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Multiple Solutions For Semilinear Elliptic Boundary Value Problems At Resonance *

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Abstract

In recent years several nonlinear techniques have been very successful in proving the existence of weak solutions for semilinear elliptic boundary value problems at resonance. One technique involves a variational approach where solutions are characterized as saddle points for a related functional. This argument requires that the Palais-Smale condition and some coercivity conditions are satisfied so that the saddle point theorem of Ambrossetti and Rabinowitz can be applied. A second technique has been to apply the topological ideas of Leray-Schauder degree. This argument typically creates a homotopy with a uniquely solvable linear problem at one end and the nonlinear problem at the other, and then an a priori bound is established so that the homotopy invariance of Leray-Schauder degree can be applied. In this paper we prove that both techniques are applicable in a wide variety of cases, and that having both techniques at our disposal gives more detailed information about solution sets, which leads to improved existence results such as the existence of multiple solutions.

1 Introduction

The fundamental question that we address in this paper is: Under what conditions are both topological and variational existence theorems applicable to the problem

$$\begin{gathered}\Delta u+\lambda_k u+g(u)+h=0\,,x\in\Omega\,,\\ u|_{\partial\Omega}=0\,,\end{gathered}\tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, λ_k is an eigenvalue of $-\Delta$, $g:\mathbb{R}\to\mathbb{R}$ is a continuous function, and $h\in L^2(\Omega)$? Of particular interest are double resonance problems where the term $(\lambda_k+g(u)/u)$ ranges between consecutive

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eigenvalues of $-\Delta$ for large $|u|$. One-sided resonance is similar except that the given term is strictly bounded away from one of the eigenvalues.

As an application we prove the existence of multiple nontrivial solutions for a class of problems where $h = 0$ and g is a C^1 function such that $g(0) = 0$ and $g'(0)$ is known. Problems such as this often arise in applications such as Population Biology, where u represents a steady-state population density, $(\lambda_k + g(u)/u)$ represents a population dependent growth rate, and $(\lambda_k + g'(0))$ represents a growth rate in the absence of certain environmental restrictions such as crowding.

Our results improve upon previous work in the following ways: Theorem 1 improves upon the basic existence result in [18] by including a saddle point characterization of at least one solution. This characterization is an important part of the subsequent multiplicity result. This improvement comes at the price of a somewhat less general solvability condition than in [18]. Theorems 2 and 3 improve upon the multiplicity results of [1] and [21]. In [1] it is assumed that g is bounded, and the argument relies on a standard Landesman-Lazer condition. In [21] the results of [1] are extended by allowing g to have linear growth, and by using a generalized Landesman-Lazer condition. However, the variational argument in [21] assumes one-sided resonance at the principal eigenvalue. In this paper g is allowed linear growth and we provide a variational argument that is valid for double resonance problems between arbitrary consecutive eigenvalues. Moreover, we rely on a Landesman-Lazer type condition that is more general than that in [21].

For purposes of clarity we consider only boundary value problems for the Laplace operator and with Dirichlet boundary conditions. However, it will be clear that our variational arguments apply to boundary value problems with more general elliptic operators, more general boundary conditions, and with nonlinear terms of the form $g(x, u)$ where g is Caratheodory.

The discussion begins in Section 2, where we state Theorem 1 along with some clarifying comments. Sections 3 and 4 provide the elements of a variational existence proof using an Ambrossetti-Rabinowitz type saddle point argument. Theorem 1 then follows as a consequence of these arguments combined with the basic degree-theoretic result of [18], and so we get a theorem that provides a better description of the solution set than either the variational or topological arguments do separately. In Section 5 we use the combined topological and variational characteristics of the solution set to prove the existence of multiple solutions for a certain class of problems. The proofs in Section 5 are similar to those in [1] and [21].

It is well known that the Landesman-Lazer condition implies coerciveness statements and the Palais-Smale condition in a natural way, see [1] for details. One consequence of the work in this paper is that generalized Landesman-Lazer conditions imply a similar structure. However, there are some interesting differences. For example, although the functional related to problem (1) will be coercive over one subspace and anticoercive over its orthogonal complement, its growth in either direction might be relatively slow. This possibility of slower growth makes it more difficult to establish a compactness condition. In fact, we will not prove the usual Palais-Smale condition in Section 4, but rather a less restrictive version often credited to G. Cerami. For a detailed discussion of this compactness condition and for additional references see [4].

Before continuing it is helpful to establish the notation that will be used throughout the paper.

$H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $L^2(\Omega)$ with respect to the norm $\|u\| = \left(\int_\Omega |\nabla u|^2\right)^{1/2}$.

$\lambda_j := j^{th}$ distinct eigenvalue of $-\Delta$, where $0 < \lambda_1 < \lambda_2 < \cdots$.

$V^j := \mathrm{Ker}(\Delta + \lambda_j)$, $V^- = \bigoplus_{j<k} V^j$, and $V^+ := \bigoplus_{j>k+1} V^j$.

Given $u \in H_0^1(\Omega)$, then u^-, u^k, u^{k+1}, and u^+ are its orthogonal components in V^-, V^k, V^{k+1}, and V^+, respectively.

$G(x,u) := \int_0^u (g(s) + h(x))ds$.

$\tilde{g}(u) := (\lambda_{k+1} - \lambda_k)u - g(u)$.

$\tilde{G}(x,u) := \int_0^u (\tilde{g}(s) - h)ds = \left(\frac{\lambda_{k+1}-\lambda_k}{2}\right) u^2 - G(x,u)$.

$f(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\lambda_k \int_\Omega u^2 - \int_\Omega G(x,u)$, for $u \in H_0^1(\Omega)$.

Notice that under reasonable conditions on g, f is a functional on $H_0^1(\Omega)$ that is twice Frechet-differentiable with

$$\begin{aligned} f'(u)v &= \int_\Omega \nabla u \cdot \nabla v - \lambda_k \int_\Omega uv - \int_\Omega (g(u)+h)v\,, \text{ and} \\ f''(u)(v,w) &= \int_\Omega \nabla v \cdot \nabla w - \lambda_k \int_\Omega vw - \int_\Omega g'(u)vw\,. \end{aligned}$$

It is a standard fact that solutions of (1) correspond to critical points of f, and that f' has the form Identity-Compact, see [17], so that Leray-Schauder techniques are applicable. We will use the notation $\deg_{LS}(f', U, 0)$ for the Leray-Schauder degree of f' with respect to the set U and the value 0.

2 A General Existence Theorem For Double Resonance Problems

Theorem 1 *If g satisfies*

(g1): $\quad 0 \le \liminf_{|s|\to\infty} \dfrac{g(s)}{s} \le \limsup_{|s|\to\infty} \dfrac{g(s)}{s} \le \lambda_{k+1} - \lambda_k,$

(g2): If $\|u_n\| \to \infty$ *such that* $\|u_n^k\|/\|u_n\| \to 1$*, then* $\exists N, \delta > 0$ *such that* $\langle g(u_n) + h, u_n^k \rangle_{L^2} \geq \delta$ *for all* $n > N$*, and*

(g3): If $\|u_n\| \to \infty$ *such that* $\|u_n^{k+1}\|/\|u_n\| \to 1$*, then* $\exists N, \delta > 0$ *such that* $\langle \tilde{g}(u_n) - h, u_n^{k+1} \rangle_{L^2} \geq \delta$ *for all* $n > N$*,*

then problem (1) has a nonempty solution set, S, and there is an $R > 0$ *such that* $S \subset B_R(0)$ *and* $\deg_{LS}(f', B_R(0), 0) = (-1)^m$*, where m represents the dimension of* $V^- \bigoplus V^k$*. Moreover, we have the following saddle point result: There are real constants* $\beta > \alpha$ *and a bounded neighborhood D of* 0 *in* $V^- \bigoplus V^k$ *such that* $f|_{\partial D} \leq \alpha$ *and* $f|_{V^{k+1} \bigoplus V^+} \geq \beta$*, and there is a critical value* $c \geq \beta$ *such that*

$$c = \inf_{h \in \Gamma} \max_{u \in \overline{D}} f(h(u)) \,,$$

where

$$\Gamma = \{h \in C(\overline{D}, H_0^1(\Omega)) | h = id \text{ on } \partial D\}.$$

The proof of the the degree computation in this theorem is given in [18] by establishing an a priori bound for the solution set of the family of equations

$$\begin{gathered}\Delta u + \tfrac{1}{2}(\lambda_k + \lambda_{k+1})u + t\left(g(u) + h - \tfrac{1}{2}(\lambda_{k+1} - \lambda_k)u\right) = 0,\ x \in \Omega,\ t \in [0,1] \\ u|_{\partial\Omega} = 0\,.\end{gathered} \tag{2}$$

The homotopy invariance of Leray-Schauder degree is then applicable, and it is straight forward to compute the degree for the linear problem at $t = 0$.

The saddle point characterization will follow from the arguments in the next two sections below. For later reference we remark that if Γ is simply a collection of curves with fixed endpoints, then we refer to the corresponding solution as a solution of *mountain pass type*. This would occur, for example, if we had $\lambda_k = \lambda_1$ so that $V^- \equiv 0$ and V^1 is one dimensional.

The existence of at least one solution is true in a much more general setting. For example, in [18] a theorem of this type is proved for a class of boundary value problems over unbounded domains. Also, in [19] it is shown that δ can be replaced by 0 in $(g2)$ and $(g3)$, although the boundedness of the solution set is lost, and so no compactness condition of Palais-Smale type is possible.

It has also been shown that many well-known solvability conditions are special cases of $(g2)$ and $(g3)$. For example the standard Landesman-Lazer condition (see [15]), the solvability conditions used by Fucik, Krbec, and Hess (see [12] and [9]), the double resonance conditions used by Berestycki and DeFigueredo (see [5]), and some cases of the density conditions at infinity (see [7]). For comparisons of solvability conditions see [20], [18], and [19]. A notable exception is the sign condition used in [13] and many other recent papers. The sign condition is a special case of the more general theorem where we replace δ by 0 in $(g2)$ and $(g3)$, see [19], but will not be included in the results of this paper. A second interesting exception is the well-known solvability condition of Ahmad, Lazer, and Paul, see [2], for problems with bounded nonlinear terms. It can be shown that the ALP condition is not a consequence of the most general form of *(g2)* and *(g3)* and vice versa.

3 Coercivity

In this section we prove

Lemma 1 *Assume that g is a continuous function satisfying (g1)-(g3). Then f is coercive on $V^{k+1} \bigoplus V^+$ and is anticoercive on $V^- \bigoplus V^k$.*

The proof of this lemma requires several technical preliminary results. We will concentrate on proving that f is coercive on $V^{k+1} \bigoplus V^+$ and remark that the anticoercive statement follows by a similar argument. In fact the second argument can be simplified by using the fact that $V^- \bigoplus V^k$ is finite dimensional.

The interested reader can verify that $f|_{V^+}$ and $f|_{V^{k+1}}$ are coercive as a direct consequence of conditions *(g1)* and *(g3)*, respectively. The technical difficulty arises when we study the functional over the combined space.

Claim 1 *Given any $r > 0$, f is coercive on the solid cone*

$$C_r := \{u \in V^{k+1} \bigoplus V^+ : \|u^+\| \geq r\|u\|\}.$$

Proof: Rewrite f as

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda_{k+1}}{2}\int_\Omega u^2 + \int_\Omega \tilde{G}(x,u)\,, \tag{3}$$

and f' as

$$f'(u)v = \int_\Omega \nabla u \cdot \nabla v - \lambda_{k+1}\int_\Omega uv + \int_\Omega (\tilde{g}(u) - h)v\,.$$

Applying the right hand side of the inequality *(g1)*, we can say that given any $\epsilon > 0$ there is a constant $\rho > 0$ such that $\tilde{g}(s)s \geq -\epsilon s^2$ for every $|s| > \rho$. Thus there is a constant, a, depending on g and ρ, such that

$$\begin{aligned} \int_\Omega \tilde{g}(u)u &\geq -\epsilon\int_\Omega u^2 - a\,, \\ &\geq -\frac{1}{\lambda_1}\epsilon\|u\|^2 - a\,, \end{aligned}$$

where we have applied Poincare's Inequality. Further,

$$\begin{aligned}\int_\Omega |\nabla u|^2 - \lambda_{k+1}\int_\Omega u^2 &= \int_\Omega |\nabla u^+|^2 - \lambda_{k+1}\int_\Omega (u^+)^2 \text{ for } u \in V^{k+1}\bigoplus V^+ \\ &\geq (1-\frac{\lambda_{k+1}}{\lambda_{k+2}})\|u^+\|^2 \text{ for } u \in V^{k+1}\bigoplus V^+ \\ &\geq r^2(1-\frac{\lambda_{k+1}}{\lambda_{k+2}})\|u\|^2 \text{ for } u \in C_r\,.\end{aligned}$$

The previous inequalities imply

$$f'(tu)u \geq t(b-\frac{\epsilon}{\lambda_1})\|u\|^2 - t\|h\|\,\|u\| - a,\ t \geq 0,$$

where $b = r^2(1-\frac{\lambda_{k+1}}{\lambda_{k+2}}) > 0$. Therefore

$$f(u) = f(0) + \int_0^1 f'(tu)u\,dt \geq f(0) + \frac{1}{2}\left[(b-\frac{\epsilon}{\lambda_1})\|u\|^2 - \|h\|\,\|u\|\right] - a\,,$$

and so an appropriate choice of ϵ finishes the proof.

Claim 2 *Given any $r > 0$ f achieves a minimum on the cylinder*

$$K_r := \{u \in V^{k+1}\bigoplus V^+ : \|u^{k+1}\| = r\}\,.$$

Proof: Applying Claim 1, it is easy to see that f is coercive when restricted to K_r . Thus if $\{u_n\}$ is a sequence in $H_0^1(\Omega)$ such that $\{f(u_n)\}$ is bounded, then $\{u_n\}$ must be bounded as well. If we also know that $f'(u_n) \to 0$, then, using the fact that f' is of the form Identity-Compact, we can show that $\{u_n\}$ must have a converging subsequence. In other words $f|_{K_r}$ satisfies the Palais-Smale condition. It is then an easy exercise to show that $f|_{K_r}$ achieves a minimum.

Proof of Lemma 1: We prove that f is coercive by examining its behavior on a sequence of "minimizers," as described in Claim 2. Let $\{u_n\} \subset V^{k+1}\bigoplus V^+$ such that $\|u_n\| \to \infty$, and such that

$$f(u_n) \leq f(u) \text{ for all } u \in K_{\|u_n^{k+1}\|}\,. \tag{4}$$

We will show that $f(u_n) \to \infty$ for some subsequence of $\{u_n\}$. It will follow that no sequence of minimizers is bounded above, and hence that

$$\lim_{\|u\|\to\infty} f(u) = \infty \text{ for } u \in V^{k+1}\bigoplus V^+\,.$$

If $\{u_n\}$ is contained in any solid cone, C_r, as in Claim 1, then $f(u_n) \to \infty$, so we need only consider the case where $\|u_n^{k+1}\|/\|u_n\| \to 1$, which brings condition *(g3)* into play. Since this implies $\|u_n^{k+1}\| \to \infty$, we may also assume that $\|u_{n+1}^{k+1}\| > \|u_n^{k+1}\|$ for all n. For any n it is clear that $(\|u_n^{k+1}\|/\|u_{n+1}^{k+1}\|)u_{n+1}^{k+1} + u_{n+1}^{+} \in K_{\|u_n^{k+1}\|}$, so property (4) implies

$$f(u_{n+1}) - f(u_n) \geq f(u_{n+1}) - f\left(\frac{\|u_n^{k+1}\|}{\|u_{n+1}^{k+1}\|}u_{n+1}^{k+1} + u_{n+1}^{+}\right) .$$

By substituting into expression (3), we get

$$f(u_{n+1}) - f(u_n) \geq \int_\Omega \left[\tilde{G}(x, u_{n+1}) - \tilde{G}\left(x, \frac{\|u_n^{k+1}\|}{\|u_{n+1}^{k+1}\|}u_{n+1}^{k+1} + u_{n+1}^{+}\right)\right] .$$

Let

$$\gamma_n(t) = t + (1-t)\frac{\|u_n^{k+1}\|}{\|u_{n+1}^{k+1}\|} ,$$

and let

$$v_n(t) = \gamma_n(t)u_{n+1}^{k+1} + u_{n+1}^{+},$$

so $v_n(0) = (\|u_n^{k+1}\|/\|u_{n+1}^{k+1}\|)u_{n+1}^{k+1} + u_{n+1}^{+}$ and $v_n(1) = u_{n+1}$. Next let

$$F_n(t) = \int_\Omega \tilde{G}(x, v_n(t)) ,$$

so

$$\begin{aligned} f(u_{n+1}) - f(u_n) &\geq F_n(1) - F_n(0) \\ &= \int_0^1 F_n'(t)\,dt \\ &= \int_0^1 \left\langle \tilde{g}(v_n(t)), \gamma_n'(t)u_{n+1}^{k+1}\right\rangle_{L^2}\,dt \\ &= \int_0^1 \left\langle \tilde{g}(v_n(t)), \gamma_n(t)u_{n+1}^{k+1}\right\rangle_{L^2}\left(\frac{\gamma_n'(t)}{\gamma_n(t)}\right)dt\,. \end{aligned}$$

We claim that, without loss of generality, there is a $\delta > 0$ such that

$$\left\langle \tilde{g}(v_n(t)), \gamma_n(t)u_{n+1}^{k+1}\right\rangle_{L^2} \geq \delta$$

for every n and every $t \in [0,1]$. If not there would be a subsequence $\{u_n\}$ and a corresponding sequence $\{t_n\} \subset [0,1]$, such that

$$\limsup_{n\to\infty} \left\langle \tilde{g}(v_n(t_n)), \gamma_n(t_n)u_{n+1}^{k+1}\right\rangle_{L^2} \leq 0 .$$

Observe that $\|v_n(t_n)^{+}\| = \|u_{n+1}^{+}\|$, and $\|v_n(t_n)^{k+1}\| = \gamma_n(t_n)\|u_{n+1}^{k+1}\|$, so

$$\|u_n^{k+1}\| \leq \|v_n(t_n)^{k+1}\| \leq \|u_{n+1}^{k+1}\|\,.$$

It follows that $\lim_{n\to\infty} \|v_n(t_n)^{k+1}\|/\|v_n(t_n)\| = 1$, but this contradicts *(g3)*.

Applying this result we have

$$\begin{aligned} f(u_{n+1}) - f(u_n) &\geq \delta \int_0^1 \frac{\gamma_n'(t)}{\gamma_n(t)}\, dt \ \ \forall\, n \\ &= \delta\left[\ln(\gamma_n(1)) - \ln(\gamma_n(0))\right] \\ &= \delta\left[\ln(1) - \ln(\frac{\|u_n^{k+1}\|}{\|u_{n+1}^{k+1}\|})\right] \\ &= \delta\left[\ln(\|u_{n+1}^{k+1}\|) - \ln(\|u_n^{k+1}\|)\right] . \end{aligned}$$

Hence

$$\begin{aligned} f(u_{n+1}) &= f(u_1) + \sum_{j=1}^{n}\left[f(u_{j+1}) - f(u_j)\right] \\ &\geq f(u_1) + \delta \sum_{j=1}^{n}\left[\ln(\|u_{j+1}^{k+1}\|) - \ln(\|u_j^{k+1}\|)\right] \\ &= f(u_1) - \delta \ln(\|u_1^{k+1}\|) + \delta \ln(\|u_{n+1}^{k+1}\|) , \end{aligned}$$

therefore $f(u_n) \to \infty$, and coerciveness is proved.

In the next section we will give more thought to the fact that f might grow only as fast as a logarithm.

4 A Compactness Condition

In this section we show that f satisfies a less restrictive form of Palais-Smale condition that is still sufficient to imply mountain pass and saddle point theorems.

It is a simple task to prove the Palais-Smale condition as a consequence of the Landesman-Lazer condition, or as a consequence of the following generalized conditions:

$(g2)'$: *If* $\|u_n\| \to \infty$ *such that* $\|u_n^k\|/\|u_n\| \to 1$*, then* $\exists N, \delta > 0$ *such that* $\left\langle g(u_n) + h, u_n^k/\|u_n^k\|\right\rangle_{L^2} \geq \delta$ *for all* $n > N$*, and*

$(g3)'$: *If* $\|u_n\| \to \infty$ *such that* $\|u_n^{k+1}\|/\|u_n\| \to 1$*, then* $\exists N, \delta > 0$ *such that* $\left\langle \tilde{g}(u_n) - h, u_n^{k+1}/\|u_n^{k+1}\|\right\rangle_{L^2} \geq \delta$ *for all* $n > N$*,*

We remark that $(g2)'$ and $(g3)'$ were satisfied in both [1] and [21], and we refer to these papers for the simple Palais-Smale argument.

A similarly easy argument based upon the conditions *(g2)* and *(g3)* does not appear to be available. In order to understand why this new situation is more delicate it is worthwhile considering a simple but instructive example. Notice that in the previous section the estimates revealed that, over the eigenspaces V^k and V^{k+1}, the functional might have only logarithmic growth. Thus we consider the following situation: Let $z : \mathbb{R}^2 \to \mathbb{R} : z(x,y) = \log(1+x^2) - \log(1+y^2)$. This function is coercive over the x-axis, anticoercive over the y-axis and satisfies conditions *(g2)* and *(g3)*, since, for example, $\langle \nabla z, (x,0)\rangle = 2x^2/(1+x^2) \to 2$ as $x \to \infty$. However, z does not satisfy the Palais-Smale condition, because $z = 0$ on the level set $|x| = |y|$, but $\nabla z = (2x/(1+x^2), -2y/(1+y^2)) \to (0,0)$ as $x, y \to \infty$.

It turns out that the functional f, as well as the example above, satisfies the following compactness condition, which is a special case of the condition used in [4].

Lemma 2 *If $\{u_n\} \subset H_0^1(\Omega)$ such that $(1 + \|u_n\|)\|f'(u_n)\| \to 0$, then $\{u_n\}$ contains a converging subsequence.*

Proof: Suppose $\{u_n\}$ is such a sequence. Once again, f' is of the form Identity-Compact, so it suffices to show that $\{u_n\}$ is bounded. We argue by contradiction, so suppose that $\|u_n\| \to \infty$. Since bounded sets in $H_0^1(\Omega)$ are weakly compact, and since $H_0^1(\Omega)$ embeds compactly in $L^2(\Omega)$, without loss of generality we may assume that there is a unit vector $w \in H_0^1(\Omega)$ and a bounded measurable $\gamma(x)$ such that

$$\begin{aligned} \frac{u_n}{\|u_n\|} &\rightharpoonup w \text{ in } H_0^1(\Omega)\,, \\ \frac{u_n}{\|u_n\|} &\to w \text{ in } L_2(\Omega)\,, \text{ and} \\ \frac{g(u_n)}{\|u_n\|} &\rightharpoonup \gamma w \text{ in } L_2(\Omega)\,, \end{aligned}$$

where $0 \le \gamma \le \lambda_{k+1} - \lambda_k$. The last statement follows directly from writing $g(u_n)/\|u_n\| = (g(u_n)/u_n)(u_n/\|u_n\|)$ for $u_n(x) \ne 0$ and then applying condition *(g1)*.

Notice that for any $v \in H_0^1(\Omega)$

$$\frac{f'(u_n)v}{\|u_n\|} = \int_\Omega \left(\frac{\nabla u_n}{\|u_n\|}\right) \cdot \nabla v - \lambda_k \int_\Omega \left(\frac{u_n}{\|u_n\|}\right) v - \int_\Omega \left(\frac{g(u_n)}{\|u_n\|}\right) v - \frac{1}{\|u_n\|} \int_\Omega hv\,.$$

Allowing $n \to \infty$ we get

$$0 = \int_\Omega \nabla w \cdot \nabla v - \lambda_k \int_\Omega wv - \int_\Omega \gamma wv \quad \forall v \in H_0^1(\Omega)\,.$$

Thus w is a nontrivial weak solution of the boundary value problem

$$\Delta w + (\lambda_k + \gamma)w = 0, \quad x \in \Omega\,,$$

$$u|_{\partial\Omega} = 0\,.$$

Since $\lambda_k \leq \lambda_k + \gamma \leq \lambda_{k+1}$, a standard argument involving the maximum principle and the unique continuation property implies that either $\lambda_k + \gamma \equiv \lambda_k$ a.e. and $w \equiv w^k$, or $\lambda_k + \gamma \equiv \lambda_{k+1}$ a.e. and $w \equiv w^{k+1}$. (The details of this argument are available in many papers, see [18] or [14], for example.) Thus we have that either $\|u_n^k\|/\|u_n\| \to 1$ or $\|u_n^{k+1}\|/\|u_n\| \to 1$, so either condition *(g2)* or *(g3)* is applicable. Suppose $w \equiv w^k$ (The argument for $w \equiv w^{k+1}$ is similar). Then by *(g2)*, and by passing to a subsequence if necessary, we can assume there is a $\delta > 0$ such that

$$f'(u_n)u_n^k = -\int_\Omega (g(u_n) + h)u_n^k \leq -\delta \;\; \forall n\,.$$

Therefore

$$\|f'(u_n)\|\,\|u_n^k\| \geq \delta \;\; \forall n\,,$$

which contradicts $(1 + \|u_n\|)\|f'(u_n)\| \to 0$, and the proof is done.

As a consequence of Lemma 2 we know that the functional f satisfies a variant of the Palais-Smale condition discussed in [4]. We refer to this paper for proofs of a deformation lemma as well as the standard mountain pass and saddle point theorems. Hence Lemmas 1 and 2 imply the variational characterization in Theorem 1.

5 Existence of Multiple Solutions

In this section we consider the following restricted version of problem (1).

$$\begin{gathered} \Delta u + \lambda_k u + g(u) = 0, \quad x \in \Omega\,, \\ u|_{\partial\Omega} = 0\,, \end{gathered} \tag{5}$$

where g is a C^1 function such that $g(0) = 0$. Thus it is given that there is a trivial solution to the problem and we are interested in proving the existence of nontrivial solutions. Observe that weak solutions are classical solutions in this case.

Most of the work for proving the following theorems has already been accomplished in the preceding sections. The general outline of the arguments is similar to that used in [1] and [21], and we will make use of the results in [3] and [11]. Ambrossetti's result in [3] states that if f has a nondegenerate critical point of mountain pass type, then the Morse index of f at this point is 1. We will use the notation $\operatorname{ind}_M(f, u)$ for the Morse index of f at a nondegenerate critical point u. Recall that this quantity is the dimension of the subspace where $f''(u)$ is negative definite. Hofer's result in [11] states that if f has an isolated critical point of mountain pass type, then the Leray-Schauder index of f' at this point is -1. We will use the notation $\operatorname{ind}_{LS}(f', u)$ for the Leray-Schauder index of f' at a critical point u. Recall that this quantity is defined as

$\lim_{r\to 0}\deg_{LS}(f', B_r(u), 0)$, if this limit exists, where $B_r(u)$ is the r-ball centered at u. The computation of $\mathrm{ind}_{LS}(f', 0)$ in the following proofs is standard, but for more detail see Theorem 2.8.1 in [16].

Both theorems of this section generalize the results in [21] by allowing double resonance rather than just one-sided resonance, and by allowing a more general Landesman-Lazer type solvability condition. Further, the first theorem of this section allows double resonance between any two consecutive eigenvalues rather than just between the first two eigenvalues.

A particular example of a nonlinear term that does not satisfy the conditions in [1] or [21] would be $g(u) = u/(1+u^2)$. Since $\lim_{|u|\to\infty} g(u) = 0$ it follows that neither the standard Landesman-Lazer condition used in [1] nor the generalization used in [21] can be satisfied. However it was shown in [19], using a simple dominated convergence argument, that this nonlinear term does satisfy *(g2)* and *(g3)*. A number of modifications can be made to this example without changing its basic characteristics, e.g. adding certain types of terms with linear growth. For a detailed discussion with examples see [19]. Finally, it is easy to see that this example can be modified for u within any specified interval $[-r, r]$ so that $g'(0)$ satisfies the hypotheses of the following theorems. For further remarks and examples on how generalized Landesman-Lazer conditions compare to solvability conditions used in other variational arguments, such as in [6], see [21].

Theorem 2 *Suppose g satisfies (g1)-(g3) and $\lambda_k + g'(0) < \lambda_1$. Then problem (5) has at least two nontrivial solutions.*

Proof: Notice that for any $v \in H_0^1(\Omega)$

$$f''(0)(v,v) = \int_\Omega |\nabla v|^2 - \lambda_k \int_\Omega v^2 - g'(0)\int_\Omega v^2 \ .$$

If $\lambda_k + g'(0) \geq 0$, then by Poincare's inequality

$$f''(0)(v,v) \geq \left(1 - \frac{\lambda_k + g'(0)}{\lambda_1}\right)\|v\|^2,$$

and if $\lambda_k + g'(0) \leq 0$, then

$$f''(0)(v,v) \geq \|v\|^2.$$

It follows that 0 is a nondegenerate critical point of f with $\mathrm{ind}_M(f,0) = 0$ and $\mathrm{ind}_{LS}(f',0) = 1$. Moreover, 0 is a strict local minimum of f, so there is an $r > 0$ and an $\alpha > 0$ such that $f|_{\partial B_r(0)} \geq \alpha$. Since $f|_{V^- \bigoplus V^k}$ is anticoercive, we can find a $w \in \overline{B_r(0)}^{\,c}$ such that $f(w) \leq 0$. Now let $\Gamma = \{h \in C([0,1], H_0^1(\Omega)) | h(0) = 0, h(1) = w\}$. The compactness condition of Lemma 2 justifies a standard deformation argument to show that

$$c = \inf_{h\in\Gamma}\max_{0\leq t\leq 1} f(h(t))$$

is a critical value for f. Thus there is at least one critical point, u_0, of mountain pass type, and, since $f(u_0) \geq \alpha > 0$, it is clear that u_0 is nontrivial.

Suppose $\{0, u_0\}$ is the entire solution set of (5). Then u_0 is an isolated critical point of mountain pass type, and it is justified to apply the result in [11] to get $\mathrm{ind}_{LS}(f', u_0) = -1$. The addition property of Leray-Schauder degree implies that for R as in Theorem 1

$$\deg_{LS}(f', B_R(0), 0) = \mathrm{ind}_{LS}(f', 0) + \mathrm{ind}_{LS}(f', u_0) = 0,$$

but this contradicts the conclusion of Theorem 1. Hence there must be at least one more nontrivial solution.

Theorem 3 *Suppose g satisfies (g1)-(g3), $k = 1$, and there is an $m \geq 2$ such that*
$\lambda_m < \lambda_1 + g'(0) < \lambda_{m+1}$. Then problem (5) has at least two nontrivial solutions.

Proof: Observe that since $k = 1$ we have

$$f''(0)(v, v) = \int_\Omega |\nabla v|^2 - \lambda_1 \int_\Omega v^2 - g'(0) \int_\Omega v^2,$$

so for $v \in \bigoplus_{j \leq m} V^j$ we have

$$f''(0)(v, v) \leq (\lambda_m - \lambda_1 - g'(0)) \int_\Omega v^2,$$

and, since the L_2 and H^1_0 norms are equivalent on the finite dimensional space $\bigoplus_{j \leq m} V^j$, there is a constant $c > 0$ such that

$$f''(0)(v, v) \leq -c\|v\|^2.$$

For $v \in \bigoplus_{j \geq m+1} V^j$ we have

$$f''(0)(v, v) \geq \left(1 - \frac{\lambda_1 + g'(0)}{\lambda_{m+1}}\right) \|v\|^2.$$

Therefore 0 is a nondegenerate critical point of f with $\mathrm{ind}_M(f, 0) = d$ and $\mathrm{ind}_{LS}(f', 0) = (-1)^d$, where d is the dimension of $\bigoplus_{j \leq m} V^j$.

Clearly $d \geq 2$ so, by the result in [3], 0 cannot be a critical point of mountain pass type, else the Morse index would be 1. But Theorem 1 for the case $k = 1$ states that f must have at least one critical point of mountain pass type, call it u_0, and so there is at least one nontrivial solution.

Suppose $\{0, u_0\}$ is the entire set of solutions. As in the proof of the previous theorem, apply Hofer's result and the additive property of Leray-Schauder degree to get

$$\deg_{LS}(f', B_R(0), 0) = \mathrm{ind}_{LS}(f', 0) + \mathrm{ind}_{LS}(f', u_0) = (-1)^d - 1.$$

Observe that the right hand side of this equality is even, which contradicts the conclusion of Theorem 1. Hence there must be at least one more nontrivial solution.

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