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EXISTENCE RESULTS FOR NON-AUTONOMOUS ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. We study solutions to the boundary value problems

$$-\Delta u(x) = \lambda f(x,u); \quad x \in \Omega$$

$$u(x) + \alpha(x)\frac{\partial u(x)}{\partial n} = 0; \quad x \in \partial\Omega$$

where $\lambda > 0$, Ω is a bounded region in $\mathbb{R}^N$; $N \geq 1$ with smooth boundary $\partial\Omega$, $\alpha(x) \geq 0$, n is the outward unit normal, and f is a smooth function such that it has either sublinear or restricted linear growth in u at infinity, uniformly in x. We also consider f such that $f(x,u)u \leq 0$ uniformly in x, when $|u|$ is large. Without requiring any sign condition on $f(x,0)$, thus allowing for both positone as well as semipositone structure, we discuss the existence of at least three solutions for given $\lambda \in (\lambda_n, \lambda_{n+1})$ where λ_k is the k-th eigenvalue of $-\Delta$ subject to the above boundary conditions. In particular, one of the solutions we obtain has non-zero positive part, while another has non-zero negative part. We also discuss the existence of three solutions where one of them is positive, while another is negative, for λ near λ_1, and for λ large when f is sublinear. We use the method of sub-super solutions to establish our existence results. We further discuss non-existence results for λ small.

1. Introduction.

We first consider the boundary value problem

$$-\Delta u(x) = \lambda f(x,u); \quad x \in \Omega \tag{1.1}$$

$$u(x) + \alpha(x)\frac{\partial u(x)}{\partial n} = 0; \quad x \in \partial\Omega \tag{1.2}$$

where Δ is the Laplacian operator, Ω is a bounded region in $\mathbb{R}^N$; $N \geq 1$ with smooth boundary $\partial\Omega$, $\alpha(x) \geq 0$, n is the unit outward normal, and f is a smooth function such that

$$\lim_{u\to\infty} \frac{f(x,u)}{u} = \sigma \text{ uniformly in } x, \tag{1.3}$$

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$$\lim_{u\to-\infty}\frac{f(x,u)}{u}=\beta \text{ uniformly in } x, \tag{1.4}$$

and

$$\frac{\partial f}{\partial u}(x,u)\geq 0. \tag{1.5}$$

Let λ_k, ϕ_k be the eigenvalues and corresponding eigenfunctions of the boundary value problem

$$-\Delta\phi_k=\lambda_k\phi_k;\quad x\in\Omega \tag{1.6}$$

$$\phi_k(x)+\alpha(x)\frac{\partial\phi_k(x)}{\partial n}=0;\quad x\in\partial\Omega. \tag{1.7}$$

Let $I=[a,b]\subset(\lambda_n,\lambda_{n+1})$ and for $\lambda\in I$ consider the unique solution Z_λ of the boundary value problem

$$-\Delta Z_\lambda-\lambda Z_\lambda=-1;\quad x\in\Omega \tag{1.8}$$

$$Z_\lambda(x)+\alpha(x)\frac{\partial Z_\lambda(x)}{\partial n}=0.;\quad x\in\partial\Omega \tag{1.9}$$

Let $\mu_1=\inf_{x\in\bar{\Omega},\,\lambda\in I}Z_\lambda(x)$, $\mu_2=\sup_{x\in\bar{\Omega},\,\lambda\in I}Z_\lambda(x)$ and $\mu=\max\{|\mu_1|,\mu_2\}$. Note that $Z_\lambda^+\not\equiv 0$ (see Appendix 1). Also, $\exists\ \delta(\Omega)>0$ such that if $I\subset(\lambda_1,\lambda_1+\delta)$ then $Z_\lambda>0$; $x\in\Omega$ (anti-maximum principle: see Clémént and Peletier in [1]).

Now assume that

$$\exists\ m>0 \text{ such that } y+m>f(x,y)>y-m\ \ \forall\ y\in[-bm\mu,bm\mu] \tag{1.10}$$

and

$$\beta,\sigma<\frac{1}{b||w_\alpha||_\infty} \tag{1.11}$$

where w_α is the unique positive solution to

$$-\Delta w_\alpha=1;\quad x\in\Omega \tag{1.12}$$

$$w_\alpha(x)+\alpha(x)\frac{\partial w_\alpha(x)}{\partial n}=0;\quad x\in\partial\Omega. \tag{1.13}$$

Then we prove:

Theorem 1.1. *Let $I=[a,b]\subset(\lambda_n,\lambda_{n+1})$ and (1.3)–(1.5), (1.10)–(1.11) hold. Then these exists at least three solutions to (1.1)–(1.2) for $\lambda\in I$. One of these solutions is a non-negative or sign changing solution, while another is a non-positive or sign changing solution.*

Remark 1.1. Note that unlike in the literature of "jumping nonlinearities" (see [2]–[3]), our results do not require (β,σ) to include a part of the spectrum of $-\Delta$, nor require, a part of the spectrum of $-\Delta$ to lie between $f'(0)$ and $f'(\pm\infty)$ for autonomous nonlinearities as in [4] and the reference within.

Theorem 1.2. *Let $I = [a, b] \subset (\lambda_1, \lambda_1 + \delta)$ and (1.3)–(1.5), (1.10)–(1.11) hold. Then there exists at least three solutions to (1.1)–(1.2) for $\lambda \in I$, where one is a positive solution while another is a negative solution.*

Next we consider the particular case when $\sigma = \beta = 0$ (sublinear case). We further assume that there exists $f_1(u) < f(x, u); \forall\, x \in \bar{\Omega}, u \geq 0$ such that

$$f_1(r_1) = 0,\ f_1'(r_1) < 0,\ \int_0^{r_1} f(s)\, ds > 0 \tag{1.14}$$

for some $r_1 > 0$, and that there exists $f_2(u) > f(x, u); \forall\, x \in \bar{\Omega}, u \leq 0$ such that

$$f_2(r_2) = 0,\ f_2'(r_2) < 0,\ \int_{r_2}^{0} f(s)\, ds < 0 \tag{1.15}$$

for some $r_2 < 0$. Then we prove:

Theorem 1.3. *Assume (1.14)–(1.15) and let (1.3)–(1.5) hold with $\sigma = \beta = 0$. Then there exists at least three solutions to (1.1)–(1.2) for λ large, where one is a positive solution while another is a negative solution.*

Remark 1.2. We refer to [4] where a positive solution for $\lambda \in I = [a, b] \subset (\lambda_1, \lambda_1 + \delta)$ was discussed in the case of autonomous, sublinear ($\sigma = 0$), and semipositone ($f(0) < 0$) problems with Dirichlet boundary conditions. In [5] it was assumed that there exists $m > 0$ such that $f(y) \geq y - m\ \forall y \in [0, bm\mu]$ to obtain a positive solution for $\lambda \in I$. Further by constructing a function $f_1(u)$ satisfying (1.14) a positive solution for λ large was discussed. See also [6] where the authors study the existence of a positive solution via degree theory arguments. Hence they allow non-autonomous problems with Robin boundary condition, but consider only the sublinear case, and study the existence of positive solutions for λ large with the assumption $f(x, 0) \leq 0\ \forall x \in \bar{\Omega}$.

We also consider f such that

$$\exists\, r > 0 \text{ for which } f(x, r) \leq 0 \text{ while } f(x, -r) \geq 0 \text{ for every } x \in \bar{\Omega}. \tag{1.16}$$

Then we prove:

Theorem 1.4. *Let $I = [a, b] \subset (\lambda_n, \lambda_{n+1})$ and (1.10), (1.16) hold. Assume $r \geq bm\mu$. Then there exists at least three solutions to (1.1)–(1.2) for $\lambda \in I$. One of these solutions is a non-negative or sign changing solution, while another is a non-positive or sign changing solution.*

Theorem 1.5. *Let $I = [a, b] \subset (\lambda_1, \lambda_1 + \delta)$ and (1.10), (1.16) hold. Assume $r \geq bm\mu$. Then there exists at least three solutions to (1.1)–(1.2) for $\lambda \in I$. One of these is a positive solution, while another is a negative solution.*

Theorem 1.6. *Let (1.16) hold, and assume that (1.14) holds for all $0 \leq u \leq r$ with $r_1 \leq r$ while (1.15) holds for all $-r \leq u \leq 0$ with $r_2 \geq -r$. Then there exists at least three solutions to (1.1)–(1.2) for λ large, where one is a positive solution while another is a negative solution.*

Remark 1.3. Consider the boundary value problem

$$-\Delta u(x) = \lambda f(||x||, u); \quad x \in B_N$$

$$u(x)+\alpha\frac{\partial u(x)}{\partial n}=0;\quad x\in\partial B_N$$

where $\alpha\geq 0$ is a constant and B_N is the unit ball in $\mathbb{R}^N$. Then under the corresponding hypothesis on $f(||x||,u)$ instead of $f(x,u)$, one can generate all the solutions obtained in Theorems 1.1–1.6 to be radial. This follows from the fact that all the sub and super solutions we will use in the proofs of these theorems will turn out to be radial.

We next discuss non-existence results under the assumption that there exists $\gamma>0$ such that

$$\frac{\partial f}{\partial u}\leq\gamma. \tag{1.17}$$

We recall that λ_1 is the principal eigenvalue and $\phi_1>0$ is a corresponding eigenfunction of $-\Delta$ subject to boundary Robin conditions (1.2). Then we prove:

Theorem 1.7. *Assume* $\int_\Omega f(x,0)\phi_1(x)\leq 0$ *and* $\lambda<\lambda_1/\gamma$. *If* $u(\not\equiv 0)$ *is any solution of (1.1)–(1.2) then* $u^-\not\equiv 0$ *(i.e. there does not exist any solution* $u(\not\equiv 0)$ *which is non-negative).*

Theorem 1.8. *Assume* $\int_\Omega f(x,0)\phi_1(x)\geq 0$ *and* $\lambda<\lambda_1/\gamma$. *If* $u(\not\equiv 0)$ *is any solution of (1.1)–(1.2) then* $u^+\not\equiv 0$ *(i.e. there does not exist any solution* $u(\not\equiv 0)$ *which is non-positive).*

Theorem 1.9. *Assume* $f(x,0)\leq 0$ *and* $\lambda<\lambda_1/\gamma$. *If* u *is any solution of (1.1)–(1.2) then* $u\leq 0$.

Theorem 1.10. *Assume* $f(x,0)\geq 0$ *and* $\lambda<\lambda_1/\gamma$. *If* u *is any solution of (1.1)–(1.2) then* $u\geq 0$.

We give detailed proofs of our results in section 2. For literature on autonomous semipositone problems with Dirichlet boundary conditions see [7], while for the positone case see [8] and the references cited in [8].

Our existence results are based on sub-super solutions. Namely, a super solution is defined as a smooth function ϕ such that

$$-\Delta\phi\geq\lambda f(x,\phi);\quad x\in\Omega \tag{1.18}$$

$$\phi(x)+\alpha(x)\frac{\partial\phi(x)}{\partial n}\geq 0;\quad x\in\partial\Omega \tag{1.19}$$

and a subsolution is a smooth function ψ that satisfies (1.18)–(1.19) with the inequalities reversed. If $\psi\leq\phi$, then it follows that (1.1)–(1.2) has a solution u such that $\psi\leq u\leq\phi$ (see [9]–[10]).

Further if ψ_1 is a subsolution, ψ_2 is a strict subsolution, ϕ_1 is a strict supersolution and ϕ_2 is a supersolution such that $\psi_1\leq\psi_2\leq\phi_2$, $\psi_1\leq\phi_1\leq\phi_2$, $\psi_2(x_0)>\phi_1(x_0)$ for some $x_0\in\bar{\Omega}$, then (1.1)–(1.2) has at least three distinct solutions u_1, u_2, u_3 such that $u_1\leq u_2\leq u_3$. See [11] for this multiplicity result which was proved for Dirichlet boundary conditions. However, it is easy to see that this result holds for the Robin boundary conditions (1.2) as well.

2. Proofs of Theorems 1.1–1.10.

Proof of Theorem 1.1. Let $v_1(x) := bmZ_\lambda(x)$ and $u_2(x) = -v_1(x)$. Then $-\Delta v_1 = bm(-\Delta Z_\lambda) = bm(\lambda Z_\lambda - 1) \leq bm(\lambda Z_\lambda - \lambda/b)($ since $\lambda \leq b) = \lambda[bmZ_\lambda - m]$ $= \lambda[v_1 - m] < \lambda f(x, v_1)$ by (1.10), and $-\Delta u_2 = \Delta v_1 \geq \lambda[-v_1 + m] = \lambda[u_2 + m]$ $> \lambda f(x, u_2)$ again by (1.10). Thus v_1 is a strict subsolution while u_2 is a strict supersolution. Note that $v_1^+ \not\equiv 0$ and $u_2^- \not\equiv 0$. Now let $u_1(x) := Jw_\alpha(x)$ where $J > 0$ is large enough so that

$$\frac{1}{b||w_\alpha||_\infty} \geq \frac{f(x, J||w_\alpha||_\infty)}{J||w_\alpha||_\infty} \tag{2.1}$$

and

$$u_1 \geq v_1, \quad u_1 \geq u_2. \tag{2.2}$$

Here (2.1) is possible since $\sigma < \frac{1}{||w_\alpha||_\infty b}$, and (2.2) is possible by the Hopf's maximum principle. Then $-\Delta u_1 = J \geq \lambda f(x, J||w_\alpha||_\infty)$; $x \in \Omega$ (by (2.1) since $\lambda \leq b$) $\geq \lambda f(x, Jw_\alpha)$ (since $\frac{\partial f}{\partial u} \geq 0) = \lambda f(x, u_1)$. Next let $v_2(x) = -\tilde{J}w_\alpha(x)$ where $\tilde{J} > 0$ is large enough so that

$$\frac{1}{b||w_\alpha||_\infty} \geq \frac{f(x, -\tilde{J}||w_\alpha||_\infty)}{-\tilde{J}||w_\alpha||_\infty} \tag{2.3}$$

and

$$v_2 \leq v_1, \quad v_2 \leq u_2. \tag{2.4}$$

Here again (2.3) is possible since $\beta < \frac{1}{||w_\alpha||_\infty b}$ and (2.4) is possible by the Hopf's maximum principle. Then $-\Delta v_2 = -\tilde{J} \leq \lambda f(x, -\tilde{J}||w_\alpha||_\infty)$; $x \in \Omega$ (by (2.3) since $\lambda \leq b) \leq \lambda f(x, -\tilde{J}w_\alpha)$ (since $\frac{\partial f}{\partial u} \geq 0) = \lambda f(x, v_2)$. Thus u_1 is a supersolution while v_2 is a subsolution such that $v_2 \leq v_1 \leq u_1$ and $v_2 \leq u_2 \leq u_1$, where v_1 is a strict subsolution, u_2 is a strict supersolution with $v_2 \leq 0$, $v_1^+ \not\equiv 0$, $u_2^- \not\equiv 0$ and $u_1 \geq 0$. Hence the result. □

Proof of Theorem 1.2. If $\lambda \in (\lambda_1, \lambda_1 + \delta)$ then $Z_\lambda > 0$; $x \in \Omega$. Then $v_1(x) > 0$; $x \in \Omega$ while $u_2 < 0$; $x \in \Omega$ and hence the result follows by the proof in Theorem 1.1. □

Proof of Theorem 1.3. Consider the autonomous Dirichlet problem

$$-\Delta v = \lambda f_1(v); \quad x \in \Omega \tag{2.5}$$

$$v = 0; \quad x \in \partial\Omega. \tag{2.6}$$

Then by (1.14) it follows that there exists $\bar{\lambda}_1 > 0$ such that for $\lambda \geq \bar{\lambda}_1$ (2.5)-(2.6) has a positive solution v_λ (see [12]). Clearly since $\frac{\partial v_\lambda}{\partial n} \leq 0$; $x \in \partial\Omega$ and $f_1(S) < f(x, S)$; $\forall x \in \bar{\Omega}$, $S \geq 0$ v_λ is a strict subsolution to (1.1)–(1.2) for $\lambda \geq \bar{\lambda}_1$.

Next consider

$$-\Delta u = \lambda f_2(u); \quad x \in \Omega \tag{2.7}$$

$$u = 0; \quad x \in \partial\Omega. \tag{2.8}$$

Then setting $w = -u$ we see that w satisfies

$$-\Delta w = \lambda[-f_2(-w)]; \quad x \in \Omega \tag{2.9}$$

$$w = 0; \quad x \in \partial\Omega. \tag{2.10}$$

Let $g(w) = -f_2(-w)$. Then $g(-r_2) = 0$, $g'(-r_2) = f_2'(r_2) < 0$ and $\int_0^{-r_2} g(s)ds = \int_0^{-r_2} -f_2(-s)ds = -\int_{\gamma_2}^0 f_2(s)ds > 0$. Hence again by [12], there exists $\bar{\lambda}_2 > 0$ such that for $\lambda \geq \bar{\lambda}_2$, (2.9)–(2.10) has a positive solution w_λ. Equivalently, (2.7)–(2.8) has a negative solution $u_\lambda = -w_\lambda$ for $\lambda \geq \bar{\lambda}_2$. Also since $\frac{\partial u_\lambda}{\partial n} \geq 0$; $x \in \partial\Omega$ and $f_2(S) > f(x, S)$; $\forall x \in \bar{\Omega}$, $S \geq 0$, u_λ is a strict super solution to (1.1)–(1.2) for $\lambda \geq \bar{\lambda}_2$. Let $\bar{\lambda} = \max\{\bar{\lambda}_1, \bar{\lambda}_2\}$ and $\lambda > \bar{\lambda}$ be fixed. Consider $u_1(x) := Jw_\alpha(x)$ where $J > 0$ is large enough so that

$$\frac{1}{\lambda||w_\alpha||_\infty} \geq \frac{f(x, J||w_\alpha||_\infty)}{J||w_\alpha||_\infty} \tag{2.11}$$

and

$$u_1 \geq v_\lambda. \tag{2.12}$$

Here (2.11) is possible since $\sigma = 0$ and (2.12) is possible by the Hopf's maximum principle. Then $-\Delta u_1 = J \geq \lambda f(x, J||w_\alpha||_\infty) \geq \lambda f(x, Jw_\alpha)$ (since $\frac{\partial f}{\partial u} \geq 0$) $= \lambda f(x, u_1)$. Next consider $v_2(x) := -\tilde{J}w_\alpha(x)$ when $\tilde{J} > 0$ is large enough so that

$$\frac{1}{\lambda||w_\alpha||_\infty} \geq \frac{f(x, -\tilde{J}||w_\alpha||_\infty)}{-\tilde{J}||w_\alpha||_\infty} \tag{2.13}$$

and

$$v_2 \leq u_\lambda. \tag{2.14}$$

Here again (2.13) is possible since $\beta = 0$ and (2.14) is possible by the Hopf's maximum principle. Then $-\Delta v_2 = -\tilde{J} \leq \lambda f(x, -\tilde{J}||w_\alpha||_\infty) \leq \lambda f(x_1, -\tilde{J}w_\alpha)$ (since $\frac{\partial f}{\partial u} \geq 0$) $= \lambda f(x, v_2)$. Thus u_1 is a supersolution while v_2 is a subsolution such that $v_2 \leq u_\lambda \leq 0 \leq v_\lambda \leq u_1$, where u_λ is a strict supersolution and v_λ is a strict subsolution. Hence the result. $\square$

Proof of Theorem 1.4. Let $v_1(x)$ and $u_2(x)$ be the strict sub and strict super solutions to (1.1)–(1.2) as in the proof of Theorem 1.1. Then since $r \geq bm\mu$, both $v_1(x)$ and $u_2(x)$ satisfy $-r \leq v_1(x) \leq r$, $-r \leq u_2(x) \leq r$. But by (1.16) $u_1(x) = r$ and $v_2(x) \equiv -r$ are super and sub solutions respectively to (1.1)–(1.2). Hence the result. $\square$

Proof of Theorem 1.5. Let $v_1(x)$ and $u_2(x)$ be as in the proof of Theorem 1.4. But $\lambda \in (\lambda_1, \lambda_1 + \delta)$. Hence $v_1(x) \geq 0$ while $u_2(x) \leq 0$. The rest of the proof is identical to the proof of Theorem 1.4. $\square$

Proof of Theorem 1.6. Let u_λ and v_λ be respectively the strict super and strict subsolutions to (1.1)–(1.2) as in the proof of Theorem 1.3. But $-r_2 \leq u_\lambda \leq 0 \leq v_\lambda \leq r_1$ (see [12]) while $u_1(x) \equiv r$ are $v_2(x) = -r$ are super and subsolutions respectively to (1.1)–(1.2). Hence the result. $\square$

Proof of Theorem 1.7. Assume $u \geq 0$, $u \not\equiv 0$. Then

$$\begin{aligned} -\Delta u &= \lambda f(x,u) \\ &= \lambda[f(x,u) - f(x,0)] + \lambda f(x,0) \\ &= \lambda \frac{\partial f}{\partial u}(x,\eta)u + \lambda f(x,0)(\text{ where } 0 \leq \eta \leq u) \\ &\leq \lambda\gamma u + \lambda f(x,0)(\text{ by } (1.17)). \end{aligned}$$

Thus u satisfies

$$-\Delta u - \lambda\gamma u \leq \lambda f(x,0); \quad x \in \Omega \tag{2.15}$$

$$u(x) + \alpha(x)\frac{\partial u(x)}{\partial n} = 0; \quad x \in \partial\Omega. \tag{2.16}$$

Multiplying (2.15) by ϕ_1 and integrating we obtain

$$\int_\Omega -\Delta u \phi_1 \, dx - \int_\Omega \lambda\gamma u\phi_1 \, dx \leq \int_\Omega \lambda f(x,0)\phi_1 \, dx \leq 0.$$

Applying Green's second identity we obtain

$$\int_{\partial\Omega} \{-\phi_1 \frac{\partial u}{\partial n} + u\frac{\partial \phi_1}{\partial n}\} \, ds + \int_\Omega u\lambda_1\phi_1 \, dx - \int_\Omega \lambda\gamma u\phi_1 \, dx \leq 0.$$

But applying the boundary conditions we see that

$$\int_{\partial\Omega} \{-\phi_1 \frac{\partial u}{\partial n} + u\frac{\partial \phi_1}{\partial n}\} \, ds = 0.$$

Thus we obtain

$$\int_\Omega u\phi_1(\lambda_1 - \lambda\gamma) \, dx \leq 0.$$

But $u \geq 0$, $\phi_1 > 0$ in Ω and this is a contradiction if $\lambda < \lambda_1/\gamma$. Hence the result. □

Proof of Theorem 1.8. Assume $u \leq 0$, $u \not\equiv 0$. Proceeding as in the proof of Theorem 1.7 we obtain

$$\begin{aligned} -\Delta u &= \lambda \frac{\partial f}{\partial u}(x,z)u + \lambda f(x,0) \\ &\geq \lambda\gamma u + \lambda f(x,0)(\text{ by } (1.17)). \end{aligned}$$

Thus u satisfies

$$-\Delta u - \lambda\gamma u \geq \lambda f(x,0); \quad x \in \Omega \tag{2.17}$$

$$u(x) + \alpha(x)\frac{\partial u(x)}{\partial n} = 0; \quad x \in \partial\Omega. \tag{2.18}$$

Multiplying (2.17) by ϕ_1, using $\int_\Omega f(x,0)\phi_1 dx \geq 0$, and proceeding as in the proof of Theorem 1.7 we obtain

$$\int_\Omega u\phi_1(\lambda_1 - \lambda\gamma) \, dx \geq 0.$$

But $u \leq 0$ while $\phi_1 > 0$ for $x \in \Omega$, thus this is a contradiction if $\lambda < \lambda_1/\gamma$. Hence the result. $\Box$

Proof of Theorem 1.9. Suppose $u > 0$ somewhere in $\bar{\Omega}$. Then $\exists$ some $\Omega_1 \subseteq \Omega$ such that $u > 0$ in Ω, and either

$$u(x) = 0 \text{ or } u(x) + \alpha(x)\frac{\partial u(x)}{\partial n} = 0; \quad x \in \partial\Omega. \tag{2.19}$$

Let $\tilde{\lambda}_1$ be the principal eigenvalue and $\tilde{\phi}_1 > 0$ be a corresponding eigenfunction of $-\Delta$ on the region Ω_1 subject to Robin conditions (1.2) on $\partial\Omega_1$. Note that $\tilde{\lambda}_1 \geq \lambda_1$. Now since $f(x,0) \leq 0$ we have $\int_\Omega f(x,0)\tilde{\phi}_1\,dx \leq 0$. Thus following the steps in the proof of Theorem 1.7 we obtain

$$\int_{\Omega_1} -\Delta u\tilde{\phi}_1\,dx - \int_{\Omega_1} \lambda\gamma u\tilde{\phi}_1\,dx \leq 0$$

and then

$$\int_{\partial\Omega_1} [-\tilde{\phi}_1\frac{\partial u}{\partial n} + u\frac{\partial\tilde{\phi}_1}{\partial n}]\,ds + \int_{\Omega_1} [\tilde{\lambda}_1 - \lambda\gamma]u\tilde{\phi}_1\,dx \leq 0.$$

But if $u = 0$ on $\partial\Omega_1$, then $\frac{\partial u}{\partial n} \leq 0$ on $\partial\Omega_1$, while if $u + \alpha\frac{\partial u}{\partial n} = 0$ on $\partial\Omega_1$, since $\tilde{\phi}_1 + \alpha\frac{\partial\tilde{\phi}_1}{\partial n} = 0$, we have $-\tilde{\phi}_1\frac{\partial u}{\partial n} + u\frac{\partial\tilde{\phi}_1}{\partial n} = 0$ on $\partial\Omega_1$. Thus in any case (see (2.19)) $-\tilde{\phi}_1\frac{\partial u}{\partial n} + u\frac{\partial\tilde{\phi}_1}{\partial n} \geq 0$. Hence

$$\int_{\Omega_1} [\tilde{\lambda}_1 - \lambda\gamma]u\tilde{\phi}_1\,dx \leq 0$$

which is a contradiction since $u > 0$, $\tilde{\phi}_1 > 0$ for $x \in \Omega_1$ while $\lambda < \lambda_1/\gamma \leq \tilde{\lambda}_1/\gamma$. Hence the result. $\Box$

Proof of Theorem 1.10. Suppose $u < 0$ somewhere in $\bar{\Omega}$. Then $\exists$ some $\Omega_1 \subseteq \Omega$ such that $u < 0$ in Ω_1 and either

$$u(x) = 0 \text{ or } u(x) + \alpha(x)\frac{\partial u(x)}{\partial n} = 0; \quad x \in \partial\Omega_1 \tag{2.20}$$

Let $\tilde{\lambda}_1$ and $\tilde{\phi}_1 > 0$ be as in the proof of Theorem 1.9. Now $f(x,0) \geq 0$ and hence $\int_{\Omega_1} f(x,0)\tilde{\phi}_1 dx \geq 0$. Thus following the steps in the proof of Theorem 1.8 we obtain

$$\int_{\Omega_1} -\Delta u\tilde{\phi}_1\,dx - \int_{\Omega_1} \lambda\gamma u\tilde{\phi}_1\,dx \geq 0,$$

and then

$$\int_{\partial\Omega} [-\tilde{\phi}_1\frac{\partial u}{\partial n} + u\frac{\partial\tilde{\phi}_1}{\partial n}]\,ds + \int_{\Omega_1} [\tilde{\lambda}_1 - \lambda\gamma]u\tilde{\phi}_1\,dx \geq 0.$$

But if $u = 0$ on $\partial\Omega_1$, then $\frac{\partial u}{\partial n} \geq 0$ on $\partial\Omega_1$, while if $u + \alpha\frac{\partial u}{\partial n} = 0$ on $\partial\Omega_1$, since $\tilde{\phi}_1 + \alpha\frac{\partial\tilde{\phi}_1}{\partial n} = 0$ on $\partial\Omega$, we have $-\tilde{\phi}_1\frac{\partial u}{\partial n} + u\frac{\partial\tilde{\phi}_1}{\partial n} = 0$ on $\partial\Omega_1$. Thus in any case (see (2.20)) $-\tilde{\phi}_1\frac{\partial u}{\partial n} + u\frac{\partial\tilde{\phi}_1}{\partial n} \leq 0$. Hence

$$\int_{\Omega_1} [\tilde{\lambda}_1 - \lambda\gamma]u\tilde{\phi}_1\,dx \geq 0$$

which is a contradiction since $u < 0$, $\tilde{\phi}_1 > 0$ for $x \in \Omega$, while $\lambda < \lambda_1/\gamma \leq \tilde{\lambda}_1/\gamma$. Hence the result. $\Box$

Appendix 1. Let $\lambda \in (\lambda_n, \lambda_{n+1})$ and consider the unique solution to the boundary value problem

$$-\Delta Z_\lambda - \lambda Z_\lambda = -1; \quad x \in \Omega$$

$$Z_\lambda(x) + \alpha(x)\frac{\partial Z_\lambda(x)}{\partial n} = 0; \quad x \in \partial\Omega.$$

Let $\phi_1 > 0$ satisfy (1.6)–(1.7) for $k = 1$. Then

$$\int_\Omega -\Delta Z_\lambda \phi_1 \, dx - \int_\Omega \lambda Z_\lambda \phi_1 \, dx = \int_\Omega -\phi_1 \, dx$$

which implies

$$\int_{\partial\Omega} \{-\phi_1 \frac{\partial Z_\lambda}{\partial n} + Z_\lambda \frac{\partial \phi_1}{\partial n}\} \, ds + \int_\Omega Z_\lambda \lambda_1 \phi_1 \, dx - \int_\Omega \lambda Z_\lambda \phi_1 dx = \int_\Omega -\phi_1 \, dx \, .$$

But

$$\int_{\partial\Omega} \{-\phi_1 \frac{\partial Z_\lambda}{\partial n} + Z_\lambda \frac{\partial \phi_1}{\partial n}\} \, ds = \int_{\partial\Omega} \{\alpha \frac{\partial \phi_1}{\partial n} \frac{\partial Z_\lambda}{\partial n} - \alpha \frac{\partial Z_\lambda}{\partial n} \frac{\partial \phi_1}{\partial n}\} \, ds = 0 \, .$$

Thus

$$\int_\Omega (\lambda_1 - \lambda) Z_\lambda \phi_1 \, dx = \int_\Omega -\phi_1 \, dx.$$

But $\lambda > \lambda_1$ and $\phi_1 > 0$ for $x \in \Omega$. Hence $Z_\lambda^+ \not\equiv 0$.

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