Weak Solutions to the One-dimensional Non-Isentropic Gas Dynamics by the Vanishing Viscosity Method *

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Abstract

In this paper we consider the non-isentropic equations of gas dynamics with the entropy preserved. Equations are formulated so that the problem is reduced into the 2 × 2 system of conservation laws with a forcing term in momentum equation. The method of compensated compactness is then applied to prove the existence of weak solution in the vanishing viscosity method.

1 Introduction

Consider the one-dimensional gas dynamics equation in the Eulerian coordinate

\begin{equation}
\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2 + p)_x &= 0 \\
s_t + us_x &= 0.
\end{aligned}
\end{equation}

where \( \rho, u, p \) and \( s \) denote the density, velocity, pressure and entropy. Other relevant quantities are the internal energy \( e \) and the temperature \( T \). We assume that the gas is ideal, so that the equation of state is given by

\[ p = R\rho T \]

and that it is polytropic, so that \( e = c_v T \) and

\begin{equation}
\begin{aligned}
p &= (\gamma - 1)e^{s/c_v} \rho^\gamma
\end{aligned}
\end{equation}

where \( \gamma = c_p/c_v > 1 \) and \( R = c_p - c_v \). Define \( \phi \) by

\begin{equation}
\begin{aligned}
\phi^{1-\gamma} &= \gamma(\gamma - 1) e^{s/c_v}.
\end{aligned}
\end{equation}

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Then, $\phi$ satisfies
\[ \phi_t + u \phi_x = 0. \]

Thus, we consider the Cauchy problem (equivalent to (1.1))
\[
\begin{align*}
\rho_t + m_x &= 0 \\
m_t + (m^2/\rho + p)_x &= 0 \\
\phi_t + u \phi_x &= 0,
\end{align*}
\]
where
\[
\begin{align*}
m &= \rho u & \text{and} & & p = \frac{1}{\gamma} \phi^{1-\gamma} \rho^\gamma,
\end{align*}
\]
with smooth initial data $(\rho_0, m_0)$ in $L^\infty(R^2)$ that approaches a constant state $(\bar{\rho}, \bar{m})$ at infinity and satisfies $\rho_0(x) \geq \delta_1 > 0$, and $\phi_0$ in $W^{1,\infty}(R)$ that satisfies $(\phi_0)_x$ converges to 0 at infinity and
\[
\phi_0(x) \geq \delta_2 > 0 \quad \text{and} \quad (\phi_0)_x(x) \geq 0 \quad \text{(or} \quad (\phi_0)_x(x) \leq 0). \]

Consider the conservation form of the gas dynamics
\[
\begin{align*}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2 + p)_x &= 0 \\
[\rho \left( \frac{1}{2} u^2 + e \right)_t + (\rho u \left[ \frac{1}{2} u^2 + e \right] + pu)_x] &= 0.
\end{align*}
\]
System (1.7) can be written as the hyperbolic system of conservation laws
\[ y_t + f(y)_x = 0 \]
where $y = y(t, x) = (\rho, \rho u, \rho \left( \frac{1}{2} u^2 + e \right)) \in R^3$ and $f$ is a smooth nonlinear mapping from $R^3$ to $R^3$. System (1.4) is equivalent to system (1.7) when solutions are smooth but not necessarily when solutions are weak (e.g.,[Sm, Chapters 16-17]). It is proved in Corollary 3.6 that the viscosity limit of solutions to (1.10) satisfies $\eta_t + q_x \leq 0$ in the sense of distributions, i.e., the third equation of (1.7), the conservation of energy $\eta_t + q_x = 0$ is replaced by the non-energy production. We also note that the isentropic solution ($\phi = \text{const}$) [Di1] is a weak solution of (1.4) but not necessarily of (1.7).

In this paper we show the existence of weak solutions to (1.4)-(1.5) using the vanishing viscosity method. The function $(\rho, m, \phi) \in L^\infty(\Omega) \times L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ with $\Omega = [0, \tau] \times R$ is a weak solution of (1.4)-(1.5) if $\phi$ satisfies the third equation of (1.4) a.e in $\Omega$ and
\[
\int_0^\tau \int_{-\infty}^\infty (v \cdot \psi_t + F(\rho, m, \phi) \cdot \psi_x) dx \ dt = 0
\]
for all $\psi \in C_c^\infty(\Omega; \mathbb{R}^2)$ where $v = (\rho, m)$ and

$$F(\rho, m, \phi) = (m, m^2/\rho + p)$$

We consider the viscous equation of (1.4) with equal diffusion rates

$$\begin{align*}
\rho_t + m_x &= \epsilon \rho_{xx} \\
m_t + (m^2/\rho + p)_x &= \epsilon m_{xx} \\
\phi_t + u \phi_x &= \epsilon \phi_{xx}.
\end{align*}$$

(1.10)

It will be shown in Theorem 3.5 that the solutions $(\rho^\epsilon, m^\epsilon, \phi^\epsilon)$ to (1.10) converge to a locally defined (in time) weak solution $(\rho, m, \phi)$ of (1.4).

Our approach is based on the following observation. Suppose $\phi$ is a constant. Then equation (1.4) reduces to the isentropic gas dynamics. For the isentropic equation it is shown in DiPerna [Di1] that (1.4) has a weak solution by the vanishing viscosity method and using the theory of compensated compactness. The key steps in [Di1] are given as follows. First, if $\phi$ is a constant and $\theta = (\gamma - 1)/2$ then

$$w = G_1(\rho, m, \phi) = \frac{m}{\rho} + \frac{1}{\theta} \phi^{-\theta} \rho^\theta$$

and

$$z = G_2(\rho, m, \phi) = -\frac{m}{\rho} + \frac{1}{\theta} \phi^{-\theta} \rho^\theta$$

(1.11)

are the Riemann invariants so that $\nabla_v G_1$ and $\nabla_v G_2$ are the two left eigenvectors of the $2 \times 2$ matrix

$$\nabla_v F = \begin{pmatrix} 0 & 1 \\ -m^2/\rho^2 + m \phi^{-\theta} \rho^\theta & 2m/\rho \end{pmatrix}.$$ 

where $\phi$ is assumed to be a positive constant. The method of invariant regions ([CCS],[Sm]) is applied to $G_1$, $G_2$ to obtain that $0 \leq \rho^\epsilon \leq \text{const}$, $|m^\epsilon/\rho^\epsilon| \leq \text{const}$. Then, there exist a subsequence of $v^\epsilon = (\rho^\epsilon, m^\epsilon)$, still denoted by $v^\epsilon$ and a Young measure $\nu_{t,x}$ such that for each $\Phi \in C(\mathbb{R}^2)$ we have $\tilde{\Phi}(v^\epsilon)$ converges weak star to $\tilde{\Phi}$ in $L^\infty(\Omega)$ where

$$\tilde{\Phi}(t, x) = \langle \nu, \Phi \rangle = \int_{\Omega} \Phi(y) \, d\nu_{t,x}(y), \text{ a.e. (}t, x\text{) } \in \Omega.$$ 

Using the entropy fields [La] and the div-curl theorem of Murat [Mu] and Tatar [Ta] for bilinear maps in the weak topology,

$$\langle \nu, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \nu, \eta_1 \rangle \langle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle$$

(1.12)

for all entropy/entropy flux pairs $(\eta_i, q_i)$ so that

$$\nabla_v \eta \nabla_v F = \nabla_v q.$$
Then, using the weak entropy pairs (i.e., \( \eta(0, \cdot) = 0 \)) it is shown that \( \nu \) reduces to a point mass, i.e., \( v_c \) converges to \( v \), a.e. in \( \Omega \).

We will apply the method described above for the non-isentropic equation (1.10). We need to overcome the two major difficulties. First, \( G_1, G_2 \) are no longer the Riemann invariants of the \( 3 \times 3 \) matrix \( M \):

\[
M = \nabla F = \begin{pmatrix}
 \frac{m^2}{\rho^2} + \rho^2 \phi^{-2\theta} & \frac{1}{\rho} & 0 \\
 -\frac{2m}{\rho} & \frac{2\theta}{\gamma} \rho^2 \phi^{-2\theta-1} & -1 \\
0 & m & \rho
\end{pmatrix}
\]

Next, equation (1.12) should be extended to the \( 3 \times 3 \) system. We resolve these difficulties by the following steps. Note (see Lemma 2.4) that \( G_i = G_i(y, m) \), \( i = 1, 2 \) satisfies

(1.13)

\[
(G_i)_t + \lambda_i \nabla G_i \cdot y_x + \frac{1}{\gamma} \begin{pmatrix}
 \rho^2 \phi^{-2\theta -1} \phi_x \\
 -\rho^2 \phi^{-2\theta -1} \phi_x \\
\end{pmatrix}, \quad i = 1, 2 = \epsilon ((G_1)_{xx} - \nabla^2 G_1(y_x, y_x))
\]

where \( y = (\rho, m, \phi) \) is a solution to (1.10) and \( \lambda_1 = u + \rho^2 \phi^{-\theta}, \lambda_2 = u - \rho^2 \phi^{-\theta} \) are the eigenvalues of the \( 2 \times 2 \) matrix \( \nabla_x F \). Here, \( G_i, i = 1, 2 \) are quasi-convex functions of \( (\rho, m, \phi) \) (see Lemma 2.5), i.e.,

\[
r \cdot \nabla G_i = 0 \text{ implies } \nabla^2 G_i(r, r) \geq 0.
\]

Note that from the third equation of (1.10) that \( \phi' \geq \delta_2 \) and \( |\phi'|_{\infty} \leq |\phi_0|_{\infty} \) (see Lemma 2.1) and moreover \( \phi_x \) satisfies

(1.14)

\[
(\phi_x^\prime)_t + (u_x^\prime \phi_x^\prime)_x = \epsilon (\phi_x^\prime)_{xx}
\]

Observing that if \( (\rho, m, \phi) \) is a solution to (1.10) then \( \xi = \log (\frac{\phi_x}{\rho}) \) satisfies

we show that if \( (\phi_0)_x \geq 0 \) (resp. \( \leq 0 \)) then \( \phi_x^\prime \geq 0 \) (resp. \( \leq 0 \)) and \( |\phi_x^\prime| \leq c \rho^\prime \) in \( \Omega \) provided that \( |(\phi_0)_x| \leq c \rho_0 \) in \( R \) (see Lemma 2.3). It thus follows from (1.13) and the quasi-convexity of \( G_i, i = 1, 2 \) that \( \max_x G_2(t, x) \leq \max_x G_2(0, x) \) (resp. \( \max_x G_1(t, x) \leq \max_x G_1(0, x) \))

\[
|\rho^2 \phi^{-2\theta -1} \phi_x| \leq c \left( \frac{\rho}{\phi} \right)^{2\theta -1/2}.
\]

By the maximum principle (see Theorem 2.6), there exists a \( \tau = \tau_c > 0 \) with \( c \to \tau_c \) monotonically decreasing and \( \tau_0 = \infty \) such that \( \max_{t \in [0, \tau], x \in R} G_1(t, x) \) is less than a constant independent of \( \epsilon > 0 \). Hence, we obtain \( 0 \leq \rho^\prime \leq \text{const}, \quad |m^\prime/\rho^\prime| \leq \text{const} \) and \( |\phi_x| \leq \text{const} \) in \( \Omega \).
Second, in contrast to the isentropic case, the system (1.10) is not endowed with a rich family of entropy-entropy flux pairs. Thus, in order to prove that the Young measures \( \nu_{t,x} \) of a weakly star convergent subsequence of \((\rho^\epsilon, m^\epsilon, \phi^\epsilon)\) reduce to a point mass, we first note that \( \{\phi^\epsilon(t,x)\} \) is precompact in \( L^2_{\text{loc}}(\Omega) \) and thus \( \phi^\epsilon \) converges \( \phi \) a.e. in \( \Omega \) (see Lemma 3.4). Also, we note that dividing the first two equations (1.10) by \( \phi \), we obtain

\[
\begin{align*}
\hat{\rho}_t + \hat{m}_x &= \epsilon (\hat{\rho}_{xx} + 2 \frac{\rho_x \phi_x}{\phi^2}) \\
\hat{m}_t + \left( \frac{\hat{m}^2}{\hat{\rho}} + \frac{1}{\gamma} \hat{\rho} \hat{\gamma} \right)_x + 
\frac{p \phi_x}{\phi^2} &= \epsilon \left( m_{xx} + 2 \frac{m_x \phi_x}{\phi^2} \right).
\end{align*}
\] (1.15)

where \( \hat{\rho} = \frac{\rho}{\phi} \) and \( \hat{m} = \frac{m}{\phi} \) (see Lemma 3.2). This implies that \( \hat{v} = (\hat{\rho}, \hat{m}) \) satisfies the (viscous) isentropic gas-dynamics with the forcing term \(-p \phi_x/\phi^2\) in the momentum equation. Since \( \frac{p^\epsilon \phi_x}{(\phi^\epsilon)^2} \in L^\infty(\Omega) \) uniformly in \( \epsilon > 0 \), thus \( \{\frac{p^\epsilon \phi_x}{(\phi^\epsilon)^2}\}_{\epsilon>0} \) is precompact in \( H_{\text{loc}}^{-1,q}(\Omega) \), \( 1 \leq q < 2 \). Hence, the method of compensated compactness in [Di1],[Ch] can be applied to the functions \( (\hat{\rho}^\epsilon, \hat{m}^\epsilon) \) to show that \( \nu_{t,x} \) is a point mass provided that \( 1 < \gamma \leq 5/3 \).

Regarding work on existence of weak solutions for conservation laws, we refer the reader to an excellent treatise by DiPerna [Di3] and references therein. Concerning basic framework on conservation laws, we refer the reader to [La],[Sm] and for the functional analytic framework of compensated compactness we refer [Mu],[Ta1],[Ev] and [Di2]. For scalar conservation laws the vanishing viscosity method is employed (e.g., in [Ol],[Kr] and references in [Sm]) to define the unique entropy solution. Also, the vanishing viscosity method is used to develop the viscosity solution to the Hamilton-Jacobi equation in [CL]. The finite-difference methods (e.g., Lax-Friedrichs and Gudunov schemes) are also used to construct weak solutions to a scalar and \( 2 \times 2 \) system of conservation laws (e.g., see [Di2],[Ch] and [Sm]).

In the case where the initial data have small total variation, Glimm [Gl] proved the global existence of BV-solutions for a general class of hyperbolic systems as the strong limit of random choice approximations. However, the problem of existence of solutions to (1.7) with large initial data is still unsolved. In [CD] the vanishing viscosity method is applied to the system (1.7) under a special class of constitutive relations in Lagrangian coordinates.

### 2 The Viscosity Method

In this section we establish the uniform \( L^\infty \) bound of \( y^\epsilon = (\rho^\epsilon, m^\epsilon, \phi^\epsilon) \).

**Lemma 2.1** If \( \phi \in C^{1,2}([0,T] \times R) \) satisfies \( \phi_t + u \phi_x = \epsilon \phi_{xx} \) then

\[
\min_x \phi_0(x) \leq \phi(t,x) \leq \max_x \phi_0(x).
\]
Proof: Using the same arguments as in the proof of Theorem 2.6, we can show that \( \max_x \phi(t, x) \leq \max_x \phi_0(x) \) and \( \min_x \phi(t, x) \geq \min_x \phi_0(x) \). □

It will be shown in Section 3 (see (3.4) and Lemma 3.1) that the normalized mechanical energy

\[
E(\rho, u, \phi) = \frac{1}{2} \rho(u - \bar{u})^2 + \frac{1}{\gamma} (\rho^\gamma - \gamma \bar{\rho}^{\gamma-1} (\gamma - \bar{\gamma}) - \bar{\rho}^\gamma) \phi^{1-\gamma}
\]

satisfies

\[
\int_{-\infty}^{\infty} E(\rho(t, x), u(t, x), \phi(t, x)) \, dx \leq \int_{-\infty}^{\infty} E(\rho_0(x), u_0(x), \phi_0(x)) \, dx.
\]

The following lemma shows the lower bound of \( \rho_x \).

Lemma 2.2 If \( \rho \in C^{1,2}([0, \tau] \times R) \) satisfies

\[
\rho_t + (u) \rho = \epsilon \rho_{xx}
\]

with \( \rho(0, \cdot) \geq 0 \) and \( u \in C^1(\Omega) \), then \( \rho(t, \cdot) \geq 0 \). Moreover, if \( \rho(0, \cdot) \geq \delta > 0 \) and

\[
\int_0^T \int_{-\infty}^{\infty} \rho |u - u_0|^2 \, dx \, dt \leq \text{const},
\]

then \( \rho(t, \cdot) \geq \delta(\epsilon, \tau) > 0 \) on \( (0, \tau) \).

Proof: Choose \( \psi = \min(\rho(t, x), 0) \). Then we have

\[
\int_{-\infty}^{\infty} \frac{1}{2} \psi(t, x)^2 + \int_0^t \int_{-\infty}^{\infty} \left( \epsilon |\psi_x|^2 - \psi u \psi_x \right) \, dx \, ds = 0.
\]

By the Hölder inequality, we obtain

\[
\int_{-\infty}^{\infty} |\psi(t, 0)|^2 \leq \frac{|u|_{\infty}}{2\epsilon} \int_0^t \int_{-\infty}^{\infty} |\psi|^2 \, dx \, ds
\]

where \( |u|_{\infty} = \sup_{(t, x) \in (0, \tau) \times R} |u(t, x)| \), and the Gronwall’s inequality implies \( \psi = 0 \). Thus, \( \rho \geq 0 \).

Next, we prove \( \rho(t, \cdot) \geq \delta = \delta(\epsilon, \tau) > 0 \) if \( \varphi(0, \cdot) \geq \delta > 0 \) by using the Stampacchia’s lemma (e.g., see [FI],[Tr]), i.e., suppose \( \chi(c) \) is a nonnegative, non-increasing function on \([c_0, \infty)\), and there exist positive constants \( K, s \) and \( t \) such that

\[
\chi(\hat{c}) \leq Ke^{s(\hat{c} - c)^{-s}} \chi(c)^{1+t} \quad \text{for all } \hat{c} > c \geq c_0,
\]

then

\[
\chi(c^*) = 0 \quad \text{for } c^* = 2c_0 \left( 1 + 2\frac{1+t}{s} K \right)^{\frac{1+t}{s}} \chi(c_0)^{\frac{1+t}{s}}.
\]
First, we establish a priori bound. We consider the class $K$ [Di1] of strictly convex $C^2$ functions $h$ with following properties:

$$h(\rho) = h'(\rho) = 0, \quad h(\rho) = \rho^{-\alpha} \text{ on } (0, \bar{\rho}/2) \text{ for some } 0 < \alpha < 1.$$ 

Premultiplying the first equation of (1.10) by $h'(\rho)$ we obtain

$$h(\rho)_t + (h'(\rho)\rho u)_x - h''(\rho)\rho_x \rho u = \epsilon (h(\rho)_{xx} - h''(\rho)\rho_x^2)$$

Integration of this over $(0, t) \times R$ yields

$$\int_{-\infty}^{\infty} h(\rho(t, x)) - h(\rho_0(x)) \, dx + \epsilon \int_0^t \int_{-\infty}^{\infty} h''(\rho)\rho_x^2 \, dx \, dt$$

$$= \int_0^t \int_{-\infty}^{\infty} h''(\rho)\rho_x \rho(u - \bar{u}) \, dx \, dt.$$ 

Note that

$$h''(\rho)\rho_x \rho(u - \bar{u}) \leq \frac{\epsilon}{2} h''(\rho)\rho_x^2 + \frac{1}{2\epsilon} h''(\rho)\rho^2(u - \bar{u})^2.$$ 

Since there exists some constant $\beta > 0$ such that

$$\rho^2 h''(\rho) \leq \beta \rho \quad \text{for } \bar{\rho}/2 \leq \rho \leq M$$

$$\rho^2 h''(\rho) \leq \beta h(\rho) \quad \text{for } 0 < \rho < \bar{\rho}/2$$

it follows that $\rho^2 h''(\rho)(u - \bar{u})^2 \leq \beta (\rho(u - \bar{u})^2 + h(\rho))$. Hence,

$$\int_{-\infty}^{\infty} h(\rho(t, x)) - h(\rho_0(x)) \, dx + \epsilon \int_0^t \int_{-\infty}^{\infty} h''(\rho)\rho_x^2 \, dx \, dt$$

$$\leq \frac{\beta}{2\epsilon} \int_0^t \int_{-\infty}^{\infty} \rho(u - \bar{u})^2 + h(\rho) \, dx \, dt.$$ 

and it follows from (2.4) and Gronwall’s inequality that

(2.5) \quad \int_{-\infty}^{\infty} h(\rho(t, x)) \, dx \leq \text{const on } [0, \tau].$$

Set $\eta = 1/\rho$. Then from (2.3) $\eta$ satisfies

$$\eta_t + u \eta_x - u_x \eta = \epsilon (\eta_{xx} - \frac{2|\eta_x|^2}{\eta}).$$

We further introduce $\hat{\eta} = \epsilon \omega t \eta$ with $\omega > 0$ to be determined later. The equation for $\hat{\eta}$ becomes

(2.6) \quad \hat{\eta}_t + \omega \hat{\eta} + u \hat{\eta}_x - u_x \hat{\eta} = \epsilon (\hat{\eta}_{xx} - \frac{2|\hat{\eta}_x|^2}{\hat{\eta}}).$
Define \( \xi = \xi_c = \max (0, \dot{\eta} - c) \) with \( c \geq c_0 = \delta^{-1} \). Pre-multiplication of (2.6) by \( \xi^3 \) and integration over \( R \) yields

\[
(2.7) \quad \int_{-\infty}^{\infty} \left( \frac{1}{4} \xi^4 + \xi^4 + \frac{3c}{4} |\xi|^{2} \right) \leq - \int_{-\infty}^{\infty} u(5 \xi^3 + 3c \xi^2) \xi_x \, dx.
\]

Note that

\[
- \int_{-\infty}^{\infty} 5u \xi^3 \xi_x \leq - \int_{-\infty}^{\infty} \frac{5}{2} u \xi^2 (\xi_x)^2 \, dx \leq \frac{\epsilon}{4} \int_{-\infty}^{\infty} |\xi_x|^2 \, dx + \frac{25}{4c} |u|^2 \int_{-\infty}^{\infty} |\xi|^4 \, dx
\]

To estimate the second term on the right hand side of (2.7), we define

\[
I_c(t) = \{ x \in R : \xi(t, x) > 0 \} = \{ x \in R : \eta(t, x) > c e^{\omega t} \}.
\]

Then, using the Hölder inequality, we have

\[
- \int_{-\infty}^{\infty} 3c \xi^2 \xi_x \, dx = - \int_{-\infty}^{\infty} \frac{3c}{2} u \xi (\xi_x)^2 \, dx
\]

\[
\leq \frac{3c}{2} |u|_\infty \left( \int_{-\infty}^{\infty} |(\xi_x)^2| \, dx \right)^{1/2} \left( \int_{-\infty}^{\infty} |\xi|^6 \, dx \right)^{1/6} |I_c(t)|^{1/3}
\]

\[
\leq \frac{\epsilon}{4} \int_{-\infty}^{\infty} |(\xi_x)^2| \, dx + \frac{9c^2}{4\epsilon} |u|^2 \left( \int_{-\infty}^{\infty} |\xi|^6 \, dx \right)^{1/3} |I_c(t)|^{2/3}.
\]

Substituting these estimates into (2.7), choosing \( \omega = \frac{\epsilon}{4} + \frac{25}{4c} |u|^2 \), and integrating on the interval \([0, t]\), we obtain

\[
\int_{-\infty}^{\infty} |\xi|^4 \, dx + \epsilon \int_0^t \int_{-\infty}^{\infty} |(\xi_x)^2| + |(\xi_x)^2| \, dx \, ds
\]

\[
\leq \frac{9c^2}{\epsilon} |u|^2_\infty \int_0^t \left( \int_{-\infty}^{\infty} |\xi|^6 \, dx \right)^{1/3} |I_c(s)|^{2/3} \, ds.
\]

Since

\[
|\xi|^2 \leq \sqrt{2} \left( \int_{-\infty}^{\infty} |(\xi_x)^2| + |(\xi_x)^2| \, dx \right)^{1/2},
\]

we have

\[
\left( \int_{-\infty}^{\infty} |\xi|^6 \, dx \right)^{1/3} \leq 2^{1/6} \left( \int_{-\infty}^{\infty} |(\xi_x)^2| + |(\xi_x)^2| \, dx \right)^{1/2}.
\]

Thus, from (2.8),

\[
\int_{-\infty}^{\infty} |\xi|^4 \, dx + \epsilon \int_0^t \int_{-\infty}^{\infty} |(\xi_x)^2| + |(\xi_x)^2| \, dx \, ds
\]

\[
\leq 2^{1/3} \frac{81c^4}{4\epsilon} |u|^4 \int_0^t |I_c(s)|^{4/3} \, ds \leq K c^4 \chi(c)^{4/3}
\]
where we define
\[ \chi(c) = \sup_{t \in [0, \tau]} |I(c)(t)| \quad \text{and} \quad K = 2^{1/3} \frac{81\tau}{4c}. \]

Clearly \( \chi(c) \) is nonnegative, non-increasing on \([c_0, \infty)\). Moreover, for \( \hat{c} > c \),
\[ \int_{-\infty}^{\infty} |\xi|^4 \, dx \geq \int_{I_0} |\xi|^4 \, dx \geq (\hat{c} - c)^4 I_0(t). \]

Hence,
\[ I_0(t) \leq K c^4 (\hat{c} - c)^{-4} \chi(c)^{4/3} \]
and by taking the sup over \( t \) we obtain
\[ \chi(\hat{c}) \leq K c^4 (\hat{c} - c)^{-4} \chi(c)^{4/3} \]

It follows from (2.5) that \( \chi(2/\hat{\rho}) < \infty \) and thus from Stampacchia’s lemma that \( \chi(c^*) = 0 \) for some \( c^* \geq c_0 \) and hence
\[ \eta(t, x) \leq c^* e^{\omega t} \quad \text{and} \quad \rho(t, x) \geq (c^*)^{-1} e^{-\omega t}, \]
where \( c^* \) depends on \( \epsilon \) and \( \tau \). □

The following lemma shows that \(|\dot{\phi}_x(t, \cdot)|\) is uniformly bounded by \( \rho(t, \cdot) \) for every \( t \in [0, \tau] \) under assumption (1.7).

**Lemma 2.3** Assume that \( \phi_0 \) satisfies (1.7) and \( |(\phi_0)_x| \leq c \rho_0 \) in \( R \). Then \( |(\phi_x(t, \cdot)| \leq c \rho(t, x) \) in \( R \) for every \( t \in [0, \tau] \).

**Proof:** If the initial condition \((\rho_0, m_0, \phi_0)\) is sufficiently smooth and \((\phi_0)_x \geq \delta_3\) then the solution to (1.10) satisfies \( \rho, u, \phi \in C^3(\Omega) \) and \( \dot{\phi}_x(t, \cdot) > 0 \). Note that \( \dot{\phi}_x \) satisfies
\[ (\phi_x)_t + (u \phi_x)_x = \epsilon (\phi_x)_{xx}. \]

Then, it is not difficult to show that if we define \( \xi = \log(\frac{\phi}{\rho}) \) then \( \xi \) satisfies
\[ \xi_t + u \xi_x = \epsilon (\xi_{xx} + |(\log \phi_x)_x|^2 - |(\log \rho)_x|^2) \]

Suppose \( \xi(t, x_0) = \max_{\Omega} \xi(t, x) \). Then
\[ \xi_x(t, x_0) = (\log \phi_x)_x(t, x_0) - (\log \rho)_x(t, x_0) = 0 \]
and \( \xi_{xx}(t, x_0) \leq 0 \). Thus, \( \partial_t(\max \xi(t, x)) \leq 0 \), which implies the lemma. Since the solution to (2.9) continuously depends on the initial data \((\phi_0)_x\) the estimate holds for when \((\phi_0)_x \geq 0 \). □

The following lemmas provide the technical properties of the functions \( G_i(t), i = 1, 2 \) defined by (1.11).
Lemma 2.4 If \( y = (\rho, m, \phi) \in C^{1,2}((0,\tau) \times R)^3 \) is a solution to (1.10), then (1.13) holds.

Proof: First, note that the \( 3 \times 3 \) matrix \( M = \nabla F \) has the eigenvalues \( \lambda_1 = \frac{m}{\rho} + \rho^n \phi - \theta, \lambda_2 = \frac{m}{\rho} - \rho^n \phi - \theta \) and \( \nabla_v G_i, \ i = 1, 2 \) are the left-eigenvectors of the sub-matrix \( \nabla_v F \) corresponding to \( \lambda_i \). Thus,

\[
(G_1)_t + \lambda_1 (\nabla G_1 \cdot y_x + \rho^n \phi^{-\theta-1} \phi_x) = \frac{2 \theta}{\gamma} \rho^{\theta-1} \phi^{-2\theta-1} + \mu \rho^n \phi^{-\theta-1} \phi_x = \epsilon \nabla G_1 \cdot y_{xx}.
\]

Since \( \nabla G_1 \cdot y_{xx} = (G_1)_{xx} - \nabla^2 G_1 (y_x, y_x) \) we obtain (1.13) for \( G_1 \). The same calculation applies to \( G_2 \).

\[\square\]

Lemma 2.5 If \( \rho > 0, \phi > 0 \) then \( G_i, \ i = 1, 2 \), are quasi-convex.

Proof: We prove \( G_1 = \frac{m}{\rho} + \frac{1}{\theta} \rho^n \phi^{-\theta} \) is quasi-convex. The same proof applies to \( G_2 \). Note that

\[
\nabla G_1 = \left( \begin{array}{c}
-\frac{m}{\rho^2} + \rho^n \phi^{-\theta} \\
\frac{1}{\rho} \\
-\rho^n \phi^{-\theta-1}
\end{array} \right)
\]

\[
\nabla^2 G_1 = \left( \begin{array}{ccc}
-\frac{2m}{\rho^3} + (\theta-1) \rho^n \phi^{-2\theta} - \frac{1}{\rho^2} - \theta \rho^n \phi^{-\theta-1} & 0 & 0 \\
-\frac{1}{\rho^2} & 0 & 0 \\
-\theta \rho^n \phi^{-\theta-1} & 0 & (\theta+1) \rho^n \phi^{-\theta-2}
\end{array} \right).
\]

If \( r = (X, Y, Z) \) satisfies \( r \cdot \nabla G_1 \) then \( Y = -\frac{m}{\rho} + \rho^n \phi^{-\theta} X + \rho^n \phi^{-\theta-1} Z \). Thus,

\[
\nabla^2 G_1 (r, r) = (\theta+1) \left( \rho^n \phi^{-2\theta} X^2 - 2 \rho^n \phi^{-\theta-1} XZ + \rho^n \phi^{-\theta-2} Z^2 \right)
\]

\[(\theta+1) \rho^n \phi^{-\theta-2} (\phi X - \rho Z)^2 \geq 0.\]

We now state the main result of this section that establishes the uniform \( L^\infty \)-bound of \( (\rho^*, m^*, \phi^*_x) \) in \( \epsilon > 0 \).

Theorem 2.6 Suppose \( \phi_0 \) satisfies (1.7) and \( |(\phi_0)_x| \leq c \rho_0 \) in \( R \). Then, there exists a \( \tau = \tau_c > 0 \) with \( c \to \tau_c \) monotonically decreasing and \( \tau_0 = \infty \) such that

\[
0 \leq \rho \leq \text{const}, |\frac{m}{\rho^*}| \leq \text{const} \text{ and } |\phi_x| \leq \text{const} \text{ in } \Omega = [0, \tau] \times R.
\]

Proof: Suppose that \( (\phi_0)_x \leq \theta \). Then, it follows from Lemmas 2.3-2.6 that \( \max_{x} G_2(t, x) \leq \max_{x} G_2(0, x) = A \). Hence, \( \frac{m}{\rho} + A \geq \frac{1}{\theta} \rho^n \phi^{-\theta} \geq 0 \). Set \( G = A + G_1 \). It then follows from Lemma 2.4 that

\[
G_t + \lambda_1 \nabla G \cdot y_x + \frac{\gamma}{\rho} \rho^n \phi^{-2\theta-1} \phi_x = \epsilon (G_{xx} - \nabla^2 G(y_x, y_x))
\]
Let $G^k = G(kh)$, $h > 0$. Then,

$$G^k - G^{k-1} + \lambda_1 \nabla G^k \cdot y^k + \frac{1}{\gamma} (\rho^k)^{2\theta} (\phi^k)^{2\theta-1} \phi^k_x = \epsilon (G^k_{xx} - \nabla^2 G^k (y^k, y^k)) + \epsilon(h)$$

where $\epsilon(h)/h \to 0$ as $h \to 0^+$. Suppose $G^k(x_0) = \max_x G^k(x)$. Then, $G^k_x(x_0) = 0$ and $G^k_{xx}(x_0) \leq 0$. It follows from Lemmas 2.3 and 2.5 that if $\psi(t) = \max_x G(t, x)$ then

$$\psi(kh) - \psi((k-1)h) - \epsilon(h) \leq \frac{c}{\gamma} \left( \frac{\rho}{\varphi}(x_0) \right)^{2\theta+1} \leq \frac{c}{\gamma} \theta^{(2\theta+1)/\theta} \psi(kh)^{(2\theta+1)/\theta}.$$ 

Taking the limit $h \to 0^+$, we obtain

$$\psi(t) - \psi(0) \leq \int_0^t \frac{c}{\gamma} \theta^{(2\theta+1)/\theta} \psi(\tau)^{(2\theta+1)/\theta} d\tau$$

and thus

$$(2.10) \quad \psi(t) \leq \left( \frac{\psi(0)^{(\theta+1)/\theta}}{1 - \frac{c}{\gamma} \left( \frac{\rho}{\varphi}(x_0) \right)^{2\theta+1}} \right)^{\theta/(\theta+1)}.$$

In fact, if

$$s(t) = \psi(0) + \int_0^t \frac{c}{\gamma} \theta^{(2\theta+1)/\theta} \psi(\tau)^{(2\theta+1)/\theta} d\tau$$

then $\psi(t) \leq s(t)$ and $s \leq \frac{c}{\gamma} \theta^{(2\theta+1)/\theta} s^{(2\theta+1)/\theta}$, which implies (2.10). Since

$$0 \leq \frac{1}{\theta} \rho^\theta \phi^{-\theta} \leq \frac{1}{2} (G_1 + G_2) \quad \text{and} \quad -G_2 \leq \frac{m}{\rho} \leq G_1,$$

the lemma follows from (2.9). □

### 3 Compensated Compactness

In this section we show that the sequence \{$(\rho^\epsilon, m^\epsilon, \phi^\epsilon)$\}$_{\epsilon > 0}$ has a subsequence that converges to a weak solution of (1.10) a.e in $\Omega$ using the method of compensated compactness. First note that the mechanical energy

$$(3.1) \quad \eta = \frac{1}{\rho} \frac{m^2}{2} + \frac{1}{\gamma(\gamma - 1)} \rho^\gamma \phi^{-\gamma+1}$$

and the corresponding entropy-flux

$$(3.2) \quad q = \frac{\rho}{2} \frac{m^3}{\rho} + \frac{1}{\gamma - 1} \frac{m}{\rho} \rho^\gamma \phi^{-\gamma+1}$$
form an entropy pair, i.e.,

$$\nabla \eta \, M = \nabla q$$

(3.3)

In order to treat solutions approaching a nonzero state at infinity, we consider a normalized entropy pair

$$\tilde{\eta} = \eta(y) - \eta((\bar{v}, \phi)) - \nabla_y \eta((\bar{v}, \phi))(v - \bar{v}),$$

$$\tilde{q} = q(y) - q((\bar{v}, \phi)) - \nabla_y \eta((\bar{v}, \phi))F(y)$$

where \(v = (\rho, m), \bar{v} = (\bar{\rho}, \bar{m})\) and \(y = (v, \phi)\). Premultiplying (1.10) by \(\tilde{\eta}\), we obtain

$$\tilde{\eta}_t + \tilde{q}_x = \epsilon (\tilde{\eta}_{xx} - \nabla^2 \eta(y, y_x)).$$

Integration over \(\Omega\) yields an energy estimate

$$\int_{-\infty}^{\infty} \tilde{\eta}(t, x) \, dx + \epsilon \int_0^t \int_{-\infty}^{\infty} \nabla^2 \eta(y, y_x) \, dx \, dt = \int_{-\infty}^{\infty} \tilde{\eta}(0, x) \, dx.$$  

(3.4)

The following lemma implies the energy estimate (2.2) where \(\tilde{\eta}(y) = E(\rho, u, \phi).

**Lemma 3.1** For \(\rho > 0, \phi > 0\), \(\nabla^2 \eta\) is non-negative.

**Proof:** Note that

$$\nabla^2 \eta = \begin{pmatrix}
\frac{m^2}{\rho^3} + \rho^{\gamma - 2}\phi^{-\gamma + 1} & -\frac{m}{\rho^2} & -\rho^{\gamma - 1}\phi^{-\gamma} \\
-\frac{m}{\rho^2} & \frac{1}{\rho} & 0 \\
-\rho^{\gamma - 1}\phi^{-\gamma} & 0 & \rho^{\gamma}\phi^{-\gamma - 1}
\end{pmatrix}.$$

Thus,

$$\nabla^2 \eta(y, y_x) = \frac{1}{\rho} \left( \frac{m}{\rho} \left( \rho_x - m_x \right)^2 + \rho^{\gamma - 2}\phi^{-\gamma - 1}(\phi \rho_x - \rho \phi_x) \right)^2 \geq 0$$

for \(y_x = (\rho_x, m_x, \phi_x)\). □

The following lemma establishes the viscosity estimate which is essential for the method of compensated compactness.

**Lemma 3.2** Assume that \(1 < \gamma \leq 2\) and \(\int_{-\infty}^{\infty} \tilde{\eta}(0, x) \, dx < \infty\). Then, if \((\rho, m, \phi)\) is a solution of (1.10)

$$\epsilon \int_0^\tau \int_{-\infty}^{\infty} (|\rho_x(t, x)|^2 + |m_x(t, x)|^2) \, dx \, dt \leq \text{const}$$

where \(\tau > 0\) is defined in Theorem 2.6.
Proof: From (2.9) and Lemma 2.2 we have
\[ \int_{-\infty}^{\infty} |\phi_x(t,x)| \, dx = \int_{-\infty}^{\infty} |\phi_x(0,x)| \, dx, \quad t \in [0, \tau]. \]
It thus follows from Theorem 2.6 that
\[ \int_0^\tau \int_{-\infty}^{\infty} |\phi_x(t,x)|^2 \, dx \leq \text{const}. \]
Since \( 0 < \rho(t,x), \phi(t,x) \leq \text{const} \) in \( \Omega \) it follows from (3.5) that
\[ \nabla^2 \eta(y_x(t,x),y_x(t,x)) + |\phi_x(t,x)|^2 \geq c_1 |y_x(t,x)|^2 \]
for some \( c_1 > 0 \). Hence, the lemma follows from (3.4). \( \square \)

We apply the method of compensated compactness for the function \( \hat{v}^\varepsilon \) defined by
\[ \hat{v}^\varepsilon = (\hat{\rho}^\varepsilon, \hat{m}^\varepsilon) = \left( \frac{\rho^\varepsilon}{\phi^\varepsilon}, \frac{m^\varepsilon}{\phi^\varepsilon} \right) \]
The function \( \hat{v}^\varepsilon \) satisfies the \( 2 \times 2 \) viscous conservation law (1.15) with the forcing term which is in \( L^\infty(\Omega) \). Based on this observation we have

**Lemma 3.3** Assume that the conditions in Theorem 2.6 are satisfied and that \( \int_{-\infty}^{\infty} \eta(0,x) \, dx < \infty \). Then, for \( 1 < \gamma \leq 2 \), the measure set
\[ \eta(\hat{v}^\varepsilon)_t + q(\hat{v}^\varepsilon)_x \]
lies in a compact subset of \( H^{-1}_{\text{loc}}(\Omega) \) for all weak entropy/entropy flux pair \((\eta, q)\) of \( \nabla_v F \), where \( \hat{v}^\varepsilon = \left( \frac{\rho^\varepsilon}{\phi^\varepsilon}, \frac{m^\varepsilon}{\phi^\varepsilon} \right) \).

**Proof:** Suppose \((\rho, m, \phi)\) is a solution to (1.10). Then, dividing the first two equations of (1.10) by \( \phi \), we obtain (1.15) for \( \hat{\rho} = \frac{\rho}{\phi} \) and \( \hat{m} = \frac{m}{\phi} \). Let \((\eta, q)\) be a weak entropy/entropy flux pair, i.e.,
\[ \nabla \eta \nabla_v F = \nabla q \quad \text{and} \quad \eta(0, \cdot) = 0. \]
It can be shown that for \( 0 < \rho \leq \text{const} \), \[ \frac{|m|}{\rho \phi} \leq \text{const} \]
\[ |\nabla \eta| \leq \text{const} \quad \text{and} \quad |\nabla^2 \eta(r, r)| \leq \text{const} \nabla^2 \eta^* (r, r) \]
where
\[ \eta^* = \frac{1}{2} \rho \left( \frac{m}{\rho} \right)^2 + \frac{1}{\gamma(\gamma - 1)} \rho^\gamma \]
is the mechanical energy, \( r \) is any vector in \( \mathbb{R}^2 \) and constant is independent of \( r \). Premultiplying (1.15) by \( \nabla \eta \), we obtain
\[ \eta(\hat{v})_t + q(\hat{v})_x = \epsilon (\eta(\hat{v})_{xx} - \nabla^2 \eta(\hat{v}_x, \hat{v}_x)) + \nabla \eta(\hat{v})A \]
where

\[ A = 2\epsilon \left( \frac{\rho_x \phi_x}{\phi^2}, \frac{m_x \phi_x}{\phi^2} \right) - \left( 0, \frac{p \phi_x}{\phi^2} \right) \]

It follows from Theorem 2.6 that \( \frac{\rho^e \phi^e}{(\phi^e)^2} \in L^\infty(\Omega) \) uniformly in \( \epsilon > 0 \). It follows from Lemma 3.2 and Theorem 2.6 that

\[ \epsilon^{1/2} \left( \frac{\rho^e \phi^e}{(\phi^e)^2}, \frac{m^e_x \phi^e_x}{(\phi^e)^2} \right) \in L^2(\Omega) \]

uniformly in \( \epsilon > 0 \). Thus, \( \{\nabla \eta(\nu^e)A^e\}_{\epsilon > 0} \) is precompact in \( W_{loc}^{-1,q}(\Omega) \), \( 1 \leq q < 2 \). Since

\[ \int_0^\tau \int_{-\infty}^\infty \epsilon |v^e_x(t,x)|^2 \, dx \, dt \leq \text{const} \]

The set \( \{\epsilon \nabla \eta^e\}_{\epsilon > 0} \) is precompact in \( L^2(\Omega) \) and so is \( \{\epsilon \eta(\nu^e)_{xx}\}_{\epsilon > 0} \) in \( H^{-1}(\Omega) \). Hence, the lemma follows from the fact that if set \( S \) is compact in \( W^{-1,q}(U) \) and bounded in \( W^{-1,r}(U) \) then \( S \) is compact in \( H^{-1}(U) \) for \( 1 \leq q < 2 < r \) and any bounded and open set \( U \) in \( R^2 \). [Ev] \( \square \)

In the next lemma we prove that the sequence \( \{\phi^e\}_{\epsilon > 0} \) is precompact in \( L^2_{loc}(\Omega) \).

**Lemma 3.4** For \( \epsilon > 0 \) and \( \tau > 0 \) defined in Theorem 2.6

\[ \int_0^\tau \int_{-\infty}^\infty (|\phi^e_x|^2 + |\phi^e|^2) \, dx \, dt \leq \text{const}. \]

Thus, the family \( \{\phi^e(t,x)\}_{\epsilon > 0} \) is compact in \( L^2(\Omega) \) for any bounded rectangle \( U = (0, \tau) \times (-L, L) \).

**Proof:** Premultiplying (1.10) by \( \phi^e_{xx} \) and integrating in \( (0, \tau) \times R \), we obtain

\[ \frac{1}{2} \int_{-\infty}^\infty |\phi_x(\tau,x)|^2 \, dx + \frac{\epsilon}{2} \int_0^\tau \int_{-\infty}^\infty |\phi_{xx}|^2 \, dx \, dt \]

where \( |u|_\infty = \sup_{(t,x) \in (0,\tau) \times R} |u(t,x)| \). Thus,

\[ \int_0^\tau \int_{-\infty}^\infty |\phi_{xx}|^2 \, dx \, dt \leq |u|_\infty^2 \int_0^\tau \int_{-\infty}^\infty |\phi_x|^2 \, dx \, dt + \epsilon \int_{-\infty}^\infty |\phi_x(0,x)|^2 \, dx \]

and

\[ \int_0^\tau \int_{-\infty}^\infty |\phi_x|^2 \, dx \, dt \leq 4|u|_\infty^2 \int_0^\tau \int_{-\infty}^\infty |\phi_x|^2 \, dx \, dt + 2\epsilon \int_{-\infty}^\infty |\phi_x(0,x)|^2 \, dx \]
which proves the lemma.

Now, we state the main result of the paper.

**Theorem 3.5** Assume that the conditions in Theorem 2.6 are satisfied and \( \int \bar{\eta}(0, x) \, dx < \infty \). Then, for \( 1 < \gamma \leq 5/3 \), there exists a subsequence of \((\rho^\varepsilon, m^\varepsilon, \phi^\varepsilon)\) such that

\[
(\rho^\varepsilon(t, x), m^\varepsilon(t, x), \phi^\varepsilon(t, x)) \to (\rho(t, x), m(t, x), \phi(t, x)) \quad \text{a.e. in } \Omega = [0, \tau] \times R.
\]

where the triple \((\rho, m, \phi) \in L_+^\infty(\Omega) \times L^\infty(\Omega) \times W^{1,\infty}(\Omega)\) is a weak solution to (1.4).

**Proof:** It follows from Lemma 3.3 that there exists a subsequence of \((\bar{\rho}^\varepsilon, \bar{m}^\varepsilon)\) such that

\[
(\bar{\rho}^\varepsilon(t, x), \bar{m}^\varepsilon(t, x)) \to (\bar{\rho}(t, x), \bar{m}(t, x)) \quad \text{a.e. in } \Omega.
\]

by applying the results of [Di1] and [Ch]. It follows from Lemma 3.4 that using a standard diagonal process, there is a subsequence of \((\bar{\rho}^\varepsilon, \bar{m}^\varepsilon)\) that converges a.e. in \( \Omega \), weakly in \( H^1(\Omega) \) and weakly-star in \( W^{1,1}(\Omega) \) to \( \bar{\rho}(t, x), \bar{m}(t, x) \). Define \( \rho(t, x) = \bar{\rho}(t, x)\phi(t, x), m(t, x) = \bar{m}(t, x)\phi(t, x) \) a.e. \((t, x) \in \Omega. \) Then, the statement (3.8) holds. It follows from the first two equations of (1.10) that

\[
\int_0^\tau \int_{-\infty}^\infty \left( (\rho^\varepsilon, m^\varepsilon) \cdot (\psi_t - \varepsilon \psi_{xx}) + F(\rho^\varepsilon, m^\varepsilon, \phi^\varepsilon) \cdot \psi_x \right) \, dx \, dt = 0
\]

for all \( \psi \in C^\infty_c(\Omega; R^2) \). It thus follows from (3.8) and the dominated convergence theorem that (1.8) is satisfied. It follows from the third equation of (1.10) that

\[
\int_0^\tau \int_{-\infty}^\infty (\phi_t^\varepsilon + u^\varepsilon \phi_x^\varepsilon) \xi \, dx \, dt = 0
\]

for all \( \xi \in C^\infty_c(\Omega; R) \). Since \( u^\varepsilon \to u \) in \( L^2(U) \) for any bounded rectangle \( U = [0, \tau] \times [-L, L] \) and \( \phi^\varepsilon \to \phi \) weakly in \( H^1(\Omega) \) it follows that

\[
\int_0^\tau \int_{-\infty}^\infty (\phi_t + u\phi_x) \xi \, dx \, dt = 0
\]

for all \( \xi \in C^\infty_c(\Omega; R) \). Hence \( \phi \) satisfies (1.4) a.e. in \( \Omega. \)

**Corollary 3.6** Suppose the entropy pair \((\eta, q)\) is defined by (3.1)-(3.2). Then

\[
\int_0^\tau \int_{-\infty}^\infty (\eta \xi_t + q \xi_x) \, dx \, dt \geq 0
\]

for all \( \xi \in C^\infty_c(\Omega; R) \) satisfying \( \xi \geq 0 \). That is, the third equation of (1.1) is replaced by the inequality \( \eta_t + q_x \leq 0 \) in the sense of distributions.
Proof: It follows from (3.3) that
\[
\int_0^T \int_{-\infty}^\infty (\eta' (\xi_t - \epsilon \eta_{xx}) + q' \eta_x) \, dx \, dt = \epsilon \int_0^T \int_{-\infty}^\infty \nabla^2 \eta(y^x, y^t) \xi \, dx \, dt
\]
for all \( \psi \in C_0^\infty(\Omega; \mathbb{R}^2) \) satisfying \( \xi \geq 0 \). It follows from Lemma 3.1 that the right hand side of this equality is nonnegative. Thus, by taking the limit as \( \epsilon \to 0^+ \) we obtain (3.9) \( \Box \)

References


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