\}$.

1.7. Verify the last sentence in the Example of Section 1.5.

1.8. Let $M_\alpha \leq V$ for each $\alpha \in A$; show $\cap\{M_\alpha : \alpha \in A\} \leq V$.

2.1. Prove parts (d) and (e) of Lemma 2.1.

2.2. If V_1, p_1 and V_2, p_2 are seminormed spaces, show $p(x) \equiv p_1(x_1) + p_2(x_2)$ is a seminorm on the product $V_1 \times V_2$.

2.3. Let V, p be a seminormed space. Show limits are unique if and only if p is a norm.

2.4. Verify all Examples in Section 2.1.

2.5. Show $\cap_{\alpha\in A}\bar{S}_\alpha = \overline{\cap_{\alpha\in A}S_\alpha}$. Verify $\bar{S}$ = smallest closed set containing S.

2.6. Show $T : V, p \to W, q$ is continuous if and only if S closed in W, q implies $T(S)$ closed in V, p. If $T \in L(V, W)$, then T continuous if and only if $K(T)$ is closed.

2.7. The composition of continuous functions is continuous; $T \in \mathcal{L}(V, W)$, $S \in \mathcal{L}(U, V) \Rightarrow T \circ S \in \mathcal{L}(U, W)$ and $|T \circ S| \leq |T|\ |S|$.

2.8. Finish proof of Theorem 2.5.

2.9. Show V' is isomorphic to $\mathcal{L}(V, \mathbb{K})$; they are equal only if $\mathbb{K} = \mathbb{R}$.

3.1. Show that a closed subspace of a seminormed space is complete.

3.2. Show that a complete subspace of a normed space is closed.

3.3. Show that a Cauchy sequence is convergent if and only if it has a convergent subsequence.

3.4. Let V, p be a seminormed space and W, q a Banach space. Let the sequence $T_n \in \mathcal{L}(V, W)$ be given *uniformly bounded*: $|T_n|_{p,q} \leq K$ for all $n \geq 1$. Suppose that D is a dense subset of V and $\{T_n(x)\}$ converges in W for each $x \in D$. Then show $\{T_n(x)\}$ converges in W for each $x \in V$ and $T(x) = \lim T_n(x)$ defines $T \in \mathcal{L}(V, W)$. Show that completeness of W is necessary above.

3.5. Let V, p and W, q be as given above. Show $\mathcal{L}(V, W)$ is isomorphic to $\mathcal{L}(V/\operatorname{Ker}(p), W)$.

3.6. Prove the remark in Section 3.3 on uniqueness of a completion.

4.1. Show that the norms p_2 and r_2 of Section 2.1 are not obtained from scalar products.

4.2. Let M be a subspace of the scalar product space $V(\cdot,\cdot)$. Then the following are equivalent: M is dense in V, $M^\perp = \{\theta\}$, and $\|f\|_{V'} = \sup\{|(f, v)_V| : v \in M\}$ for every $f \in V'$.

4.3. Show $\lim x_n = x$ in V, $(\cdot,\cdot)$ if and only if $\lim \|x_n\| = \|x\|$ and $\lim f(x_n) = f(x)$ for all $f \in V'$.

4.4. If V is a scalar product space, show V' is a Hilbert space. Show that the Riesz map of V into V' is surjective only if V is complete.

5.1. Prove Theorem 5.2.

5.2. Prove Corollary 5.3.

5.3. Verify $T = i' \circ R \circ i$ in Section 5.3.

5.4. In the situation of Theorem 5.2, prove the following are equivalent: $\operatorname{Rg}(T)$ is closed, $\operatorname{Rg}(T^*)$ is closed, $\operatorname{Rg}(T) = K(T^*)^\perp$, and $\operatorname{Rg}(T^*) = K(T)^\perp$.

7.1. Let $G = (0,1)$ and $H = L^2(G)$. Show that the sequence $v_n(x) = 2\sin(n\pi x)$, $n \geq 1$ is orthonormal in H.

7.2. In Theorem 7.1, show that $\{u_n\}$ is a Cauchy sequence.

7.3. Show that the eigenvalues of a non-negative self-adjoint operator are all non-negative.

7.4. In the situation of Theorem 7.5, show $K(T)$ is the orthogonal complement of the linear span of $\{v_1, v_2, v_3, \ldots\}$.