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LIE GENERATORS FOR SEMIGROUPS OF TRANSFORMATIONS ON A POLISH SPACE

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Abstract. Let X be a separable complete metric space. We characterize completely the infinitesimal generators of semigroups of linear transformations in $C_b(X)$, the bounded real-valued continuous functions on X, that are induced by strongly continuous semigroups of continuous transformations in X. In order to do this, $C_b(X)$ is equipped with a locally convex topology known as the *strict topology.*

Introduction. A strongly continuous semigroup of transformations on a topological space X is a function T from $[0,\infty)$ into the collection of continuous transformations from X into X such that
(1) $T(0) = I$, the identity transformation on X,
(2) $T(t) \circ T(s) = T(t+s)$ for all $t, s \geq 0$, and
(3) if $x \in X$, then the function $T(\cdot)x$ is continuous from $[0,\infty)$ into X.
We will follow the standard practice writing the semigroup as a collection and denoting such a semigroup as $\{T(t) : t \geq 0\}$ or just $\{T(t)\}$. Since at least the time of Sophus Lie, mathematicians have been investigating generators, often called *infinitesimal generators*, of such semigroups. The case in which X is a Banach space and $T(t)$ is a bounded linear operator on X for each $t \geq 0$ has become particularly well understood. A series of results, now called Hille-Yosida theorems, or Hille-Yosida-Phillips theorems, forms the heart of this understanding. The *infinitesimal generator* of a strongly continuous semigroup of (bounded) linear operators on X is defined by

$$Ax = \lim_{t\to 0} \frac{1}{t}\left[T(t)x - x\right], \tag{4}$$

with domain $\mathcal{D}(A)$ consisting of all x for which this limit exists. The simplest Hille-Yosida theorem characterizes infinitesimal generators of strongly continuous semigroups of nonexpansive linear transformations. If A is the infinitesimal generator of a strongly continuous nonexpansive semigroup $\{T(t)\}$, then $\mathcal{D}(A)$ is dense in X, and

(5) $I - \varepsilon A$ has a nonexpansive inverse defined on all of X for each $\varepsilon > 0$.

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Furthermore, the semigroup is constructed from its infinitesimal generator by

$$T(t)x = \lim_{n\to\infty} (I - (t/n)A)^{-n}\, x \quad \text{for} \quad t > 0,\ x \in X. \tag{6}$$

Conversely, any densely defined operator A in X that satisfies (5) is the infinitesimal generator of a strongly continuous nonexpansive semigroup, which is given by (6).

All Hille-Yosida theorems deal with a collection $\mathcal{G}$ of generators and a collection $\mathcal{S}$ of semigroups. Each $A \in \mathcal{G}$ is obtained from a $T \in \mathcal{S}$ by means of a formula like (4), and each $T \in \mathcal{S}$ is obtained from an $A \in \mathcal{G}$ by a formula like (6). The main motivation for this activity is to solve initial value problems for abstract ordinary differential equations, which, by allowing A to be discontinuous, include partial differential equations (with boundary conditions incorporated into the domain of A).

Considerable effort was made in the late 1960's and early 1970's to find analogous results for semigroups of nonlinear transformations. The only complete success was in the case of strongly continuous semigroups of nonexpansive transformations on convex sets in a Hilbert space; see [Br]. The infinitesimal generator of such a semigroup was characterized as a single-valued selection (the element of minimum norm) of a possibly multivalued maximal dissipative operator. There similar results for Banach spaces with smooth norm ([Ba],[R]). Another very important result was the Crandall-Liggett theorem, [CL], which gave sufficient conditions on a possibly multivalued operator in a general Banach space X in order that the formula (6) give a strongly continuous semigroup of nonexpansive transformations on a convex subset of X, but there is no proof that every strongly continuous nonexpansive semigroup arises in this way. Progress toward a fairly complete nonlinear Hille-Yosida theory in general Banach spaces (even for nonexpansive semigroups) has been minimal.

A second approach to the semigroup-generator problem goes back to Sophus Lie himself in his search for a theory of ordinary differential equations in terms of integrating factors; see [In]. This second approach shares common ground with Koopman and von Neumann on representation by means of unitary groups of groups generated by Hamiltonian systems; see [Ko], [Nm]. For a general strongly continuous semigroup $\{T(t)\}$ of continuous transformations in a metric space X, one may consider the induced linear semigroup $\{U(t)\}$ in the space $C_b(X)$ of bounded real-valued continuous functions on X given by $U(t)f = f \circ T(t)$. It was essentially shown in [Nb] that the semigroup $\{U(t)\}$, while not generally strongly continuous, has a generator A that is dense in the topology of pointwise convergence (the limit in the definition of generator is also taken pointwise), and that $I - \varepsilon A$ has a nonexpansive inverse defined on all of $C_b(X)$ for each $\varepsilon > 0$. Furthermore, $\{U(t)\}$, and hence $\{T(t)\}$, may be recovered from A by means of

$$U(t)f = \lim_{n\to\infty} (I - (t/n)A)^{-n}\, f \quad \text{for} \quad t > 0, f \in C_b(X),$$

where the limit is taken pointwise on X. This is at best half the Hille-Yosida result, since a characterization of such Lie generators was not found.

It is the purpose of this paper to give a complete characterization of the Lie generators of strongly continuous semigroups of continuous transformations on a separable complete metric space. We hope that this characterization will be a useful substitute for a still nonexistent Hille-Yosida characterization of such semigroups in terms of ordinary generators. Let X denote a separable complete metric space, and let $\{T(t) : t \geq 0\}$ denote a strongly continuous semigroup of continuous transformations on X. That is, each $T(t)$ is a continuous transformation from X into X, and for each $x \in X$, the mapping $t \to T(t)x$ is continuous from $[0, \infty)$ into X. It follows that $\{T(t)\}$ is *jointly continuous*, that is, the mapping $(t, x) \to T(t)x$ is continuous from $[0, \infty) \times X$ into X. Now let $\{U(t)\}$ denote the semigroup of linear transformations in $C_b(X)$, the linear space of bounded real-valued continuous functions on X, given by $U(t)f = f \circ T(t)$. It is clear that each $U(t)$ is a multiplicative homomorphism. We describe a topology β on $C_b(X)$ such that $\{U(t)\}$ is a strongly continuous semigroup of continuous transformations in $(C_b(X), \beta)$, such that $\{U(t) : 0 \leq t \leq b\}$ is β-equicontinuous for each $b > 0$, and such that if $\alpha > 0$, then the semigroup $\{e^{-\alpha t}U(t)\}$ is β-equicontinuous. There is a version of the Hille-Yosida-Phillips Theorem for such semigroups of linear transformations in topological vector spaces. The semigroup consists of homomorphisms if and only if the generator is a derivation. We also show that every β-strongly continuous semigroup of β-continuous homomorphisms in $C_b(X)$ is induced by a strongly continuous semigroup of continuous transformations in X. This gives a correspondence between the strongly continuous semigroups of continuous transformations in X and a well-described class of linear derivations in $C_b(X)$. Probably the most interesting case is that in which X is a separable G_δ set in a Banach space, equipped with a complete metric that induces the topology inherited from the Banach space. A similar characterization was obtained by the first author [D1], [D2] for strongly continuous semigroups of transformations in a locally compact Hausdorff space.

Section 1. Preliminaries. In this section we give some facts about topological vector spaces that will be needed in order to establish our main results. We begin by stating a simple and straight-forward generalization of the usual Hille-Yosida-Phillips Theorem to the setting of topological vector spaces. Many extensions have been given, and many complications can occur. However, the following theorem, which is essentially given in [Y, Chapter IX, Section 7], will suit our purposes.

1.1 Theorem. *Let E be a sequentially complete locally convex topological vector space, and let A be a linear operator in E. Then the following two statements are equivalent:*

i) A is the infinitesimal generator of a strongly continuous equicontinuous semigroup of transformations in E.

ii) The domain of A is dense in E and

$$\left\{ \left(I - n^{-1}A\right)^{-m} : m, n = 1, 2, \ldots \right\}$$

is an equicontinuous collection of linear operators on E.

Remark. Formula (6) of the introduction for obtaining the semigroup from its generator is not established for this setting in Yosida's book; rather, the related formula

$$U(t)x = \lim_{n\to\infty} \exp\left(tn\left[(I - n^{-1}A)^{-1} - I\right]\right)x \quad \text{for } t > 0, x \in E$$

is proven, where $\{U(t)\}$ is the semigroup generated by A.

The following proposition characterizes the infinitesimal generators of strongly continuous semigroups of continuous multiplicative homomorphisms in a topological vector space that is also an algebra. The proof is completely routine and is omitted.

1.2 Proposition. *Let $\{U(t)\}$ be a strongly continuous semigroup of continuous linear transformations in the sequentially complete locally convex topological vector space E, and let A be its infinitesimal generator. Suppose E is also an algebra and that multiplication is jointly continuous. Then $\{U(t)\}$ is a semigroup of homomorphisms if and only if A is a derivation; that is, $f, g \in \mathcal{D}(A)$ implies that $fg \in \mathcal{D}(A)$ and $A(fg) = f(Ag) + (Af)g$.*

Now let $E = C_b(X)$, the linear space of all bounded real-valued continuous functions on the separable complete metric space X. For $r > 0$, let B_r denote the closed ball $\{f \in E : \|f\| \leq r\}$, where $\|\cdot\|$ denotes the supremum norm. Let κ denote the compact-open topology on E, and let β denote the strongest locally convex topology on E that agrees with κ on each set B_r. We call β the strict topology on E. Throughout this paper, "norm" will mean the supremum norm, and E^* will denote the space of β-continuous linear functionals on E. The rest of this section is devoted to establishing the properties of the topology β that are needed for our main results. Mainly, the properties we need are documented in the paper [S], and what we do here is to provide a guide for finding the documentation. Sentilles defines three "strict" topologies [S, p 315] in the case that X is a completely regular Hausdorff space, and then establishes that the three topologies coincide if X is a complete separable metric space [S, Theorem 9.1, p 332].

1.3 Proposition. *$\varphi \in E^*$ if and only if there is a bounded Borel measure μ on X such that $\varphi(f) = \int_X f\, d\mu$ for all $f \in E$.*

Proof. This is (c) of [S, Theorem 9.1, p 332], together with the fact that every bounded Borel measure on a complete separable metric space is compact regular; see for example [P, Thm. 3.2, p 29], or [K, Cor. to Thm. 3.3, p 147] (applied to the total variation of a signed measure). ∎

1.4 Proposition. *Let V be an absolutely convex absorbent set in E having the property ($\mathcal{P}$): for each $r > 0$, there is a β-neighborhood V_r of 0 such that $V \supset V_r \cap B_r$. Then V is a β-neighborhood of 0.*

Proof. Since β is the strongest locally convex topology on E agreeing with κ on each set B_r, then β is the strongest locally convex topology on E agreeing with itself on each set B_r. The collection of all absolutely convex absorbent sets V having property ($\mathcal{P}$) is a base for a locally convex topology γ on E by [RR, Thm. 2, p 10]. Clearly, γ is stronger than β, and the restriction of β to any set B_r coincides with the restriction of γ to B_r. Therefore, $\beta = \gamma$. ∎

1.5 Proposition. *The strict topology has a base of 0-neighborhoods of the form*

$$W\{(K_n, a_n)\} = \{f \in E : \|f\|_{K_n} \le a_n \text{ for all } n\},$$

where K_n is a compact set in X for each n, $\{a_n\}$ is a sequence of positive numbers converging to ∞, and $\|f\|_K = \sup_{x\in K} |f(x)|$ for $f \in E$ and $K \subset X$.

Proof. See [S, Thm. 2.4(a), p 316]. ∎

1.6 Proposition. *If m is a nonzero β-continuous multiplicative linear functional on E, then there is a point $\hat{m} \in E$ such that $m(f) = f(\hat{m})$ for all $f \in E$.*

Proof. By Proposition 1.3, m is given by $m(f) = \int_X f\,d\mu$ for some bounded Borel measure μ on X. The fact that m is a norm continuous multiplicative linear functional implies that there is a point z of the Stone-Čech compactification $\widehat{X}$ of X such that $m(f) = \bar{f}(z)$ for all $f \in E$, where $\bar{f}$ denotes the continuous extension of f to $\widehat{X}$. Therefore, $\int_X f\,d\mu = \bar{f}(z)$ for all $f \in E$. By [K, Thm. 2.1, p 142], $z \in X$. ∎

Section 2. The Induced Linear Semigroup. Let $\{T(t)\}$ denote a strongly continuous semigroup of continuous transformations from X into X. That is, we assume that the mapping $(t, x) \to T(t)x$ is separately continuous. It follows that this mapping is jointly continuous; see [CM, Thm. 4]. Now let $\{U(t)\}$ denote the semigroup of linear transformations in $C_b(X)$ defined by $U(t)f = f \circ T(t)$. $\{U(t)\}$ is clearly a semigroup of multiplicative homomorphisms.

2.1 Theorem. *If $b > 0$, then the collection $\{U(t) : 0 \le t \le b\}$ is β-equicontinuous.*

Proof. Let $V = W\{(K_n, a_n)\}$ (see Proposition 1.5), and define the transformation $\Phi : [0,\infty) \times X \to X$ by $\Phi(t, x) = T(t)x$. Let $K'_n = \Phi([0, b] \times K_n)$ for each n, and let $V' = W\{(K'_n, a_n)\}$. Then $U(t)V' \subset V$ for $0 \le t \le b$. ∎

2.2 Theorem. *$\{U(t)\}$ is β-strongly continuous; that is, for each $f \in E$, $U(\cdot)f$ is β-continuous from $[0,\infty)$ to E.*

Proof. We prove strong continuity from the right at 0. Two-sided strong continuity everywhere follow from this and Proposition 1.1. Let $f \in E$ and $V = W\{(K_n, a_n)\}$. Choose N so that $a_n > 2\|f\|$ for $n > N$, for $n = 1, \dots, N$, choose $\delta_n > 0$ so that $|f(T(t)x)-f(x)| < a_n$ for $0 \le t \le \delta_n$ and $x \in K_n$. Then $U(t)f - f \in V$ for $0 \le t \le \min\{\delta_1, \dots \delta_n\}$. ∎

2.3 Theorem. *If $\alpha > 0$ then $\{e^{-\alpha t}U(t) : t \ge 0\}$ is β-equicontinuous.*

Proof. Let $V = W\{(K_n, a_n)\}$, $a = \min a_n$. For each $r > 0$, choose $b_r > 0$ so that $re^{-\alpha b_r} < a$, and choose a β-neighborhood V_r of 0 such that $U(t)V_r \subset V$ for $0 \le t \le b_r$. Then $e^{-\alpha t}U(t)\,(V_r \cap B_r) \subset V$ for all $t \ge 0$. Therefore,

$$\bigcap_{t\ge 0} e^{\alpha t}U(t)^{-1}V \supset V_r \cap B_r$$

for all $r > 0$ and is this a β-neighborhood of 0 by Proposition 1.4. ∎

Now let A denote the infinitesimal generator of $U(t)$. Then A is a derivation by Proposition 1.2, and for each $\alpha > 0$, the collection

$$\left\{ \left[(\frac{n+\alpha}{n}) \left(I - (n+\alpha)^{-1} A \right) \right]^{-m} : m, n = 1, 2, \ldots \right\}$$

is a β-equicontinuous collection, by Theorem 1.1.

Section 3. The Induced Nonlinear Semigroup. Now let $\{U(t)\}$ be an arbitrary β-strongly continuous semigroup of linear β-continuous multiplicative homomorphisms in E. We want to prove that there is a strongly continuous semigroup $\{T(t)\}$ of continuous mappings in X such that $U(t)f = f \circ T(t)$ for all f and t. Incidentally, this will prove that $\{e^{-\alpha t} U(t)\}$ is β-equicontinuous for all $\alpha > 0$, by Theorem 2.3. If $t \geq 0$, and $x \in X$, then $m \in E^*$ defined by $m(f) = [U(t)f](x)$ is a nonzero multiplicative linear functional, and therefore, by Proposition 1.6, there is a point $y \in X$ such that $m(f) = f(y)$ for all $f \in E$. Since $C_b(X)$ separates points, this point y is unique. Thus, for each $t \geq 0$, we can define the transformation $T(t)$ from X into X by

$$f(T(t)x) = [U(t)f](x)$$

for $f \in E$ and $x \in X$. From the fact that $C_b(X)$ separates points, it also follows that $\{T(t)\}$ is a semigroup of transformations. We need to prove that each transformation $T(t)$ is continuous and that the semigroup $\{T(t)\}$ is strongly continuous. Let ρ denote the metric on X.

3.1 Lemma. *If $t \geq 0$, then $T(t)$ is continuous.*

Proof. Suppose not, and choose $x \in X$, $\{x_n\} \subset X$ converging to x, and $r > 0$ so that $\rho(T(t)x_n, T(t)x) \geq r$ for all n. Now choose $f \in E$ so that $f(x) = 1$ and $f(y) = 0$ for $\rho(y, x) \geq r$. Then $[U(t)f](x_n) = 0$ for all n, which contradicts the continuity of $U(t)f$. ∎

3.2 Lemma. *If $x \in X$, $\{t_n\} \subset (0, \infty)$, $\{x_n\} \subset X$, $x_n \to x$, and $t_n \to 0$, then $T(t_n)x_n \to x$.*

Proof. Suppose not, and choose subsequences $\{y_n\}$ of $\{x_n\}$ and $\{u_n\}$ of $\{t_n\}$ such that for some $r > 0$, $\rho(T(u_n)y_n, x) \geq r$ for all n. Now choose $f \in E$ such that $f(x) = 1$ and $f(y) = 0$ for $\rho(y, x) \geq r$. The sequence $\{U(u_n)f\}$ converges to f in the topology β, and therefore, it converges uniformly on the compact set $\{x, y_1, y_2, \ldots\}$. This yields a contradiction. ∎

3.3 Theorem. *The transformation $(t, x) \to T(t)x$ is separately continuous from $[0, \infty) \times X$ into X.*

Proof. Define Φ on $[0, \infty) \times X$ by $\Phi(t, x) = T(t)x$. We want to show that Φ is separately continuous. By Lemma 1.2, Φ is separately continuous at $(0, x)$ for each $x \in X$. Now, let $t > 0$ and $x \in X$. To show that Φ is separately continuous at (t, x) it is sufficient to prove that if $\{h_n\} \subset (0,t)$, $\{x_n\} \subset X$, $h_n \to 0$, and $x_n \to x$, then $\Phi(t+h_n, x_n) \to \Phi(t,x)$ and $\Phi(t-h_n, x_n) \to \Phi(t,x)$. The first conclusion follows from Lemmas 3.1 and 3.2, since $\Phi(t+h_n, x_n) = T(t)\,(T(h_n)x_n)$. To prove that $\Phi(t-h_n, x_n) \to \Phi(t,x)$, we use the fact that if $f \in E$, then $\{U(t-h_n)f\}$ converges to $U(t)f$ uniformly on the compact set $\{x, x_1, x_2, \ldots, T(h_1)x, T(h_2)x, \ldots\}$ and that

$$f\left(\Phi(t-h_n, x_n)\right) - f\left(\Phi(t,x)\right) = \left[U(t-h_n)f\right](x_n) - \left[U(t-h_n)f\right](T(h_n)x_n)\,. \quad \blacksquare$$

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ADDENDUM

November 17, 1994. We have stated that a strongly continuous semigroup T of continuous transformations on a Polish space X is necessarily *jointly continuous*; that is, the mapping $(t, x) \to T(t)x$ is jointly continuous from $[0, \infty) \times X$ into X. This fact was attributed to [CM, Thm. 4], but in fact [CM, Thm. 4] asserts joint continuity only on $(0, \infty) \times X$, and in fact, an example is given in [C] to show that joint continuity at $t = 0$ does not follow in this general a setting. Therefore, in Section 2, we must assume that the semigroup T is jointly continuous in order to establish the necessary properties of the induced linear semigroup U on $C_b(X)$. Fortunately, the argument for Theorem 3.3 proves that the semigroup T on X induced by be an arbitrary β-strongly continuous semigroup U of linear β-continuous multiplicative homomorphisms in $C_b(X)$ is jointly continuous on $[0, \infty) \times X$.

Thus, in the statement of Theorem 3.3, the word "separately" should be changed to "jointly", and our paper characterizes Lie generators of jointly continuous semigroups of maps on a Polish space.

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