Chapter VI

Second Order Evolution Equations

1 Introduction

We shall find well-posed problems for evolution equations which contain the second order time derivative of the solution. These arise, for example, when we attempt to use the techniques of the preceding chapters to solve a Cauchy problem for the *wave equation*

$$\partial_t^2 u(x,t) - \Delta_n u(x,t) = F(x,t) \ . \tag{1.1}$$

The corresponding abstract problem will contain the second order evolution equation

$$u''(t) + \mathcal{A}u(t) = f(t) \ , \tag{1.2}$$

where $\mathcal{A}$ is an operator which contains $-\Delta_n$ in some sense. Wave equations with damping or friction occur in practice, e.g., the *telegraphists equation*

$$\partial_t^2 u(x,t) + R \cdot \partial_t u(x,t) - \Delta_n u(x,t) = F(x,t) \ ,$$

so we shall add terms to (1.2) of the form $\mathcal{B}u'(t)$. Finally, certain models in fluid mechanics lead to equations, for example,

$$\partial_t^2(\Delta_n u(x,t)) + \partial_n^2 u(x,t) = 0 \ , \qquad x = (x_1, \ldots, x_n) \ , \tag{1.3}$$

which contain spatial derivatives in the terms with highest (= second) order time derivatives. These motivate us to consider abstract evolution equations of the form

$$\mathcal{C}u''(t) + \mathcal{B}u'(t) + \mathcal{A}u(t) = f(t)\ , \qquad t > 0\ . \tag{1.4}$$

We consider in Section 2 equations of the form (1.4) in which $\mathcal{C}$ is invertible; this situation is similar to that of Section V.2, so we call (1.4) a *regular equation* then. The equation (1.3) is known as Sobolev's equation, so we call (1.4) a *Sobolev equation* when $\mathcal{C}$ is invertible and both $\mathcal{C}^{-1}\mathcal{B}$ and $\mathcal{C}^{-1}\mathcal{A}$ are continuous. This situation is studied in Section 3 and is the analogue of (first-order) pseudoparabolic problems. Section 4 will be concerned with (1.4) when $\mathcal{C}$ is *degenerate* in the sense of Section V.4. Such equations arise, for example from a system described by appropriately coupled wave and heat equations

$$\partial_t^2 u(x,t) - \Delta_n u(x,t) = 0\ , \qquad x \in G_1\ ,$$

$$\partial_t u(x,t) - \Delta_n u(x,t) = 0\ , \qquad x \in G_2\ .$$

Here the operator $\mathcal{C}$ is multiplication by the characteristic function of G_1 and $\mathcal{B}$ is multiplication by the characteristic function of G_2. G_1 and G_2 are disjoint open sets whose closures intersect in an $(n-1)$-dimensional manifold or *interface*. Additional examples will be given in Section 5.

2 Regular Equations

Let V and W be Hilbert spaces with V a dense subspace of W for which the injection is continuous. Thus, we identify $W' \leq V'$ by duality. Let $\mathcal{A} \in \mathcal{L}(V, V')$ and $\mathcal{C} \in \mathcal{L}(W, W')$ be given. Suppose $D(B) \leq V$ and $B : D(B) \to V'$ is linear. If $u_0 \in V$, $u_1 \in W$ and $f \in C((0,\infty), W')$ are given, we consider the problem of finding $u \in C([0,\infty), V) \cap C^1((0,\infty), V) \cap C^1([0,\infty), W) \cap C^2((0,\infty), W)$ such that $u(0) = u_0$, $u'(0) = u_1$, and

$$\mathcal{C}u''(t) + Bu'(t) + \mathcal{A}u(t) = f(t) \tag{2.1}$$

for all $t > 0$. Note that for any such solution of (2.1) we have $u'(t) \in D(B)$ and $Bu'(t) + \mathcal{A}u(t) \in W'$ for all $t > 0$.

We shall solve (2.1) by reducing it to a first order equation on a product space and then applying the results of Section V.2. The idea is to write (2.1) in the form

$$\begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{C} \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}' + \begin{pmatrix} 0 & -\mathcal{A} \\ \mathcal{A} & B \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}\ .$$

Define $V_m = V \times W$, the product Hilbert space with scalar-product given by

$$([x_1, x_2], [y_1, y_2])_{V_m} = (x_1, y_1)_V + (x_2, y_2)_W \ , \qquad [x_1, x_2], [y_1, y_2] \in V \times W \ .$$

We have then $V'_m = V' \times W'$, and we define $\mathcal{M} \in \mathcal{L}(V_m, V'_m)$ by

$$\mathcal{M}([x_1, x_2]) = [\mathcal{A}x_1, \mathcal{C}x_2] \ , \qquad [x_1, x_2] \in V_m \ .$$

Define $D = \{[x_1, x_2] \in V \times D(B) : \mathcal{A}x_1 + Bx_2 \in W'\}$ and $L \in L(D, V'_m)$ by

$$L([x_1, x_2]) = [-\mathcal{A}x_2, \mathcal{A}x_1 + Bx_2] \ , \qquad [x_1, x_2] \in D \ .$$

If $u(\cdot)$ is a solution of (2.1), then the function defined by $w(t) = [u(t), u'(t)]$, $t \geq 0$, satisfies the following: $w \in C([0, \infty), V_m) \cap C^1((0, \infty), V_m)$, $w(0) = [u_0, u_1] \in V_m$, and

$$\mathcal{M}w'(t) + Lw(t) = [0, f(t)] \ , \qquad t > 0 \ . \tag{2.2}$$

This is precisely the situation of Section V.2, so we need only to find conditions on the data in (2.1) so that Theorems 2.1 or 2.2 of Chapter V are applicable. This leads to the following.

Theorem 2.1 *Let V and W be Hilbert spaces with V dense and continuously imbedded in W. Assume $\mathcal{A} \in \mathcal{L}(V, V')$ and $\mathcal{C} \in \mathcal{L}(W, W')$ are the Riesz maps of V and W, respectively, and let B be linear from the subspace $D(B)$ of V into V'. Assume that B is monotone and that $\mathcal{A} + B + \mathcal{C} : D(B) \to V'$ is surjective. Then for every $f \in C^1([0, \infty), W')$ and $u_0 \in V$, $u_1 \in D(B)$ with $\mathcal{A}u_0 + Bu_1 \in W'$, there exists a unique solution $u(t)$ of* (2.1) *(on $t \geq 0$) with $u(0) = u_0$ and $u'(0) = u_1$.*

Proof: Since $\mathcal{A}$ and $\mathcal{C}$ are Riesz maps of their corresponding spaces, we have

$$\begin{aligned} \mathcal{M}([x_1, x_2])([y_1, y_2]) &= \mathcal{A}x_1(y_1) + \mathcal{C}x_2(y_2) \\ &= (x_1, y_1)_V + (x_2, y_2)_W \\ &= ([x_1, x_2], [y_1, y_2])_{V_m} \ , \qquad [x_1, x_2], [y_1, y_2] \in V_m \ , \end{aligned}$$

so $\mathcal{M}$ is the Riesz map of V_m. Also we have for $[x_1, x_2] \in D$

$$L([x_1, x_2])([y_1, y_2]) = -\mathcal{A}x_2(y_1) + (\mathcal{A}x_1 + Bx_2)(y_2) \ , \qquad [y_1, y_2] \in V_m \ ,$$

hence, $L([x_1,x_2])([x_1,x_2]) = -\overline{\mathcal{A}x_1(x_2)} + \mathcal{A}x_1(x_2) + Bx_2(x_2)$ since $\mathcal{A}$ is symmetric. From this we obtain

$$\operatorname{Re} L([x_1,x_2])([x_1,x_2]) = \operatorname{Re} Bx_2(x_2) \ , \qquad [x_1,x_2] \in D \ ,$$

so B being monotone implies L is monotone. Finally, if $f_1 \in V'$ and $f_2 \in W'$, then we can find $x_2 \in D(B)$ such that $(\mathcal{A} + B + \mathcal{C})x_2 = f_2 - f_1$. Setting $x_1 = x_2 + \mathcal{A}^{-1} f_1 \in V$, we have a pair $[x_1,x_2] \in D$ for which $(\mathcal{M} + L)[x_1,x_2] = [f_1,f_2]$. (Note that $\mathcal{A}x_1 + Bx_2 = f_2 - \mathcal{C}x_2 \in W'$ as required.) Thus $\mathcal{M} + L$ is a surjection of D onto V'_m. Theorem 2.1 of Chapter V asserts the existence of a solution $w(t) = [u(t), v(t)]$ of (2.2). Since $\mathcal{A}$ is a norm-preserving isomorphism, $v(t) = u'(t)$ and the result follows.

A special case of Theorem 2.1 that occurs frequently in applications is that $D(B) = V$ and $B = \mathcal{B} \in \mathcal{L}(V, V')$. Then one needs only to verify that $\mathcal{B}$ is monotone, for then $\mathcal{A}+\mathcal{B}+\mathcal{C}$ is V-coercive, hence surjective. Furthermore, in this case we may define $\mathcal{L} \in \mathcal{L}(V_\ell, V'_\ell)$ and $V_\ell = V \times V$ by

$$\mathcal{L}([x_1,x_2])([y_1,y_2]) = -\mathcal{A}x_2(y_1) + (\mathcal{A}x_1 + \mathcal{B}x_2)(y_2) \ , \qquad [x_1,x_2],[y_1,y_2] \in V_\ell \ .$$

Thus, Theorem 2.2 of Chapter V applies if we can show that $\mathcal{L}(\cdot)(\cdot)$ is V_ℓ-elliptic. Of course we need only to verify that $(\lambda\mathcal{M} + \mathcal{L})(\cdot)(\cdot)$ is V_ℓ-elliptic for some $\lambda > 0$ (Exercise V.2.3), and this leads us to the following.

Theorem 2.2 *Let $\mathcal{A}$ and $\mathcal{C}$ be the Riesz maps of the Hilbert spaces V and W, respectively, where V is dense and continuously imbedded in W. Let $\mathcal{B} \in \mathcal{L}(V, V')$ and assume $\mathcal{B} + \lambda\mathcal{C}$ is V-elliptic for some $\lambda > 0$. Then for every Hölder continuous $f : [0,\infty) \to W'$, $u_0 \in V$ and $u_1 \in W$, there is a unique solution $u(t)$ of* (2.1) *on $t > 0$ with $u(0) = u_0$ and $u'(0) = u_1$.*

Theorem 2.2 applies to evolution equations of second order which are parabolic, i.e., those which can be solved for more general data u_0, u_1 and $f(\cdot)$, and whose solutions are smooth for all $t > 0$. Such problems occur when energy is strongly dissipated; we give examples below. The situation in which energy is conserved is described in the following result. We leave its proof as an exercise, as it is a direct consequence of either Theorem 2.2 above or Section IV.5.

Theorem 2.3 *In addition to the hypotheses of Theorem 2.1, assume that* $\operatorname{Re} Bx(x) = 0$ *for all $x \in D(B)$ and that both $\mathcal{A} + B + \mathcal{C}$ and $\mathcal{A} - B + \mathcal{C}$ are* *surjections of* $D(B)$ *onto* V'. *Then for every* $f \in C^1(\mathbb{R}, W')$ *and* $u_0 \in V$, $u_1 \in D(B)$ *with* $\mathcal{A}u_0 + Bu_1 \in W'$, *there exists a unique solution of* (2.1) *on* $\mathbb{R}$ *with* $u(0) = u_0$ *and* $u'(0) = u_1$.

We shall describe how Theorems 2.1 and 2.3 apply to an *abstract wave equation*. Examples will be given afterward. Assume we are given Hilbert spaces $V \le H$, and B, and a linear surjection $\gamma : V \to B$ with kernel V_0 such that γ factors into an isomorphism of V/V_0 onto B, the injection $V \hookrightarrow H$ is continuous, V_0 is dense in H, and H is identified with its dual H' by the Riesz map. We thereby obtain continuous injections $V_0 \hookrightarrow H \hookrightarrow V_0'$ and $V \hookrightarrow H \hookrightarrow V'$.

Let $a_1 : V \times V \to \mathbb{K}$ and $a_2 : B \times B \to \mathbb{K}$ be continuous symmetric sesquilinear forms and define $a : V \times V \to \mathbb{K}$ by

$$a(u, v) = a_1(u, v) + a_2(\gamma(u), \gamma(v)) \ , \qquad u, v \in V \ . \tag{2.3}$$

Assume $a(\cdot, \cdot)$ is V-elliptic. Then $a(\cdot, \cdot)$ is a scalar-product on V which gives an equivalent norm on V, so we hereafter consider V with this scalar-product, i.e., $(u, v)_V \equiv a(u, v)$ for $u, v \in V$. The form (2.3) will be used to prescribe an abstract boundary value problem as in Section III.3. Thus, we define $A : V \to V_0'$ by

$$Au(v) = a_1(u, v) \ , \qquad u \in V \ , \ v \in V_0$$

and $D_0 = \{u \in V : Au \in H\}$. Then Theorem III.2.3 gives the abstract boundary operator $\partial_1 \in L(D_0, B')$ for which

$$a_1(u, v) - (Au, v)_H = \partial_1 u(\gamma v) \ , \qquad u \in D_0 \ , \ v \in V \ .$$

We define $D = \{u \in V : \mathcal{A}u \in \mathcal{H}\}$, where $\mathcal{A}$ is the Riesz map of V given by

$$\mathcal{A}u(v) = a(u, v) \ , \qquad u, v \in V \ ,$$

and $\mathcal{A}_2 : B \to B'$ is given by

$$\mathcal{A}_2\varphi(\psi) = a_2(\varphi, \psi) \ , \qquad \varphi, \psi \in B \ .$$

Then, we recall from Corollary III.3.2 that $u \in D$ if and only if $u \in D_0$ and $\partial_1 u + \mathcal{A}_2(\gamma u) = 0$.

Let $(\cdot, \cdot)_W$ be a scalar-product on H whose corresponding norm is equivalent to that of $(\cdot, \cdot)_H$, and let W denote the Hilbert space consisting of H with the scalar-product $(\cdot,\cdot)_W$. Then the Riesz map $\mathcal{C}$ of W satisfies $\mathcal{C} \in \mathcal{L}(H)$ (and $\mathcal{C}^{-1} \in \mathcal{L}(H)$). Suppose we are also given an operator $\mathcal{B} \in \mathcal{L}(V,H)$ which is monotone (since $H \leq V'$).

Theorem 2.4 *Assume we are given the Hilbert spaces V, H, B, V_0, W and linear operators γ, ∂_1, $\mathcal{A}_2$, A, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as above. Then for every $f \in C^1([0,\infty),H)$, $u_0 \in D$ and $u_1 \in V$, there is a unique solution $u(\cdot)$ of* (2.1), *and it satisfies*

$$\left.\begin{array}{ll} \mathcal{C}u''(t) + \mathcal{B}u'(t) + Au(t) = f(t)\ , & t \geq 0\ , \\ u(t) \in V\ , \quad \partial_1 u(t) + \mathcal{A}_2\gamma(u(t)) = 0\ , & t \geq 0\ , \\ u(0) = u_0\ , \quad u'(0) = u_1\ , & \end{array}\right\} \tag{2.4}$$

Proof: Since $\mathcal{A}+\mathcal{B}+\mathcal{C} \in \mathcal{L}(V,V')$ is V-elliptic, it is surjective so Theorem 2.1 (with $B = \mathcal{B}$) asserts the existence of a unique solution. Also, since each of the terms $\mathcal{C}u''(t)$, $\mathcal{B}u'(t)$ and $f(t)$ of the equation (2.1) are in H, it follows that $\mathcal{A}u(t) \in H$ and, hence, $u(t) \in D$. This gives the middle line in (2.4).

In each of our examples below, the first line in (2.4) will imply an abstract wave equation, possibly with damping, and the second line will imply boundary conditions.

2.1

Let G be open in $\mathbb{R}^n$ and take $H = L^2(G)$. Let $\rho \in L^\infty(G)$ satisfy $\rho(x) \geq c > 0$ for $x \in G$, and define

$$(u,v)_W \equiv \int_G \rho(x)u(x)\overline{v(x)}\,dx\ , \qquad u,v \in H\ .$$

Then $\mathcal{C}$ is just multiplication by $\rho(\cdot)$.

Suppose further that ∂G is a C^1 manifold and Γ is a closed subset of ∂G. We define $V = \{v \in H^1(G) : \gamma_0(v)(s) = 0, \text{ a.e., } s \in \Gamma\}$, $\gamma = \gamma_0|_V$ and, hence, $V_0 = H_0^1(G)$ and B is the range of γ. Note that $B \hookrightarrow L^2(\partial G \sim \Gamma) \hookrightarrow B'$. We define

$$a_1(u,v) = \int_G \nabla u \cdot \nabla \bar{v}\,dx\ , \qquad u,v \in V\ ,$$

and it follows that $A = -\Delta_n$ and ∂_1 is the normal derivative

$$\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$$

on ∂G. Let $\alpha \in L^\infty(\partial G)$ satisfy $\alpha(s) \geq 0$, a.e. $s \in \partial G$, and define

$$a_2(\varphi, \psi) = \int_{\partial G \sim \Gamma} \alpha(s)\varphi(s)\overline{\psi(s)}\, ds\ , \qquad \varphi, \psi \in B\ .$$

Then $\mathcal{A}_2$ is multiplication by $\alpha(\cdot)$.

Assume that for each $t \in [0, T]$ we are given $F(\cdot, t) \in L^2(G)$, that $\partial_t F(x, t)$ is continuous in t for almost every $x \in G$, and $|\partial_t F(x, t)| \leq g(x)$ for some $g \in L^2(G)$. It follows that the map $t \mapsto F(\cdot, t) \equiv f(t)$ belongs to $C^1([0, T], L^2(G))$. Finally, let $U_0(\cdot) \in D$ (see below) and $U_1(\cdot) \in V$ be given. Then, if $u(\cdot)$ denotes the solution of (2.4) it follows from Theorem IV.7.1 that we can construct a function $U \in L^2(G \times [0, T])$ such that $U(\cdot, t) = u(t)$ in $L^2(G)$ for each $t \in [0, T]$ and this function satisfies the partial differential equation

$$\rho(x)\partial_t^2 U(x, t) - \Delta_n U(x, t) = F(x, t)\ , \qquad x \in G\ ,\ 0 \leq t \leq T \tag{2.5}$$

and the initial conditions

$$U(x, 0) = U_0(x)\ , \quad \partial_t U(x, 0) = U_1(x)\ , \quad \text{a.e.} \quad x \in G\ .$$

Finally, from the inclusion $u(t) \in D$ we obtain the boundary conditions for $t \geq 0$

$$\left.\begin{aligned} &U(s, t) = 0\ , && \text{a.e.} \quad s \in \Gamma \quad \text{and} \\ &\frac{\partial U(s, t)}{\partial \nu} + \alpha(s) U(s, t) = 0\ , && \text{a.e.} \quad s \in \partial G \sim \Gamma\ . \end{aligned}\right\} \tag{2.6}$$

The first equation in (2.6) is the boundary condition of *first type*. The second is the boundary condition of *second type* where $\alpha(s) = 0$ and of *third type* where $\alpha(s) > 0$. (Note that U_0 necessarily satisfies the conditions of (2.6) with $t = 0$ and that U_1 satisfies the first condition in (2.6). If $F(\cdot, t)$ is given as above but for each $t \in [-T, T]$, then Theorem 2.3 (and Theorem III.7.5) give a solution of (2.5) on $G \times [-T, T]$.

2.2

In addition to all the data above, suppose we are given $R(\cdot) \in L^\infty(G)$ and a vector field $\mu(x) = (\mu_1(x), \ldots, \mu_n(x))$, $x \in G$, with each $\mu_j \in C^1(\bar{G})$. We define $\mathcal{B} \in \mathcal{L}(V, H)$ (where $V \leq H^1(G)$ and $H = L^2(G)$) by

$$\mathcal{B}u(v) = \int_G \left(R(x)u(x) + \frac{\partial u(x)}{\partial \mu}\right)\overline{v(x)}\, dx \tag{2.7}$$

the indicated directional derivative being given by

$$\frac{\partial u(x)}{\partial \mu} \equiv \sum_{j=1}^{n} \partial_j u(x) \mu_j(x) \ .$$

From the Divergence Theorem we obtain

$$2\,\mathrm{Re} \int_G \frac{\partial u(x)}{\partial \mu}\, \overline{u(x)}\, dx + \int_G \Big(\sum_{j=1}^{n} \partial_j \mu_j(x) \Big) |u(x)|^2 \, dx = \int_{\partial G} (\mu \cdot \nu) |u(x)|^2 \, ds \ ,$$

where $\mu \cdot \nu = \sum_{j=1}^{n} \mu_j(s)\nu_j(s)$ is the indicated euclidean scalar-product. Thus, $\mathcal{B}$ is monotone if

$$-\left(\tfrac{1}{2}\right) \sum_{j=1}^{n} \partial_j \mu_j(x) + \mathrm{Re}\{R(x)\} \geq 0 \ , \qquad x \in G$$

$$\mu(s) \cdot \nu(s) \geq 0 \ , \qquad s \in \partial G \sim \Gamma \ .$$

The first equation represents friction or energy dissipation distributed throughout G and the second is friction distributed over ∂G. Note that these are determined by the divergence of μ and the normal component of μ, respectively. If $u(\cdot)$ is a solution of (2.4) and the corresponding $U(\cdot,\cdot)$ is obtained as before from Theorem IV.7.1, then $U(\cdot,\cdot)$ is a generalized solution of the initial-boundary value problem

$$\begin{cases} \rho(x)\partial_t^2 U(x,t) + R(x)\partial_t U(x,t) + \partial_t \dfrac{\partial U(x,t)}{\partial \mu} - \Delta_n U(x,t) = F(x,t) \ , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in G \ , \ t \geq 0 \\ U(s,t) = 0 \ , \qquad \text{a.e.} \quad s \in \Gamma \ , \\ \dfrac{\partial U(s,t)}{\partial \nu} + \alpha(s) U(s,t) = 0 \ , \qquad \text{a.e.} \quad s \in \partial G \sim \Gamma \\ U(x,0) = U_0(x) \ , \quad \partial_t U(x,0) = U_1(x) \end{cases}$$

One could similarly solve problems with the fourth boundary condition, oblique derivatives, transition conditions on an interface, etc., as in Section III.4. We leave the details as exercises.

We now describe how Theorem 2.2 applies to an *abstract viscoelasticity equation*.

Theorem 2.5 *Assume we are given the Hilbert spaces V, H, B, V_0, W and linear operators γ, ∂_1, $\mathcal{A}_2$, A, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as in Theorem 2.4. Then for every $f : [0, \infty) \to H$ which is Hölder continuous, $u_0 \in V$ and $u_1 \in H$, there is a unique solution $u(t)$ of (2.1) with $B = \mathcal{B} + \varepsilon\mathcal{A}$ and $\varepsilon > 0$. This solution satisfies*

$$\left.\begin{aligned} &\mathcal{C}u''(t) + (\mathcal{B} + \varepsilon A)u'(t) + Au(t) = f(t)\ , && t > 0\ ,\\ &u(t) \in V\ , && t \geq 0\ ,\\ &\partial_1(\varepsilon u'(t) + u(t)) + \mathcal{A}_2\gamma(\varepsilon u'(t) + u(t)) = 0\ , && t > 0\ ,\\ &u(0) = u_0\ , \quad u'(0) = u_1\ . && \end{aligned}\right\} \tag{2.8}$$

Proof: This follows immediately from

$$\operatorname{Re} Bx(x) \geq \varepsilon\mathcal{A}x(x) = \varepsilon\|x\|_V^2\ , \qquad x \in V\ ,$$

(since $\mathcal{B}$ is monotone) and the observation that $\varepsilon u'(t) + u(t) \in D$ for $t > 0$.

2.3

Let all spaces and operators be chosen just as in Section 2.1 above. Suppose $U_0 \in V$, $U_1 \in H$ and $f(t) = F(t, \cdot)$, $t \geq 0$, where $F(\cdot, \cdot)$ is given as in Theorem IV.7.3. Then we obtain a generalized solution of the initial-boundary value problem

$$\left.\begin{aligned} &\rho(x)\partial_t^2 U(x,t) - \varepsilon\partial_t\Delta_n U(x,t) - \Delta_n U(x,t) = F(x,t)\ ,\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } x \in G\ ,\ t > 0\ ,\\ &U(s,t) = 0\ , \qquad \text{a.e. } s \in \Gamma\ ,\ t \geq 0\ ,\\ &\frac{\partial}{\partial\nu}(\varepsilon\partial_t U(s,t) + U(s,t)) + \alpha(s)(\varepsilon\partial_t U(s,t) + U(s,t)) = 0\ ,\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } s \in \partial G \sim \Gamma\ ,\ t > 0\ ,\\ &U(x,0) = U_0(x)\ , \quad \partial_t U(x,0) = U_1(x)\ , \qquad x \in G\ . \end{aligned}\right\} \tag{2.9}$$

In certain applications the coefficient $\varepsilon > 0$ corresponds to *viscosity* in the model and it distinguishes the preceding parabolic problem from the corresponding hyperbolic problem in Section 2.1. Problems with viscosity result in very strong damping effects on solutions. Dissipation terms of lower order like (2.7) could easily be added to the system (2.9), and other types of boundary conditions could be obtained.

3 Sobolev Equations

We shall give sufficient conditions for a certain type of evolution equation to have either a weak solution or a strong solution, a situation similar to that for pseudoparabolic equations. The problems we consider here have the strongest operator as the coefficient of the term in the equation with the second order derivative.

Theorem 3.1 *Let V be a Hilbert space and $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{L}(V, V')$. Assume that the sesquilinear form corresponding to $\mathcal{C}$ is V-elliptic. Then for every $u_0, u_1 \in V$ and $f \in C(\mathbb{R}, V)$ there is a unique $u \in C^2(\mathbb{R}, V)$ such that*

$$\mathcal{C}u''(t) + \mathcal{B}u'(t) + \mathcal{A}u(t) = f(t) \ , \qquad t \in \mathbb{R} \ , \tag{3.1}$$

and $u(0) = u_0$, $u'(0) = u_1$.

Proof: The change of variable $v(t) \equiv e^{-\lambda t} u(t)$ gives an equivalent problem with $\mathcal{A}$ replaced by $\mathcal{A} + \lambda\mathcal{B} + \lambda^2\mathcal{C}$, and this last operator is V-coercive if λ is chosen sufficiently large. Hence, we may assume $\mathcal{A}$ is V-elliptic. If we define $\mathcal{M}$ and $\mathcal{L}$ as in Section 2.2, then $\mathcal{M}$ is $V \times V \equiv V_m$-elliptic, and Theorem V.3.1 then applies to give a solution of (2.2). The desired result then follows.

A solution $u \in C^2(\mathbb{R}, V)$ of (3.1) is called a *weak solution*. If we are given a Hilbert space H in which V is continuously imbedded and dense, we define $D(C) = \{v \in V : \mathcal{C}v \in H\}$ and $C = \mathcal{C}|_{D(C)}$. The corresponding restrictions of $\mathcal{B}$ and $\mathcal{A}$ to H are denoted similarly. A (weak) solution u of (3.1) for which each term belongs to H at each $t \in \mathbb{R}$ is called a *strong solution*, and it satisfies

$$Cu''(t) + Bu'(t) + Au(t) = f(t) \ , \qquad t \in \mathbb{R} \ . \tag{3.2}$$

Theorem 3.2 *Let the Hilbert space V and operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be given as in Theorem 3.1. Let the Hilbert space H and corresponding operators A, B, C be defined as above, and assume $D(C) \subset D(A) \cap D(B)$. Then for every pair $u_0 \in D(A)$, $u_1 \in D(C)$, and $f \in C(\mathbb{R}, H)$, there is a unique strong solution $u(\cdot)$ of (3.2) with $u(0) = u_0$, $u'(0) = u_1$.*

Proof: We define $M[x_1, x_2] = [Ax_1, Cx_2]$ on $D(A) \times D(C) = D(M)$ and $L[x_1, x_2] = [-Ax_2, Ax_1 + Bx_2]$ on $D(A) \times D(A) \cap D(B)$ and apply Theorem V.3.2.

Corollary 3.3 *In the situation of Theorem 3.2, the weak solution $u(\cdot)$ is strong if and only if, for some $t_0 \in \mathbb{R}$, $u(t_0) \in D(A)$ and $u'(t_0) \in D(C)$.*

We give an example which includes the classical Sobolev equation from fluid mechanics and an evolution equation of the type used to describe certain vibration problems. Let G be open in $\mathbb{R}^n$ and suppose that ∂G is a C^1 manifold and that Γ is a closed subset of ∂G. Let $V = \{v \in H^1(G) : \gamma v(s) = 0, \text{ a.e. } s \in \Gamma\}$ and

$$\mathcal{C}u(v) = (u,v)_{H^1(G)} \ , \qquad u,v \in V \ .$$

Suppose $a_j(\cdot) \in L^\infty(G)$ for $j = 1, 2, \ldots, n$, and define

$$\mathcal{A}u(v) = \sum_{j=1}^{n} \int_G a_j(x)\partial_j u(x)\partial_j \overline{v(x)}\, dx \ , \qquad u,v \in V \ .$$

Let the functions $t \mapsto F(\cdot,t) : \mathbb{R} \to L^2(G)$ and $t \mapsto g(\cdot,t) : \mathbb{R} \to L^2(\partial G)$ be continuous and define $f \in C(\mathbb{R}, V')$ by

$$f(t)(v) = \int_G F(x,t)\overline{v(x)}\, dx + \int_{\partial G} g(s,t)\gamma\overline{v(s)}\, ds \ , \qquad v \in V \ .$$

Then for each pair $U_0, U_1 \in V$, we obtain from Theorems 3.1 and IV.7.1 a unique generalized solution of the problem

$$\left.\begin{aligned}
&\partial_t^2 U(x,t) - \Delta_n \partial_t^2 U(x,t) - \sum_{j=1}^{n} \partial_j(a_j(x)\partial_j U(x,t)) = F(x,t) \ , \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in G \ , \ t > 0 \ , \\
&U(s,t) = 0 \ , \qquad s \in \Gamma \ , \\
&\partial_\nu \partial_t^2 U(s,t) + \sum_{j=1}^{n} a_j(s)\partial_j U(s,t) = g(s,t) \ , \qquad s \in \partial G \sim \Gamma \ , \\
&U(x,0) = U_0(x) \ , \quad \partial_t U(x,0) = U_1(x) \ .
\end{aligned}\right\} \tag{3.3}$$

In the special case of $a_j \equiv 0$, $1 \le j \le n-1$, and $a_n(x) \equiv 1$, the partial differential equation in (3.3) is *Sobolev's equation* which describes inertial waves in rotating fluids. Terms due to temperature gradients will give (3.3) with $a_j(x) \equiv a > 0$, $1 \le j \le n-1$, and $a_n(x) \equiv 1$. Finally, if $a_j(x) \equiv a > 0$, $1 \le j \le n$, then the partial differential equation in (3.3) is *Love's equation* for longitudinal vibrations with lateral inertia.

Suppose now that $g \equiv 0$ in the above, hence, $f \in C(\mathbb{R}, H)$, where $H = L^2(G)$. If we assume $\Gamma = \partial G$, hence, $V = H_0^1(G)$, then $D(C) = H_0^1(G) \cap H^2(G) \subset D(A)$, so Theorem 3.2 gives a smoother solution of (3.3) whenever $U_0, U_1 \in D(C)$. If instead we assume $a_j(x) \equiv a$, $1 \leq j \leq n$, then $D(C) = D(A)$, and Theorem 3.2 gives a smoother solution of (3.3) whenever $U_0, U_1 \in D(C)$.

Similar problems containing dissipation effects can easily be added, and we leave these to the exercises. In particular, there is motivation to consider problems like (3.3) with viscosity.

4 Degenerate Equations

We shall consider evolution equations of the form (2.1) wherein the leading operator $\mathcal{C}$ may not necessarily be the Riesz map of a Hilbert space. In particular, certain applications lead to (2.1) with $\mathcal{C}$ being symmetric and monotone. Our plan is to first solve a first order system like (2.2) by using one of Theorems V.4.1 or V.4.2. Then the first and second components will be solutions (of appropriate modifications) of (2.1). Also we shall obtain well-posed problems for a first order evolution equation in which the leading operator is not necessarily symmetric. (The results of Section V.4 do not apply to such a situation.)

4.1

Let $\mathcal{A}$ be the Riesz map of a Hilbert space V to its dual V'. Let $\mathcal{C} \in \mathcal{L}(V, V')$ and suppose its sesquilinear form is symmetric and non-negative on V. Then it follows (cf., Section V.4) that $x \mapsto \mathcal{C}x(x)^{1/2}$ is a seminorm on V; let W denote the corresponding seminorm space. Finally, suppose $D(B) \leq V$ and $B \in L(D(B), V')$ are given. Now we define V_m to be the product $V \times W$ with the seminorm induced by the symmetric and non-negative sesquilinear form

$$m(x, y) = \mathcal{A}x_1(y_1) + \mathcal{C}x_2(y_2) \ , \qquad x, y \in V_m \equiv V \times W \ .$$

The identity $\mathcal{M}x(y) = m(x, y)$, $x, y \in V_m$, defines $\mathcal{M} \in \mathcal{L}(V_m, V_m')$. Finally we define $D \equiv \{[x_1, x_2] \in V \times D(B) : \mathcal{A}x_1 + Bx_2 \in W'\}$ and the linear map $L : D \to V_m'$ by

$$L[x_1, x_2] = [-\mathcal{A}x_2, \mathcal{A}x_1 + Bx_2] \ .$$

We shall apply Theorem V.4.1 to obtain the following result.

Theorem 4.1 *Let $\mathcal{A}$ be the Riesz map of the Hilbert space V and let W be the seminorm space obtained from a symmetric and monotone $\mathcal{C} \in \mathcal{L}(V, V')$. Let $D(B) \leq V$ and $B \in L(D(B), V')$ be monotone. Assume $B + \mathcal{C}$ is strictly monotone and $\mathcal{A}+B+\mathcal{C} : D(B) \to V'$ is a surjection. Let $f \in C^1([0,\infty), W')$ and $g \in C^1([0,\infty), V')$. If V_m and D are the spaces denoted above, then for every pair $[u_0, u_1] \in D$ there exists a unique function $w(\cdot) : [0,\infty) \to D$ such that $\mathcal{M}w(\cdot) \in C^1([0,\infty), V_m')$, $\mathcal{M}w(0) = \mathcal{M}[u_0, u_1]$, and*

$$(\mathcal{M}w)'(t) + Lw(t) = [-g(t), f(t)] \ , \qquad t \geq 0 \ . \tag{4.1}$$

Proof: We need to verify that the hypotheses of Theorem V.4.1 are valid in this situation. First note that $K(\mathcal{M}) \cap D = \{[0, x_2] : x_2 \in D(B),\ Bx_2 \in W',\ \mathcal{C}x_2 = 0\}$. But if $y \in D(B)$ with $By \in W'$, then there is a $K \geq 0$ such that

$$|By(x)| \leq K|\mathcal{C}x(x)|^{1/2} \ , \qquad x \in V \ ,$$

hence, $|By(y)| \leq K|\mathcal{C}y(y)|^{1/2} = 0$ if $\mathcal{C}y = 0$. Thus, we have shown that

$$\mathrm{Re}(B + \mathcal{C})x_2(x_2) = 0 \ , \qquad x = [0, x_2] \in K(\mathcal{M}) \cap D \ ,$$

so $B+\mathcal{C}$ being strictly monotone implies that $K(\mathcal{M}) \cap D = \{[0, 0]\}$. Finally, just as in the proof of Theorem 2.1, it follows from $\mathcal{A}+B+\mathcal{C}$ being surjective that $\mathcal{M} + L$ is surjective, so all the hypotheses of Theorem V.4.1 are true.

Let $w(\cdot)$ be the solution of (4.1) from Theorem 4.1 and set $w(t) = [u(t), v(t)]$ for each $t \geq 0$. If we set $g \equiv 0$ and eliminate $v(\cdot)$ from the system (4.1), then we obtain an equivalent second order evolution equation which $u(\cdot)$ satisfies and, thereby, the following result.

Corollary 4.2 *Let the spaces and operators be given as in Theorem 4.1. For every $f \in C^1([0,\infty), W')$ and every pair $u_0 \in V$, $u_1 \in D(B)$ with $\mathcal{A}u_0 + Bu_1 \in W'$ there exists a unique $u(\cdot) \in C^1([0,\infty), V)$ such that $\mathcal{C}u'(\cdot) \in C^1([0,\infty), W')$, $u(0) = u_0$, $\mathcal{C}u'(0) = \mathcal{C}u_1$, and for each $t \geq 0$, $u'(t) \in D(B)$, $\mathcal{A}u(t) + Bu'(t) \in W'$, and*

$$(\mathcal{C}u'(t))' + Bu'(t) + \mathcal{A}u(t) = f(t) \ . \tag{4.2}$$

Similarly, the function $v(\cdot)$ obtained from a solution of (4.1) satisfies a second order equation.

Corollary 4.3 *Let the spaces and operators be given as in Theorem 4.1. If* $F \in C([0,\infty), W')$, $g \in C^1([0,\infty), V')$, $u_1 \in D(B)$ *and* $U_2 \in W'$, *then there exists a unique* $v(\cdot) : [0,\infty) \to D(B)$ *such that* $\mathcal{C}v(\cdot) \in C^1([0,\infty), W')$, $(\mathcal{C}v)' + Bv(\cdot) \in C^1([0,\infty), V')$, $\mathcal{C}v(0) = \mathcal{C}u_1$, $(\mathcal{C}v' + Bv)(0) = U_2 + Bu_1$, *and for each* $t \geq 0$,

$$((\mathcal{C}v)'(t) + Bv(t))' + \mathcal{A}v(t) = F(t) + g(t) \ . \tag{4.3}$$

Proof: Given $F(\cdot)$ as above, define $f(\cdot) \in C^1([0,\infty), W')$ by $f(t) = \int_0^t F$. With u_1 and U_2 as above, there is a unique $u_0 \in V$ for which $\mathcal{A}u_0 = -Bu_1 - U_2$. Thus, $\mathcal{A}u_0 + Bu_1 \in W'$ so Theorem 4.1 gives a unique $w(\cdot)$ as indicated. Letting $w(t) \equiv [u(t), v(t)]$ for $t \geq 0$, we have immediately $v(t) \in D(B)$ for $t \geq 0$, $\mathcal{C}v \in C^1([0,\infty), W')$ and $\mathcal{C}v(0) = \mathcal{C}u_1$. The second line of (4.1) shows

$$(\mathcal{C}v)' + Bv = f - \mathcal{A}u \in C^1([0,\infty), V')$$

and the choice of u_0 above gives $(\mathcal{C}v)'(0)+Bv(0) = U+Bu_1$. Eliminating $u(\cdot)$ from (4.1) gives (4.3). This establishes the existence of $v(\cdot)$. The uniqueness result follows by defining $u(\cdot)$ by the second line of (4.1) and then noting that the function defined by $w(t) \equiv [u(t), v(t)]$ is a solution of (4.1).

Finally, we record the important special case of Corollary 4.3 that occurs when $\mathcal{C} = 0$. This leads to a well-posed problem for a first order equation whose leading operator is not necessarily symmetric.

Corollary 4.4 *Let the spaces* V, $D(B)$ *and operators* B, $\mathcal{A}$ *be given as in Theorem 4.1 but with* $\mathcal{C} = 0$, *hence,* $W' = \{0\}$. *Then for every* $g \in C^1([0,\infty), V')$ *and* $u_1 \in D(B)$, *there exists a unique* $v : [0,\infty) \to D(B)$ *such that* $Bv(\cdot) \in C^1([0,\infty), V')$, $Bv(0) = Bu_1$, *and for each* $t \geq 0$,

$$(Bv)'(t) + \mathcal{A}v(t) = g(t) \ . \tag{4.4}$$

4.2

Each of the preceding results has a parabolic analogue. We begin with the following.

Theorem 4.5 *Let* $\mathcal{A}$ *be the Riesz map of the Hilbert space* V *and let* W *be the seminorm space obtained from a symmetric and monotone* $\mathcal{C} \in \mathcal{L}(V, V')$. *Let* $\mathcal{B} \in \mathcal{L}(V, V')$ *be monotone and assume that* $\mathcal{B} + \lambda\mathcal{C}$ *is* V*-elliptic for some*

$\lambda > 0$. Then for every pair of Hölder continuous functions $f : [0,\infty) \to W'$, $g : [0,\infty) \to V'$ and each pair $u_0 \in V$, $U_1 \in W'$, there exists a unique function $w : [0,\infty) \to V_m$ such that $\mathcal{M}w(\cdot) \in C([0,\infty), V'_m) \cap C^1((0,\infty), V'_m)$, $\mathcal{M}w(0) = [\mathcal{A}u_0, U_1]$, and for all $t > 0$,

$$(\mathcal{M}w)'(t) + \mathcal{L}w(t) = [-g(t), f(t)] \ ,$$

where $\mathcal{L} \in \mathcal{L}(V \times V, V' \times V')$ is defined by $\mathcal{L}[x_1, x_2] = [-\mathcal{A}x_2, \mathcal{A}x_1 + \mathcal{B}x_2]$, and $\mathcal{M}$ is given as in Theorem 4.1.

Proof: By introducing a change-of-variable, if necessary, we may replace $\mathcal{L}$ by $\lambda\mathcal{M} + \mathcal{L}$. Since for $x \equiv [x_1, x_2] \in V \times V$ we have

$$\mathrm{Re}(\lambda\mathcal{M} + \mathcal{L})x(x) = \lambda\mathcal{A}x_1(x_1) + (\mathcal{B} + \lambda\mathcal{C})x_2(x_2) \ ,$$

we may assume $\mathcal{L}$ is $V \times V$-elliptic. The desired result follows from Theorem V.4.2.

Corollary 4.6 *Let the spaces and operators be given as in Theorem 4.5. For every Hölder continuous $f : [0,\infty) \to W'$, $u_0 \in V$ and $U_1 \in W'$, there exists a unique $u(\cdot) \in C([0,\infty), V) \cap C^1((0,\infty), V)$ such that $\mathcal{C}u'(\cdot) \in C((0,\infty), W') \cap C^1((0,\infty), W')$, $u(0) = u_0$, $\mathcal{C}u'(0) = U_1$, and*

$$(\mathcal{C}u'(t))' + \mathcal{B}u'(t) + \mathcal{A}u(t) = f(t) \ , \qquad t > 0 \ . \tag{4.5}$$

Corollary 4.7 *Let the spaces and operators be given as in Theorem 4.5. Suppose $F : (0,\infty) \to W'$ is continuous at all but a finite number of points and for some $p > 1$ we have $\int_0^T \|F(t)\|^p_{W'}\,dt < \infty$ for all $T > 0$. If $g : [0,\infty) \to V'$ is Hölder continuous, $u_1 \in V$ and $U_2 \in V'$, then there is a unique function $v(\cdot) : [0,\infty) \to V$ such that $\mathcal{C}v \in C([0,\infty), W') \cap C^1((0,\infty), W')$, $(\mathcal{C}v)' + \mathcal{B}v \in C([0,\infty), V')$ and is continuously differentiable at all but a finite number of points, $\mathcal{C}v(0) = \mathcal{C}u_1$, $(\mathcal{C}v' + \mathcal{B}v)(0) = U_2 + \mathcal{B}u_1$, and*

$$((\mathcal{C}v)'(t) + \mathcal{B}v(t))' + \mathcal{A}v(t) = F(t) + g(t) \tag{4.6}$$

at those points at which the derivative exists.

Proof: Almost everything follows as in Corollary 4.3. The only difference is that we need to note that with $F(\cdot)$ as given above, the function $f(t) = \int_0^t F$

satisfies

$$\|f(t) - f(\tau)\|_{W'} \leq \int_\tau^t \|F\|_{W'} \leq |t-\tau|^{1/q}\left(\int_\tau^t \|F\|_{W'}^p\right)^{1/p}$$
$$\leq |t-\tau|^{1/q}\left(\int_0^T \|F\|_{W'}^p\right)^{1/p} , \qquad 0 \leq \tau \leq t \leq T ,$$

where $1/q = 1 - 1/p \geq 0$. Hence, f is Hölder continuous.

5 Examples

We shall illustrate some applications of our preceding results by various examples of initial-boundary value problems. In each such example below, the operator $\mathcal{A}$ will correspond to one of the elliptic boundary value problems described in Section III.4, and we refer to that section for the computations as well as occasional notations. Our emphasis here will be on the *types* of operators that can be chosen for the remaining coefficients in either of (4.2) or (4.3).

We begin by constructing the operator $\mathcal{A}$ from the abstract boundary value problem of Section III.3. Let V, H and B be Hilbert spaces and $\gamma : V \to B$ a linear surjection with kernel V_0, and assume γ factors into a norm-preserving isomorphism of V/V_0 onto B. Assume the injection $V \hookrightarrow H$ is continuous, V_0 is dense in H, and H is identified with H'. Then we obtain the continuous injections $V_0 \hookrightarrow H \hookrightarrow V_0'$ and $V \hookrightarrow H \hookrightarrow V'$ and

$$(f, v)_H = f(v) , \qquad f \in H , \ v \in V .$$

Let $a_1 : V \times V \to \mathbb{K}$ and $a_2 : B \times B \to \mathbb{K}$ be continuous, sesquilinear and symmetric forms and define

$$a(u,v) \equiv a_1(u,v) + a_2(\gamma u, \gamma v) , \qquad u, v \in V . \tag{5.1}$$

We shall assume $a(\cdot,\cdot)$ is V-elliptic; thus, $a(\cdot,\cdot)$ is a scalar-product on V whose norm is equivalent to the original one on V. Hereafter, we shall take $a(\cdot,\cdot)$ as the scalar-product on V; the corresponding Riesz map $\mathcal{A} \in \mathcal{L}(V, V')$ is given by

$$\mathcal{A}u(v) = a(u,v) , \qquad u, v \in V .$$

Similarly, we define $A \in \mathcal{L}(V, V_0')$ by

$$Au(v) = a_1(u,v) , \qquad u \in V , \ v \in V_0 , \tag{5.2}$$

Let $D_0 \equiv \{u \in V : Au \in H\}$, and denote by $\partial \in L(D_0, B')$ the abstract Green's operator constructed in Theorem III.2.3 and characterized by the identity

$$a_1(u,v) - (Au,v)_H = \partial u(\gamma(v)) \ , \qquad u \in D_0 \ , \ v \in V \ . \tag{5.3}$$

Finally, we denote by $\mathcal{A}_2 \in \mathcal{L}(B, B')$ the operator given by

$$\mathcal{A}_2\varphi(\psi) = a_2(\varphi, \psi) \ , \qquad \varphi, \psi \in B \ .$$

It follows from (5.1), (5.2) and (5.3) that

$$\mathcal{A}u(v) - (Au,v)_H = (\partial u + \mathcal{A}_2(\gamma u))(\gamma v) \ , \qquad u \in D_0 \ , \ v \in V \ , \tag{5.4}$$

and this identity will be used to characterize the weak or variational boundary conditions below.

Let $c : H \times H \to \mathbb{K}$ be continuous, non-negative, sesquilinear and symmetric; define the monotone $\mathcal{C} \in \mathcal{L}(H)$ by

$$\mathcal{C}u(v) = c(u,v) \ , \qquad u, v \in H \ ,$$

where $\mathcal{C}u \in H$ follows from $H' = H$. Note that the inclusion $W' \subset H$ follows from the continuity of the injection $H \hookrightarrow W$, where W is the space H with seminorm induced by $c(\cdot,\cdot)$. Finally let $\mathcal{B} \in \mathcal{L}(V, H)$ be a given monotone operator

$$\mathrm{Re}\, \mathcal{B}u(v) \geq 0 \ , \qquad u \in V \ , \ v \in H \ ,$$

and assume $\mathcal{C} + \mathcal{B}$ is strictly monotone:

$$(\mathcal{C} + \mathcal{B})u(u) = 0 \ \text{ only if } \ u = 0 \ .$$

Theorem 5.1 *Let the Hilbert spaces and operators be given as above. For every $f \in C^1([0,\infty), W')$ and every pair $u_0, u_1 \in V$ with $\mathcal{A}u_0 + \mathcal{B}u_1 \in W'$, there exists a unique $u \in C^1([0,\infty), V)$ such that $\mathcal{C}u' \in C^1([0,\infty), W')$, $u(0) = u_0$, $\mathcal{C}u'(0) = \mathcal{C}u_1$, and for each $t \geq 0$,*

$$(\mathcal{C}u'(t))' + \mathcal{B}u'(t) + Au(t) = f(t) \ , \tag{5.5}$$

$$u(t) \in D_0 \subset V \ , \quad \partial u(t) + \mathcal{A}_2\gamma(u(t)) = 0 \ . \tag{5.6}$$

Proof: The existence and uniqueness of $u(\cdot)$ follows from Corollary 4.2. With $\mathcal{C}$ and $\mathcal{B}$ as above (4.2) shows that $\mathcal{A}u(t) \in H$ for all $t \geq 0$, so (5.6) follows from Corollary III.3.2. (Cf. (5.4).) To be sure, the pair of equations (5.5), (5.6), is equivalent to (4.2).

We illustrate Theorem 5.1 in the examples following in Sections 5.1 and 5.2.

5.1

Let G be open in $\mathbb{R}^n$, $H = L^2(G)$, $\Gamma \subset \partial G$ and $V = \{v \in H^1(G) : \gamma_0(v)(s) = 0$, a.e. $s \in \Gamma\}$. Let $p \in L^\infty(G)$ with $p(x) \geq 0$, $x \in G$, and define

$$c(u,v) = \int_G p(x)u(x)\overline{v(x)}\,dx\ , \qquad u, v \in H\ . \tag{5.7}$$

Then $\mathcal{C}$ is multiplication by p and $W' = \{p^{1/2}v : v \in L^2(G)\}$. Let $R \in L^\infty(G)$ and the real vector field $\mu(x) = (\mu_1(x), \ldots, \mu_n(x))$ be given with each $\mu_j \in C^1(\bar{G})$; assume

$$-\left(\tfrac{1}{2}\right)\sum_{j=1}^{n} \partial_j \mu_j(x) + \mathrm{Re}\{R(x)\} \geq 0\ , \qquad x \in G\ ,$$

$$\left(\tfrac{1}{2}\right)\mu(s)\cdot\nu(s) \geq 0\ , \qquad s \in \partial G \sim \Gamma\ .$$

Then $\mathcal{B} \in \mathcal{L}(V, H)$ given by (2.7) is monotone. Furthermore, we shall assume

$$p(x) - \left(\tfrac{1}{2}\right)\sum_{j=1}^{n} \partial_j \mu_j(x) + \mathrm{Re}\{R(x)\} > 0\ , \qquad x \in G\ ,$$

and this implies $\mathcal{C} + \mathcal{B}$ is strictly-monotone.

Let $a_0, a_{ij} \in L^\infty(G)$, $1 \leq i, j \leq n$, and assume $a_0(x) \geq 0$, $a_{ij}(x) = \overline{a_{ji}(x)}$, $x \in G$, and that

$$a(u,v) \equiv \int_G \left\{\sum_{i,j=1}^{n} a_{ij}(x)\partial_i u(x)\partial_j \overline{v(x)} + a_0(x)u(x)\overline{v(x)}\right\} dx \tag{5.8}$$

is V-coercive (cf. Section III.5). Then (5.8) is a scalar product on V whose norm is equivalent to that of $H^1(G)$ on V.

Let $F(\cdot, t) \in L^2(G)$ be given for each $t \geq 0$ such that $t \mapsto F(\cdot, t)$ belongs to $C^1([0,\infty), L^2(G))$ (cf. Section 2.1). Then $f(t) \equiv p^{1/2}F(\cdot, t)$ defines $f \in C^1([0,\infty), W')$. Finally, let $U_0, U_1 \in V$ satisfy $\mathcal{A}U_0 + \mathcal{B}U_1 \in W'$. (This can be translated into an elliptic boundary value problem.) Using Theorem IV.7.1, we can obtain a (measurable) function $U(\cdot,\cdot)$ on $G \times [0,\infty)$ which is a solution of the initial-boundary value problem

$$\partial_t(p(x)\partial_t U(x,t)) + R(x)\partial_t U(x,t) + \frac{\partial}{\partial\mu}(\partial_t U(x,t)) \tag{5.9}$$

$$
\begin{aligned}
&-\sum_{j=1}^{n} \partial_j a_{ij}(x) \partial_i U(x,t) + a_0(x) U(x,t) \\
&= p^{1/2}(x) F(x,t) \ , \qquad x \in G \ , \ t \geq 0 \ ;
\end{aligned}
$$

$$
\left.\begin{aligned}
&U(s,t) = 0 \ , \qquad s \in \Gamma \ , \\
&\frac{\partial U(s,t)}{\partial \nu_A} = 0 \ , \qquad s \in \partial G \sim \Gamma \ ;
\end{aligned}\right\} \tag{5.10}
$$

$$
\left.\begin{aligned}
&U(x,0) = U_0(x) \ , \\
&p(x) \partial_t U(x,0) = p(x) U_1(x) \ , \qquad x \in G \ .
\end{aligned}\right\} \tag{5.11}
$$

We refer to Section III.4.1 for notation and computations involving the operators associated with the form (5.8).

The partial differential equation (5.9) is of mixed hyperbolic-parabolic type. Note that the initial conditions (5.11) imposed on the solution at $x \in G$ depend on whether $p(x) > 0$ or $p(x) = 0$. Also, the equation (5.9) is satisfied at $t = 0$, thereby imposing a compatibility condition on the initial data U_0, U_1. Finally, we observe that (5.10) contains the boundary condition of *first type* along Γ and the boundary condition of *second type* on $\partial G \sim \Gamma$.

5.2

Let H and $\mathcal{C}$ be given as in Section 5.1; let $V = H^1(G)$ and define $\mathcal{B}$ by (2.7) with $\mu \equiv 0$ and assume

$$
\begin{aligned}
\mathrm{Re}\{R(x)\} &\geq 0 \ , \qquad p(x) \geq 0 \ , \\
p(x) + \mathrm{Re}\{R(x)\} &> 0 \ , \qquad x \in G \ ,
\end{aligned}
$$

as before. Define

$$
\begin{aligned}
a_1(u,v) &= \int_G \nabla u \cdot \nabla \bar{v} \qquad & u, v \in V \ , \\
a_2(\varphi,\psi) &= \int_{\partial G} \alpha(s) \varphi(s) \overline{\psi(x)} \, ds \ , \qquad & \varphi, \psi \in L^2(\partial G)
\end{aligned}
$$

where $\alpha \in L^\infty(\partial G)$, $\alpha(s) \geq 0$, a.e. $s \in \partial G$. Then $\mathcal{A}_2$ is multiplication by α. We assume that $a(\cdot,\cdot)$ given by (5.1) is V-coercive (cf. Corollary III.5.5). With $F(\cdot,\cdot)$, U_0, and U_1 as above, we obtain a unique generalized solution of the problem

$$
\partial_t (p(x) \partial_t U(x,t)) + R(x) \partial_t U(x,t) - \Delta_n U(x,t) \tag{5.12}
$$

$$= p^{1/2}(x)F(x,t) \;, \qquad x \in G \;,\; t \geq 0 \;,$$

$$\frac{\partial U(s,t)}{\partial \nu} + \alpha(s)U(s,t) = 0 \;, \qquad s \in \partial G \;,\; t \geq 0 \;, \tag{5.13}$$

and (5.11). We note that at those $x \in G$ where $p(x) > 0$, (5.12) is a (hyperbolic) wave equation and (5.11) specifies initially U and $\partial_t U$, whereas at those $x \in G$ where $p(x) = 0$, (5.12) is a homogeneous (parabolic) diffusion equation and only U is specified initially. The condition (5.13) is the boundary condition of *third type*.

If we choose $V = \{v \in H^1(G) : \gamma_0(v) \text{ is constant}\}$ as in Section III.4.2 and prescribe everything else as above, then we obtain a solution of (5.12), (5.11) and the boundary condition of *fourth type*

$$\left.\begin{aligned} &U(s,t) = h(t) \;, \qquad s \in \partial G \;, \\ &\int_{\partial G} \frac{\partial U(s,t)}{\partial \nu}\, ds + \int_{\partial G} \alpha(s)\, ds \cdot h(t) = 0 \;. \end{aligned}\right\} \tag{5.14}$$

Note that $h(\cdot)$ is an unknown in the problem. Boundary value problems with *periodic boundary conditions* can be put in the form of (5.14).

5.3

Let $H = L^2(G)$, $V = H^1(G)$, and define $\mathcal{A}$ as in Section 5.2. Set $\mathcal{B} \equiv 0$ and define

$$c(u,v) = \int_G p(x)u(x)\overline{v(x)}\, dx + \int_{\partial G} \sigma(s)u(s)\overline{v(s)}\, ds \;, \qquad u,v \in V$$

when $p \in L^\infty(G)$ and $\sigma \in L^\infty(\partial G)$ satisfy $p(x) > 0$, $x \in G$, and $\sigma(s) \geq 0$, $s \in \partial G$. Let $t \mapsto F(\cdot,t)$ be given in $C^1([0,\infty), L^2(G))$ and $t \mapsto g(\cdot,t)$ be given in $C^1([0,\infty), L^2(\partial G))$; then define $f \in C^1([0,\infty), W')$ by

$$f(t)(v) = \int_G p^{1/2}(x)F(x,t)\overline{v(x)}\, dx + \int_{\partial G} \sigma^{1/2}(s)g(s,t)\overline{v(s)}\, ds \;, \quad v \in V \;,\; t \geq 0 \,.$$

Let $U_0, U_1 \in V$ with $\mathcal{A}U_0 \in W'$. (This last inclusion is equivalent to requiring $\Delta_n U_0 = p^{1/2}H$ for some $H \in L^2(G)$ and $\partial_\nu U_0 + \alpha U_0 = \sigma^{1/2}h$ for some $h \in L^2(\partial G)$.) Then Corollary 4.2 applies to give a unique solution u of (4.2) with initial conditions. From this we obtain a solution U of the problem

$$\begin{cases} \partial_t(p(x)\partial_t U(x,t)) - \Delta_n U(x,t) = p^{1/2}(x)F(x,t) \ , \\ \hfill x \in G \ , \ t \geq 0 \ , \\ \partial_t(\sigma(s)\partial_t U(s,t)) + \partial_\nu U(s,t) + \alpha(s)U(s,t) = \sigma^{1/2}(s)g(s,t) \ , \\ \hfill s \in \partial G \ , \ t \geq 0 \ , \\ U(x,0) = U_0(x) \ , \quad \partial_t U(x,0) = U_1(x) \ . \end{cases}$$

The boundary condition is obtained *formally* since we do not know $\Delta_n U(\cdot,t) \in L^2(G)$ for all $t > 0$; hence, (5.3) is not directly applicable. Such boundary conditions arise in models of vibrating membranes (or strings) with boundaries (or ends) loaded with a mass distribution, thereby introducing an inertia term. Such problems could also contain mass distributions (or point loads) on internal regions. Similarly, internal or boundary damping can be included by appropriate choices of $\mathcal{B}$, and we illustrate this in the following example.

5.4

Let H, V, $\mathcal{A}$ and $\mathcal{C}$ be given as in Section 5.3. Assume $R \in L^\infty(G)$, $r \in L^\infty(\partial G)$ and that $\mathrm{Re}\{R(x)\} \geq 0$, $x \in G$, $\mathrm{Re}\{r(s)\} \geq 0$, $s \in \partial G$. We define $\mathcal{B} \in \mathcal{L}(V,V')$ by

$$\mathcal{B}u(v) = \int_G R(x)u(x)\overline{v(x)}\,dx + \int_{\partial G} r(s)u(s)\overline{v(s)}\,ds \ , \qquad u,v \in V \ .$$

We need only to assume $p(x) + \mathrm{Re}\{R(x)\} > 0$ for $x \in G$; then Corollary 4.3 is applicable. Let $t \mapsto F_1(\cdot,t)$ in $C([0,\infty),L^2(G))$, $t \mapsto G_1(\cdot,t)$ in $C^1([0,\infty),L^2(\partial G))$, and $t \mapsto G_2(t)$ in $C^1([0,\infty),L^2(G))$ be given. We then define $F \in C([0,\infty),W')$ and $g \in C^1([0,\infty),V')$ by

$$\begin{aligned} F(t) &= p^{1/2}F_1(\cdot,t) \ , \\ g(t)(v) &= \int_{\partial G} \sigma^{1/2}(s)G_1(s,t)\overline{v(s)}\,ds + \int_G G_2(x,t)\overline{v(x)}\,dx \ , \qquad v \in V \ . \end{aligned}$$

If $U_1 \in V$ and $V_1 \in L^2(G)$, and $V_2 \in L^2(\partial G)$, then $U_2 \in W'$ is defined by

$$U_2(v) = \int_G p^{1/2}(x)V_1(x)\overline{v(x)}\,dx + \int_{\partial G} \sigma^{1/2}(s)V_2(s)\overline{v(s)}\,ds \ , \qquad v \in V \ ,$$

and Corollary 4.3 gives a generalized solution of the following problem:

$$\begin{cases} \partial_t^2(p(x)U(x,t)) + \partial_t(R(x)U(x,t)) - \Delta_n U(x,t) \\ \qquad\qquad = p^{1/2}(x)F_1(x,t) + G_2(x,t) \;, \qquad x \in G \;, \\ \partial_t^2(\sigma(s)U(s,t)) + \partial_t(r(s)U(s,t)) + \partial_\nu U(s,t) + (s)U(s,t) \\ \qquad\qquad = \sigma^{1/2}(s)G_1(s,t) \;, \qquad s \in \partial G \;,\; t > 0 \;, \\ p(x)U(x,0) = p(x)U_1(x) \;, \\ \sigma(s)U(s,0) = \sigma(s)U_1(s) \;, \qquad s \in \partial G \\ \partial_t(p(x)U(x,0)) + R(x)U(x,0) = p^{1/2}(x)V_1(x) \;, \\ \partial_t(\sigma(s)U(s,0)) + r(s)U(s,0) = \sigma^{1/2}(s)V_2(s) \;. \end{cases}$$

The right side of the partial differential equation could contain singularities in x as well. When $\mathrm{Re}\{R(x)\} > 0$ in G, the preceding problem with $p \equiv 0$ and $\sigma \equiv 0$ is solved by Corollary 4.4.

Similarly one can obtain generalized solutions to boundary value problems containing partial differential equations of the type (3.3); that is, equations of the form (5.9) plus the fourth-order term $-\partial_t(\Delta_n \partial_t U(x,t))$. Finally, we record an abstract parabolic boundary value problem which is solved by using Corollary 4.6. Such problems arise in classical models of linear viscoelasticity (cf. (2.9)).

Theorem 5.2 *Let the Hilbert spaces and operators be given as in Theorem 5.1, except we do not assume $\mathcal{B} + C$ is strictly monotone. If $\varepsilon > 0$, $f : [0,\infty) \to W'$ is Hölder continuous, $u_0 \in V$ and $U_1 \in W'$, there exists a unique $u \in C([0,\infty), V) \cap C^1([0,\infty), V)$ such that $\mathcal{C}u' \in C([0,\infty), W') \cap C^1((0,\infty), W')$, $u(0) = u_0$, $\mathcal{C}u'(0) = U_1$, and (5.5), (5.6) hold for each $t > 0$.*

Exercises

1.1. Use the separation-of-variables technique to obtain a series representation for the solution u of (1.1) with $u(0,t) = u(\pi,t) = 0$, $u(x,0) = u_0(x)$ and $\partial_t u(x,0) = u_1(x)$.

1.2. Repeat the above for the viscoelasticity equation

$$\partial_t^2 u - \varepsilon \partial_t \Delta_n u - \Delta_n u = F(x,t) \;, \qquad \varepsilon > 0 \;.$$

1.3. Compare the convergence rates of the two series solutions obtained above.

2.1. Explain the identification $V_m' = V' \times W'$ in Section 2.1.

2.2. Use Theorem 2.1 to prove Theorem 2.3.

2.3. Use the techniques of Section 2 to deduce Theorem 2.3 from IV.5.

2.4. Verify that the function f in Section 2.1 belongs to $C^1([0,T], L^2(G))$.

2.5. Use Theorem 2.1 to construct a solution of (2.5) satisfying the fourth boundary condition. Repeat for each of the examples in Section III.4.

2.6. Add the term $\int_{\partial G} r(s)u(s)\overline{v(s)}\,ds$ to (2.7) and find the initial-boundary value problem that results.

2.7. Show that Theorem 2.1 applies to appropriate problems for the equation

$$\partial_t^2 u(x,t) + \partial_x^3 \partial_t u(x,t) - \partial_x^2 u(x,t) = F(x,t) \ .$$

2.8. Find some well-posed problems for the equation

$$\partial_t^2 u(x,t) + \partial_x^4 u(x,t) = F(x,t) \ .$$

3.1. Complete the proofs of Theorem 3.2 and Corollary 3.3.

3.2. Verify that (3.3) is the characterization of (3.1) with the given data.

4.1. Use Corollary 4.4 to solve the problem

$$\begin{aligned} &\partial_t \partial_x u(x,t) - \partial_x^2 u(x,t) = F(x,t) \\ &u(0,t) = cu(1,t) \\ &u(x,0) = u_0(x) \end{aligned}$$

for $|c| \le 1$, $c \ne 1$.

4.2. For each of the Corollaries of Section 4, give an example which illustrates a problem solved by that Corollary only.

5.1. In the proof of Theorem 5.1, verify that (4.2) is equivalent to the pair (5.5), (5.6).

5.2. In Section 5.1, show $\mathcal{C}+\mathcal{B}$ is strictly monotone, give sufficient conditions for (5.8) to be V-elliptic, and characterize the condition $\mathcal{A}U_0 + \mathcal{B}U_1 \in W'$ as requiring that U_0 satisfy an elliptic boundary value problem (cf. Section 5.3).

5.3. In Section 5.2, give sufficient conditions for $a(\cdot,\cdot)$ to be V-elliptic.

5.4. Show the following problem with periodic boundary conditions is well-posed: $\partial_t^2 u - \partial_x^2 u = F(x,t)$, $u(x,0) = u_0(x)$, $\partial_t u(x,0) = u_1(x,0)$, $u(0,t) = u(1,t)$, $\partial_x u(0,t) = \partial_x u(1,t)$. Generalize this to higher dimensions.

5.5. A vibrating string loaded with a point mass m at $x = \frac{1}{2}$ leads to the following problem: $\partial_t^2 u = \partial_x^2 u$, $u(0,t) = u(1,t) = 0$, $u(x,0) = u_0(x)$, $\partial_t u(x,0) = u_1(x)$, $u((\frac{1}{2})^-,t) = u((\frac{1}{2})^+,t)$, $m\partial_t^2 u(\frac{1}{2},t) = \partial_x u((\frac{1}{2})^+,t) - \partial_x u((\frac{1}{2})^-,t)$. Use the methods of Section 5.3 to show this problem is well-posed.