A Remark on $\infty$-harmonic Functions on Riemannian Manifolds *  

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Abstract

In this note we prove an equality for $\infty$-harmonic functions on Riemannian manifolds. As a corollary, there is no non-constant $\infty$-harmonic function on positively (or negatively) curved manifolds.

1 Introduction

In [1], [2], Aronsson studied solutions of the boundary value problem for the degenerate elliptic equation

$$\sum_{i,j} \nabla_i u \nabla_j u \nabla_i \nabla_j u = 0$$

in a bounded subdomain $D$ of $\mathbb{R}^n$ with the boundary condition $u = \varphi$ on $\partial D$. His motivation is to consider the absolutely minimizing Lipschitz extension problem, which means the problem of finding an extension $u$ in $W^{1,\infty}(D)$ of any given Lipschitz function $\varphi$ on $\partial D$ satisfying the minimization property

$$\|\nabla u\|_{L^{\infty}(U)} \leq \|\nabla v\|_{L^{\infty}(U)}$$

for any open set $U \subset D$ and for $v \in W^{1,\infty}(U)$ such that $v - u \in W^{1,\infty}_0(U)$. The equation (1) is the Euler-Lagrange equation of the functional $F_\infty(u) = \|\nabla u\|_{L^{\infty}}$ in the following sense. A $p$-harmonic function $u$ is a solution of

$$\text{div}(\|\nabla u\|^{p-2} \nabla u) = 0,$$

which is the Euler-Lagrange equation of the functional $F_p(u) = \|\nabla u\|_{L^p}$. Rewrite (2) to read

$$\frac{1}{p-2} \|\nabla u\|^2 \Delta u + \sum_{i,j} \nabla_i u \nabla_j u \nabla_i \nabla_j u = 0.$$
Formally passing to the limit as \( p \) tends to infinity, the Euler-Lagrange equation (2) of the functional \( F_p \) converges in some sense to the Euler-Lagrange equation (1) of the functional \( F_\infty \). From the point of view by Aronsson, Jensen [6] obtained existence and uniqueness results. (See also Bhattacharya, DiBenedetto and Manfredi [4].) He proved

1. any solution of the absolutely minimizing Lipschitz extension problem is a viscosity solution of (1), and
2. there exists a unique viscosity solution of (1). Any bounded such solution is locally Lipschitz continuous.


The absolutely minimizing Lipschitz extension problem is considered also on subdomains of Riemannian manifolds \( M \). Then the associated equation corresponding to (1) is

\[
g^{ij}g^{jq}r_{i}u^{r}u_{j}^{r}u_{p}^{q}u_{s}^{r}u = 0, \quad (3)
\]

where \( g_{ij} \) (resp. \( g^{ij} \)) is the metric of \( M \) (resp. the inverse matrix of \( g_{ij} \)), and \( \nabla \) denotes the Levi-Civita connection of \( g \). (Throughout this note, we use the Einstein summation convention; if the same index appears twice, once as a superscript and once as a subscript, then the index is summed over all possible values.) In this note we are concerned with \( W^{2,2+\varepsilon}_{\text{loc}} \)-solutions of (3) \( \varepsilon > 0 \). We say that \( u \) is a \( W^{2,2+\varepsilon}_{\text{loc}} \)-solution of (3) in \( D \) if the following two conditions hold:

1. \( u \) is locally Lipschitz continuous, and
2. \( u \in W^{2,2+\varepsilon}_{\text{loc}}(D) \), and \( u \) satisfies (3) a.e.,

where \( W^{2,2+\varepsilon}_{\text{loc}}(D) \) denotes the Sobolev space of functions whose second derivatives belong to \( L^{2+\varepsilon}_{\text{loc}}(D) \). On this general setting, the curvature of \( M \) provides an obstruction on existence of nontrivial \( W^{2,2+\varepsilon}_{\text{loc}} \)-solutions of (3). The purpose of this note is to prove the following equality.

**Theorem 1** Let \( M \) be a Riemannian manifold, and let \( D \) be a domain in \( M \). Let \( u \) be a \( W^{2,2+\varepsilon}_{\text{loc}} \)-solution of the equation (3) in \( D \). Then

\[
g^{ip}g^{jq}g^{kr}g^{is}R_{ijkl}u_{p}^{r}u_{j}^{r}u_{l}^{r}u_{k}^{s}u = 0 \quad \text{a.e. in } D, \quad (4)
\]

where \( R_{ijkl} \) is the Riemannian curvature tensor of \( M \).

Note that when \( M = \mathbb{R}^n \), \( R_{ijkl} \equiv 0 \); hence the equality (4) holds automatically in this case. From equality (4), we have \( \nabla u = 0 \) at any point where the curvature is positive (or negative). So we have:
Corollary 1 Suppose that the sectional curvature of $M$ is positive (or negative) in $D$. Then any $W^{2,2+\varepsilon}_{loc}$-solution of (3) in $D$ is a constant function.

We mention a related fact on harmonic functions. Let $u$ be a harmonic function on a Riemannian manifold $M$. Then $u$ is a constant function if one of the following two conditions holds:

1. $M$ is compact (the maximum principle).
2. $M$ is complete and non-compact, the Ricci curvature of $M$ is nonnegative, and $u$ is bounded on $M$ (Yau [7]).

These results need the assumption that $u$ is globally defined on compact or complete manifolds. On the other hand, the above equality (4) holds when an $\infty$-harmonic function $u$ is defined on a subdomain of $M$; the structure of $\infty$-Laplace gives a restriction on local existence of solutions.

The author thinks that our theorem holds without the assumption that solutions belong to the class $W^{2,2+\varepsilon}_{loc}(D)$, though we use this assumption. Then Aronsson’s minimization approach of the Lipschitz extension problem does not seem to work on any positively (or negatively) curved manifold.

2 A Bochner type formula

In this section we prove the following formula of Bochner type.

Lemma 1 Let $u$ be a $C^3_{loc}$-solution of (3) on a subdomain $D$ of a Riemannian manifold $M$. Then the following equality holds.

\[
g^{ip}g^{jq}\nabla_i u \nabla_j u \nabla_p \nabla_q \|\nabla u\|^2 + \frac{1}{2} \|\nabla u\|^4 + 2g^{ip}g^{jq}g^{kr}g^{ts}R_{ikjl} \nabla_p u \nabla_q u \nabla_r u \nabla_s u = 0 \quad \text{in } D,
\]

where $\|\nabla u\|^2 = g^{ij}\nabla_i u \nabla_j u$ and $\|\nabla u\| \|\nabla^2 u\| = g^{ij}\nabla_i \nabla_j \nabla u$.

Proof. Note $\nabla g_{ij} = \nabla g^{ij} = 0$, since $\nabla$ is the Levi-Civita connection. Applying $\nabla_r$ to both sides of (3), we have

\[
g^{ip}g^{jq}\nabla_i u \nabla_j u \nabla_p \nabla_q u + 2g^{ip}g^{jq}\nabla_i u \nabla_j u \nabla_r u \nabla_p \nabla_q u = 0. \quad (6)
\]

We see that

\[
\nabla_p \nabla_q \nabla_r u = \nabla_p \nabla_r \nabla_q u \quad (7)
\]

by the Ricci formula.
Then we have

\[ g^{ip}g^{jq}g^{kr}\nabla_i u \nabla_j u \nabla_p \nabla_k u \nabla_q \nabla_r u \]

\[ = \frac{1}{2} g^{kr} \nabla_k (g^{ip} \nabla_i u \nabla_p u) \frac{1}{2} \nabla_r (g^{jq} \nabla_j u \nabla_q u) \]

\[ = \frac{1}{4} g^{kr} \nabla_k \|\nabla u\|^2 \nabla_r \|\nabla u\|^2 \]

\[ = \frac{1}{4} \|\nabla \|\nabla u\|^2 \|^2 \].

Then we have

\[ g^{ip}g^{jq} \nabla_i u \nabla_j u \nabla_p \nabla_q \|\nabla u\|^2 \]

\[ = g^{ip}g^{jq} \nabla_i u \nabla_j u \nabla_p \nabla_q (g^{kr} \nabla_r u \nabla_k u) \]

\[ = 2 g^{ip}g^{jq} g^{kr} \nabla_i u \nabla_j u \nabla_p \nabla_q \nabla_r u \nabla_k u \]

\[ + 2 g^{ip}g^{jq} g^{kr} \nabla_i u \nabla_j u \nabla_q \nabla_r u \nabla_p \nabla_k u \]

\[ = 2 g^{ip}g^{jq} g^{kr} \nabla_i u \nabla_j u \nabla_r \nabla_p \nabla_q u \nabla_k u \]

\[ - 2 g^{ip}g^{jq} g^{kr} \nabla_i u \nabla_j u \nabla^s R_{prqs} \nabla_i u \nabla_k u \]

\[ + 2 g^{ip}g^{jq} g^{kr} \nabla_i u \nabla_j u \nabla_q \nabla_r u \nabla_p \nabla_k u \]

\[ = - 4 g^{ip}g^{jq} g^{kr} \nabla_i u \nabla_k u \nabla_r \nabla_j u \nabla_p \nabla_q u \]

\[ - 2 g^{ip}g^{jq} g^{kr} g^{ls} R_{prqs} \nabla_i u \nabla_j u \nabla_k u \nabla_l u \]

\[ + 2 g^{ip}g^{jq} g^{kr} \nabla_i u \nabla_j u \nabla_q \nabla_r u \nabla_p \nabla_k u \]

\[ = - 2 g^{ip}g^{jq} g^{kr} g^{ls} R_{prqs} \nabla_i u \nabla_j u \nabla_k u \nabla_l u \]

\[ - 2 g^{ip}g^{jq} g^{kr} \nabla_i u \nabla_j u \nabla_q \nabla_r u \nabla_p \nabla_k u \]

\[ = - 2 g^{ip}g^{jq} g^{kr} g^{ls} R_{prqs} \nabla_i u \nabla_j u \nabla_k u \nabla_l u - \frac{1}{2} \|\nabla \|\nabla u\|^2 \|^2 \quad \text{(by (8))}. \]

### 3 Proof of Theorem 1 for \( C^3_{loc}\)-solutions

Take any \( \eta \in C_0^\infty(D) \). Then from (5), we have

\[ \int_D \eta g^{ip} g^{jq} \nabla_i u \nabla_j u \nabla_p \nabla_q \|\nabla u\|^2 \frac{1}{2} \int_D \|\nabla \|\nabla u\|^2 \|^2 \eta \]

\[ + 2 \int_D \eta g^{ip} g^{jq} g^{kr} g^{ls} R_{ikjl} \nabla_p u \nabla_q u \nabla_r u \nabla_s u = 0. \]

Note here

\[ g^{ij} \nabla_j u \nabla_q \|\nabla u\|^2 = g^{ij} \nabla_j u \nabla_q (g^{ip} \nabla_i u \nabla_p u) \]

\[ = 2 g^{ip} g^{jq} \nabla_j u \nabla_i u \nabla_q \nabla_p u = 0. \]
Using integration by parts, we get
\begin{equation}
\int_D \eta g^{ip}g^{jq} \nabla_i u \nabla_j u \nabla_p \nabla_q \| \nabla u \|^2 = - \int_D g^{ip}g^{jq} \eta \nabla_i u \nabla_j u \nabla_q \| \nabla u \|^2 \\
- \int_D \eta g^{ip}g^{jq} \nabla_i u \nabla_j u \nabla_q \| \nabla u \|^2 \\
- \int_D \eta g^{ip}g^{jq} \nabla_i u \nabla_p \nabla_j u \nabla_q \| \nabla u \|^2 \\
= - \int_D \eta g^{ip}g^{jq} \nabla_i u \nabla_p \nabla_j u \nabla_q \| \nabla u \|^2 \quad \text{(by (10))}
\end{equation}

\begin{equation}
= - \int_D \eta \frac{1}{2} g^{ip} \nabla_j (g^{ip} \nabla_i u \nabla_p u) \nabla_q \| \nabla u \|^2 \\
= - \frac{1}{2} \int_D \eta g^{jq} \nabla_j \| \nabla u \| \nabla_q \| \nabla u \|^2 \\
= - \frac{1}{2} \int_D \| \nabla \| \nabla u \|^2 \|^2 \eta.
\end{equation}

From (9) and (11), we have
\begin{equation}
\int_D \eta g^{ip}g^{jq}g^{kr}g^{ls} R_{ikjl} \nabla_p u \nabla_q u \nabla_r u \nabla_s u = 0. \quad (12)
\end{equation}

Since \( \eta \) is an arbitrary test function in \( C_0^\infty(D) \), we have
\[ g^{ip}g^{jq}g^{kr}g^{ls} R_{ikjl} \nabla_p u \nabla_q u \nabla_r u \nabla_s u = 0 \quad \text{a.e. in } D. \quad \Box \]

4 Proof of Theorem 1

In this section we complete our proof of Theorem 1 using an approximation. For any \( W^{2,2+\varepsilon}_{loc} \)-solution \( u \) of (3), we take an approximating sequence \( \{ u^{(\nu)} \}_{\nu=1}^\infty \subset C_0^\infty(D) \) such that for any compact set \( K \) in \( D \),

1. \( \varphi^{(\nu)} := u^{(\nu)} - u \) approaches zero in \( W^{2,2+\varepsilon}_{loc}(D) \) as \( \nu \) tends to infinity, and

2. the Lipschitz constants of \( u^{(\nu)} \) (\( \nu = 1, 2, ... \)) are uniformly bounded on \( K \) : hence \( \| \nabla u^{(\nu)} \|_{L^\infty(K)} \) and \( \| \nabla \varphi^{(\nu)} \|_{L^\infty(K)} \) (\( \nu = 1, 2, ... \)) are uniformly bounded on \( K \).

Since \( u = u^{(\nu)} - \varphi^{(\nu)} \) satisfies (3), we have
\[ g^{ip}g^{jq} \nabla_i (u^{(\nu)} - \varphi^{(\nu)}) \nabla_j (u^{(\nu)} - \varphi^{(\nu)}) \nabla_p \nabla_q (u^{(\nu)} - \varphi^{(\nu)}) = 0 \]
i.e.,

\[g^p g^q \nabla_i u^{(v)} \nabla_j u^{(v)} \nabla_p \nabla_q u^{(v)} + F(\varphi^{(v)}, u^{(v)}) = 0 \]  

(13)

where

\[
F(\varphi^{(v)}, u^{(v)}) = -g^p g^q \nabla_i \varphi^{(v)} \nabla_j u^{(v)} \nabla_p \nabla_q u^{(v)} - g^p g^q \nabla_i u^{(v)} \nabla_j \varphi^{(v)} \nabla_p \nabla_q u^{(v)} - g^p g^q \nabla_i u^{(v)} \nabla_j \varphi^{(v)} \nabla_p \nabla_q \varphi^{(v)} + g^p g^q \nabla_i \varphi^{(v)} \nabla_j u^{(v)} \nabla_p \nabla_q \varphi^{(v)} + g^p g^q \nabla_i \varphi^{(v)} \nabla_j \varphi^{(v)} \nabla_p \nabla_q \varphi^{(v)} - g^p g^q \nabla_i \varphi^{(v)} \nabla_j \varphi^{(v)} \nabla_p \nabla_q \varphi^{(v)}. \]

Let \( \psi \in W^{1,1}_0(D) \). Multiply by \(-\nabla_r \psi \) both sides of (13) and use integration by parts, then we have

\[
\int_D \psi g^p g^q \nabla_i u^{(v)} \nabla_j u^{(v)} \nabla_p \nabla_q u^{(v)} - 2 \int_D \psi g^p g^q \nabla_i u^{(v)} \nabla_j u^{(v)} \nabla_p \nabla_q u^{(v)} + \int_M F(\varphi^{(v)}, u^{(v)}) \nabla_r \psi = 0.
\]

Let \( \psi = \eta g^r \nabla_k u^{(v)} \) and sum them up with respect to \( r \). Then we get

\[
\int_D \eta g^p g^q g^r \nabla_i u^{(v)} \nabla_j u^{(v)} \nabla_p \nabla_q u^{(v)} \nabla_k u^{(v)} + 2 \int_D \eta g^p g^q g^r \nabla_i u^{(v)} \nabla_k u^{(v)} \nabla_j u^{(v)} \nabla_p \nabla_q u^{(v)} - \int_M F(\varphi^{(v)}, u^{(v)}) g^r \nabla_r (\eta \nabla_k u^{(v)}) = 0
\]

(14)

We see

\[
\int_D \eta g^p g^q \nabla_i u^{(v)} \nabla_j u^{(v)} \nabla_p \nabla_q \|u^{(v)}\|^2 = \int_D \eta g^p g^q \nabla_i u^{(v)} \nabla_j u^{(v)} \nabla_p \nabla_q (g^{kr} \nabla_r \nabla_k u)
\]

\[= 2 \int_D \eta g^p g^q g^{kr} \nabla_i u^{(v)} \nabla_j u^{(v)} \nabla_p \nabla_q \nabla_r u^{(v)} \nabla_k u^{(v)} + 2 \int_D \eta g^p g^q g^{kr} \nabla_i u^{(v)} \nabla_j u^{(v)} \nabla_q \nabla_r u^{(v)} \nabla_p \nabla_k u^{(v)}
\]

\[= 2 \int_D \eta g^p g^q g^{kr} \nabla_i u^{(v)} \nabla_j u^{(v)} \nabla_r \nabla_p \nabla_q u^{(v)} \nabla_k u^{(v)} - 2 \int_D \eta g^p g^q g^{kr} \nabla_i u^{(v)} \nabla_j u^{(v)} g^{ls} R_{prqs} \nabla_i u^{(v)} \nabla_k u^{(v)}
\]

\[= 2 \int_D \eta g^p g^q g^{kr} \nabla_i u^{(v)} \nabla_j u^{(v)} \nabla_r \nabla_p \nabla_q u^{(v)} \nabla_k u^{(v)} - 2 \int_D \eta g^p g^q g^{kr} \nabla_i u^{(v)} \nabla_j u^{(v)} g^{ls} R_{prqs} \nabla_i u^{(v)} \nabla_k u^{(v)}
\]
Then using integration by parts, we get

$$+2 \int_D \eta g^{ip} g^{jq} g^{kr} \nabla_i u^{(\nu)} \nabla_j u^{(\nu)} \nabla_q \nabla_r u^{(\nu)} \nabla_p \nabla_k u^{(\nu)} \quad \text{(by (7))}$$

$$= -4 \int_D \eta g^{ip} g^{jq} g^{kr} \nabla_i u^{(\nu)} \nabla_k u^{(\nu)} \nabla_r \nabla_j u^{(\nu)} \nabla_p \nabla_q u^{(\nu)}$$

$$+2 \int_D F(\varphi^{(\nu)}, u^{(\nu)}) g^{kr} \nabla_k (\eta \nabla_r u^{(\nu)})$$

$$-2 \int_D \eta g^{ip} g^{jq} g^{kr} g^{ls} R_{pqrs} \nabla_i u^{(\nu)} \nabla_j u^{(\nu)} \nabla_k u^{(\nu)} \nabla_l u^{(\nu)}$$

$$+2 \int_D \eta g^{ip} g^{jq} g^{kr} \nabla_i u^{(\nu)} \nabla_j u^{(\nu)} \nabla_q \nabla_r u^{(\nu)} \nabla_p \nabla_k u^{(\nu)} \quad \text{(by (14))}$$

$$= -2 \int_D \eta g^{ip} g^{jq} g^{kr} g^{ls} R_{pqrs} \nabla_i u^{(\nu)} \nabla_j u^{(\nu)} \nabla_k u^{(\nu)} \nabla_l u^{(\nu)}$$

$$-\frac{1}{2} \int_M \|\nabla \nabla u^{(\nu)}\|^2 + 2 \int_M F(\varphi^{(\nu)}, u^{(\nu)}) g^{kr} \nabla_k (\eta \nabla_r u^{(\nu)}) \quad \text{(by (8))}.$$ 

Therefore we obtain an integral form of the Bochner equality for $u^{(\nu)}$:

$$\int_M \eta g^{ip} g^{jq} g^{kr} \nabla_i u^{(\nu)} \nabla_j u^{(\nu)} \nabla_p \nabla q \|\nabla u^{(\nu)}\|^2 + \frac{1}{2} \int_M \|\nabla \nabla u^{(\nu)}\|^2 \eta$$

$$-2 \int_M F(\varphi^{(\nu)}, u^{(\nu)}) g^{kr} \nabla_k (\eta \nabla_r u^{(\nu)}) \quad \text{(15)}$$

$$+2 \int_M \eta g^{ip} g^{jq} g^{kr} \nabla_i u^{(\nu)} \nabla_j u^{(\nu)} \nabla_q \nabla_r u^{(\nu)} \nabla_u^{(\nu)} \quad = 0.$$ 

for any $\eta \in C_0^\infty(D)$. Note here

$$g^{jq} \nabla_j u^{(\nu)} \nabla_q \|\nabla u^{(\nu)}\|^2 = g^{jq} \nabla_j u^{(\nu)} \nabla_q (g^{ip} \nabla_i u^{(\nu)} \nabla_p u^{(\nu)}) \quad \text{(16)}$$

$$= 2 g^{ip} g^{jq} \nabla_j u^{(\nu)} \nabla_i u^{(\nu)} \nabla_q \nabla_p u^{(\nu)}$$

$$= -2 F(\varphi^{(\nu)}, u^{(\nu)}) \quad \text{(by (13))}.$$ 

Then using integration by parts, we get

$$\int_D \eta g^{ip} g^{jq} g^{kr} \nabla_i u^{(\nu)} \nabla_j u^{(\nu)} \nabla_p \nabla q \|\nabla u^{(\nu)}\|^2 \quad \text{(17)}$$

$$= -\int_D g^{ip} g^{jq} \nabla_p \eta \nabla_r u^{(\nu)} \nabla_j u^{(\nu)} \nabla_q \|\nabla u^{(\nu)}\|^2.$$
Since $kr$, then by (15) and (17), we obtain

$$-\int_D \eta g^{ip} g^{jq} \nabla_i u^{(\nu)} \nabla_j u^{(\nu)} \nabla_q \|\nabla u^{(\nu)}\|^2$$

$$-\int_D \eta g^{ip} g^{jq} \nabla_i u^{(\nu)} \nabla_j u^{(\nu)} \nabla_q \|\nabla u^{(\nu)}\|^2$$

$$= 2 \int_D \eta g^{ip} \nabla_i u^{(\nu)} \nabla_p \nabla u^{(\nu)} \nabla_q \|\nabla u^{(\nu)}\|^2$$

because

$$g^{ip} \nabla_i u^{(\nu)} \nabla_p \nabla_j u^{(\nu)} = \frac{1}{2} \nabla_j (g^{ip} \nabla_i u^{(\nu)} \nabla_p u^{(\nu)}) = \frac{1}{2} \nabla_j \|\nabla u^{(\nu)}\|^2.$$  

Then by (15) and (17), we obtain

$$2 \int_D \eta g^{ip} g^{jq} g^{kr} g^{ls} R_{ijkl} \nabla_i u^{(\nu)} \nabla_j u^{(\nu)} \nabla_k u^{(\nu)} \nabla_l u^{(\nu)}$$

$$= -2 \int_D \eta g^{ip} \nabla_i u^{(\nu)} \nabla_p \nabla F(\varphi^{(\nu)}, u^{(\nu)})$$

$$-2 \int_D \eta \nabla u^{(\nu)} F(\varphi^{(\nu)}, u^{(\nu)})$$

$$+ 2 \int_D g^{kr} \nabla_r (\eta \nabla_k u^{(\nu)}) F(\varphi^{(\nu)}, u^{(\nu)}).$$

Since $\|\nabla u^{(\nu)}\|$ is bounded uniformly on $K$, we get

| the right hand side of (18) |

$$\leq C \int_K \|F(\varphi^{(\nu)}, u^{(\nu)})\| + C \int_K \|\nabla \nabla u^{(\nu)}\| \|F(\varphi^{(\nu)}, u^{(\nu)})\|$$

$$\leq C \int_K \|\nabla \varphi^{(\nu)}\| \|\nabla \nabla u^{(\nu)}\| + C \int_K \|\nabla \varphi^{(\nu)}\|$$

$$+ C \int_K \|\nabla \varphi^{(\nu)}\|^2 \|\nabla \nabla u^{(\nu)}\| + C \int_K \|\nabla \varphi^{(\nu)}\| \|\nabla \varphi^{(\nu)}\|$$

$$+ C \int_K \|\nabla \varphi^{(\nu)}\|^2 \|\nabla \varphi^{(\nu)}\| + C \int_K \|\nabla \varphi^{(\nu)}\| \|\nabla \nabla u^{(\nu)}\|^2$$

$$+ C \int_K \|\nabla \varphi^{(\nu)}\| \|\nabla \varphi^{(\nu)}\| \|\nabla \nabla u^{(\nu)}\|$$

$$+ C \int_K \|\nabla \varphi^{(\nu)}\| \|\nabla \varphi^{(\nu)}\| \|\nabla \varphi^{(\nu)}\| \|\nabla \nabla u^{(\nu)}\|.$$
\[ + C \int_K \|\nabla \varphi(\nu)\|^2 \|\nabla \nabla \varphi(\nu)\| \|\nabla \nabla u(\nu)\| \]
\[ =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10}. \]

Since \( \varphi(\nu) \) converges to in \( W^{2,2+\varepsilon}(K) \) as \( \nu \) tends to infinity, \( \nabla \varphi(\nu) \) and \( \nabla \nabla \varphi(\nu) \) approaches zero in \( L^2(K) \). Then

\[ I_1 \leq C \left\{ \int_K \|\nabla \varphi(\nu)\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla \nabla u(\nu)\|^2 \right\}^{1/2} \to 0, \]

\[ I_2 \leq C \left\{ \int_K \|\nabla \nabla \varphi(\nu)\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla \nabla u(\nu)\|^2 \right\}^{1/2} \to 0, \]

\[ I_3 \leq C \sup_K \|\nabla \varphi(\nu)\| \left\{ \int_K \|\nabla \varphi(\nu)\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla \nabla u(\nu)\|^2 \right\}^{1/2} \to 0, \]

\[ I_4 \leq C \left\{ \int_K \|\nabla \varphi(\nu)\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla \nabla \varphi(\nu)\|^2 \right\}^{1/2} \to 0, \]

\[ I_5 \leq C \sup_K \|\nabla \varphi(\nu)\| \left\{ \int_K \|\nabla \varphi(\nu)\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla \nabla u(\nu)\|^2 \right\}^{1/2} \to 0, \]

\[ I_7 \leq C \left\{ \int_K \|\nabla \varphi(\nu)\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla \nabla u(\nu)\|^2 \right\}^{1/2} \to 0, \]

\[ I_8 \leq C \sup_K \|\nabla \varphi(\nu)\| \left\{ \int_K \|\nabla \varphi(\nu)\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla \nabla u(\nu)\|^2 \right\}^{1/2} \to 0, \]

\[ I_{10} \leq C \sup_K \|\nabla \varphi(\nu)\| \left\{ \int_K \|\nabla \varphi(\nu)\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla \nabla u(\nu)\|^2 \right\}^{1/2} \to 0. \]

Furthermore, since \( \varphi(\nu) \) converges to zero in \( W^{2,2+\varepsilon}(K) \), we have

\[ I_6 \leq C \sup_K \|\nabla \varphi(\nu)\|^{1-\varepsilon} \left\{ \int_K \|\nabla \varphi(\nu)\|^{2+\varepsilon} \right\}^{2/(2+\varepsilon)} \}
\[ \left\{ \int_K \|\nabla \nabla u(\nu)\|^{2+\varepsilon} \right\}^{2/(2+\varepsilon)} \to 0, \]

\[ I_8 \leq C \sup_K \|\nabla \varphi(\nu)\|^{2-\varepsilon} \left\{ \int_K \|\nabla \varphi(\nu)\|^{2+\varepsilon} \right\}^{2/(2+\varepsilon)} \to 0. \]

Thus the right hand side of (18) converges to zero as \( \nu \) tends to infinity. Then, letting \( \nu \) go to infinity in (18), we have (12). This completes the proof. \( \square \)
References


[2] Aronsson, G., *On the partial differential equation* \( u_x^2 u_{xx} + 2 u_x u_y u_{xy} + u_y^2 u_{yy} = 0 \), Ark. Mat. 7 (1968), 395-425.


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In this article, Theorem 1 follows from general properties of the Riemannian curvature tensor, and Corollary 1 is incorrect. The Bochner formula does not seem to work in this situation.

Lemma 1 can be used in proving the following Liouville theorem for $C^3$-solutions.

**Theorem A.** Let $M$ be a complete noncompact Riemannian manifold of non-negative (sectional) curvature. Let $u$ be a bounded $\infty$-harmonic function of $C^3$-class on $M$. Then $u$ is a constant function.

The curvature assumption in Theorem A is necessary only for applying the Hessian comparison theorem in the proof (Here we use the operator $Q^{ij} = g^{ip}g^{jq}\nabla_p\nabla_q$). Theorem A also follows from arguments in [Cheng, S.Y., Liouville theorem for harmonic maps, Proc. Symp. Pure Math. 36(1980), 147-151]. See also the article [Hong, N.C., Liouville theorems for exponentially harmonic functions on Riemannian manifolds, Manuscripta Math. 77(1992), 41-46].

Sincerely yours,

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