Electronic Journal of Differential Equations, Vol. 1997(1997), No. 01, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp (login: ftp) 147.26.103.110 or 129.120.3.113

SUB-ELLIPTIC BOUNDARY VALUE PROBLEMS FOR QUASILINEAR ELLIPTIC OPERATORS

DIAN K. PALAGACHEV & PETER R. POPIVANOV

Abstract. Classical solvability and uniqueness in the Hölder space $C^{2+\alpha}(\overline{\Omega})$ is proved for the oblique derivative problem

$$\begin{cases} a^{ij}(x)D_{ij}u + b(x,\, u,\, Du) = 0 \quad \text{in } \Omega, \\ \partial u/\partial \ell = \varphi(x) \quad \text{on } \partial\Omega \end{cases}$$

in the case when the vector field $\ell(x) = (\ell^1(x), \dots, \ell^n(x))$ is tangential to the boundary $\partial\Omega$ at the points of some non-empty set $S \subset \partial\Omega$, and the nonlinear term $b(x,\, u,\, Du)$ grows quadratically with respect to the gradient Du.

0. Introduction

The paper is devoted to the study of so-called *oblique derivative problem* firstly posed by H. Poincaré ([Poi]): *given a domain Ω, find a solution in Ω of an elliptic differential equation that satisfies boundary condition in terms of directional derivative with respect to a vector field ℓ defined on the boundary $\partial\Omega$.* More precisely, we shall be concerned with the problem

$$\begin{cases} a^{ij}(x)D_{ij}u + b(x,\, u,\, Du) = 0 \quad \text{in } \Omega, \\ \partial u/\partial \ell \equiv \ell^i(x)D_i u = \varphi(x) \quad \text{on } \partial\Omega \end{cases} \tag{0.1}$$

in the degenerate (tangential) case, i.e. a situation when the vector field $\ell(x) = (\ell^1(x), \dots, \ell^n(x))$ prescribing the boundary operator becomes tangential to $\partial\Omega$ at the points of some non-empty set S. This way, the well-known Shapiro–Lopatinskii complementary condition is violated on the set S and the classical theory (cf. [G-T]) cannot be applied to the problem (0.1).

The linear tangential problem ($b(x,\, z,\, p) = b^i(x)p_i + c(x)z$) has been very well studied in the last three decades. The pioneering works of Bicadze [B] and Hörmander [H] indicated how the solvability and uniqueness properties depend on the way in which the normal component of $\ell(x)$ changes its sign across S. More precisely, suppose S to be a submanifold of $\partial\Omega$ of co-dimension one, and let $\ell(x) =$

1991 *Mathematics Subject Classification.* 35J65, 35R25.
Key words and phrases. Quasilinear elliptic operator, degenerate oblique derivative problem, sub-elliptic estimates.

Submitted October 28, 1996. Published January 8, 1997.
Partially supported by the Bulgarian Ministry of Education, Science and Technologies under Grant MM–410.

$\tau(x)+\gamma(x)\nu(x)$. Here $\nu(x)$ is the unit outward normal to $\partial\Omega$ and $\tau(x)$ is a tangential field to $\partial\Omega$ such that $|\ell(x)| = 1$. There are three possible behaviors of $\ell(x)$ near the set $S = \{x \in \partial\Omega\colon \gamma(x) = 0\}$:

a) *$\ell(x)$ is of neutral type:* $\gamma(x) \geq 0$ or $\gamma(x) \leq 0$ on $\partial\Omega$;

b) *$\ell(x)$ is of emergent type:* the sign of $\gamma(x)$ changes from $-$ to $+$ in the positive direction on τ-integral curves through the points of S;

c) *$\ell(x)$ is of submergent type:* the sign of $\gamma(x)$ changes from $+$ to $-$ along the τ-integral curves through S.

Hörmander's results were refined by Egorov and Kondrat'ev [E-K] who proved that the linear problem (0.1) is of Fredholm type in the neutral case *a)*. Moreover, they showed that either the values of u should be prescribed on S in order to get uniqueness in the case *b)*, or to accept jump discontinuity on S in order to have existence in the case *c)*. What is the universal property of the linear problem (0.1), however, no matter the type of $\ell(x)$, is that a loss of regularity of the solution occurs in contrast to the regular ($S = \emptyset$) oblique derivative problem.

Later, precise studies were carried out in order to indicate the exact regularity that a solution of the linear problem (0.1) gains on the data both in Sobolev and Hölder spaces. We refer the reader to [E], [M], [M-Ph], [Gu], [Sm], [W1]–[W4], and most recently to [Gu-S1] and [Gu-S2].

The investigations on the quasilinear problem (0.1) (especially, in the weak nonlinear case $b(x, z, p) = b^i(x, z)p_i + c(x, z)$) were initiated by the papers [P-K1] and [P-K2]. In our previous study [P-Pa], classical solvability results were obtained for (0.1) both in the cases of neutral and emergent C^∞-vector field $\ell(x)$ supposing C^∞ structure of the elliptic operator. Moreover, we assumed in [P-Pa] that $\ell(x)$ has a contact of order $k < \infty$ with $\partial\Omega$, and $|b|$, $|b_x| = O(|p|^2)$, $|b_z| = o(|p|^2)$, $|b_p| = o(|p|)$ as $|p| \to \infty$, uniformly on x and z.

The general aim of the present article is to improve the results of [P-Pa] weakening the growth assumptions on $b(x, z, p)$ with respect to p. Let us note that although our results here hold true both for neutral and emergent fields $\ell(x)$, for the sake of simplicity we have restricted ourselves to consider the case of emergent field only. (Detailed exposition of the study on degenerate problem with a neutral vector field ℓ can be found in [Pa-P].) That is why, according to the above mentioned result of Egorov and Kondrat'ev, we consider the problem (0.1) supplied with the extra condition

$$u = \psi(x) \quad \text{on the set of tangency } S. \tag{0.2}$$

Concerning the problem (0.1), (0.2), we prove its solvability and uniqueness in the Hölder space $C^{2+\alpha}(\overline{\Omega})$ assuming $a^{ij} \in C^\alpha(\overline{\Omega})$, $b(x, z, p) \in C^\alpha(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $\ell^i \in C^{2+\alpha}(\partial\Omega)$ and $|b(x, z, p)| \leq \mu(|u|)(1 + |p|^2)$ with a non-decreasing function μ (no growth assumptions on the derivatives of b are required!). Further on, suitable conditions due to P. Guan and E. Sawyer [Gu-S2] and concerning behavior of $\ell(x)$ on $\partial\Omega$ are imposed. It is worth noting that our growth condition on $b(x, z, p)$ includes these in [P-Pa], as well as the natural structural conditions in the treatment of regular oblique derivative problems for nonlinear elliptic equations (see [L-T]).

The main tool in proving our results is the Leray–Schauder fixed point theorem, that reduces solvability of (0.1), (0.2) to the establishment of an a priori $C^{1+\beta}(\overline{\Omega})$-estimate for the solutions of related problems. The bound for $\|u\|_{C^0(\overline{\Omega})}$ is a simple consequence of the maximum principle. In order to estimate the $C^\beta(\overline{\Omega})$-norm of the gradient Du, we use an approach due to F. Tomi [To] (see [A-C] also) that imbeds the problem (0.1), (0.2) into a family of similar problems depending on a parameter $\rho \in [0,1]$ and having solutions $u(\rho;x)$. Then the norm $\|Du\|_{C^\beta(\overline{\Omega})} = \|D_x u(1;x)\|_{C^\beta(\overline{\Omega})}$ can be estimated in terms of $\|D_x u(0;x)\|_{C^\beta(\overline{\Omega})}$ after iterations on ρ, assuming the difference $u(\rho_1;x) - u(\rho_2;x)$ to be under control for small $\rho_1 - \rho_2$. To realize this strategy, we use the refined sub-elliptic estimates in Sobolev and Hölder spaces proved very recently by Guan and Sawyer [Gu-S2]. At the end, uniqueness for the solutions of (0.1), (0.2) follows by the maximum principle.

Acknowledgements. The authors are indebted to Professor Pengfei Guan for supplying them with the text of manuscript [Gu-S2] before its publication.

1. Statement of the problem and main results

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. On the boundary $\partial\Omega$ a unit vector field $\ell(x) = (\ell^1(x), \dots, \ell^n(x))$ is defined, which can be decomposed into

$$\ell(x) = \tau(x) + \gamma(x)\nu(x) \qquad x \in \partial\Omega,$$

where $\nu(x)$ is the unit outward normal to $\partial\Omega$ and $\tau(x)$ is the tangential projection of $\ell(x)$ on $\partial\Omega$. Let

$$S = \{x \in \partial\Omega\colon\ \gamma(x) = 0\}$$

be the set of tangency between $\ell(x)$ and $\partial\Omega$. Throughout the paper we consider the case $S \not\equiv \emptyset$. In order to describe our technique, we shall consider the case of emergent field $\ell(x)$ only. In other words, we suppose that $\gamma(x)$ changes its sign from $-$ to $+$ in the positive direction on the τ-integral curves passing through the points of S. Moreover, to avoid unessential complications, we assume that S is a closed submanifold of $\partial\Omega$, $\mathrm{codim}_{\partial\Omega} S = 1$, $\partial\Omega = \partial\Omega_+ \cup \partial\Omega_- \cup S$ where $\partial\Omega_\pm = \{x \in \partial\Omega\colon \gamma(x) \gtrless 0\}$, and let the field $\ell(x)$ be strictly transversal to S at each point $x \in S$ (indeed, $\ell \equiv \tau$ there).

We aimed to study the classical solvability of the degenerate oblique derivative problem:

$$\begin{cases} a^{ij}(x)D_{ij}u + b(x,u,Du) = 0 & \text{in } \Omega, \\ \partial u/\partial\ell \equiv \ell^i(x)D_i u = \varphi(x) \quad \text{on } \partial\Omega, & u = \psi(x) \quad \text{on } S. \end{cases} \tag{1.1}$$

Hereafter, the standard summation convention is adopted and Du denotes the gradient $(D_1u, \dots, D_nu)$ of $u(x)$ with $D_i \equiv \partial/\partial x_i$. Further on, the symbol $C^{k+\alpha}(\overline{\Omega})$, $k \geq 0$ integer, stands for the Hölder functional space equipped with the norm $\|\cdot\|_{C^{k+\alpha}(\overline{\Omega})}$ (see [G-T]). The letter C will denote a constant, independent of u, that may vary from a line into another.

In order to state our result, we give a list of assumptions.

Uniform ellipticity: there exists a positive constant λ such that

$$a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2 \quad \forall x \in \overline{\Omega},\ \forall \xi \in \mathbb{R}^n, \qquad a^{ij} = a^{ji}; \tag{1.2}$$

Regularity conditions: for some $\alpha \in (0,1)$

$$\begin{cases} a^{ij} \in C^\alpha(\overline{\Omega}),\ b(x,z,p) \in C^\alpha(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n), \\ b(x,z,p) \text{ is continuously differentiable with respect to } z \text{ and } p, \\ \ell^i(x) \in C^{2+\alpha}(\partial\Omega), \quad \partial\Omega \in C^{3+\alpha}, \quad S \in C^{2+\alpha}; \end{cases} \tag{1.3}$$

Monotonicity condition: there exists a positive constant b_0 such that

$$b_z(x,\,z,\,p) \le -b_0 < 0 \quad \forall (x,\,z,\,p) \in \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n \quad (b_z = \partial b/\partial z); \tag{1.4}$$

Quadratic growth with respect to the gradient: there exists a positive and non-decreasing function $\mu(t)$ such that

$$|b(x,\,z,\,p)| \le \mu(|z|)\Big(1+|p|^2\Big) \quad \forall (x,\,z,\,p) \in \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n. \tag{1.5}$$

Denote by $\omega(t,\,x)$ the parameterization of the τ-integral curve passing through the point $x \in \partial\Omega$, i.e. $\frac{d}{dt}\omega(t,\,x) = \tau(\omega(t,\,x))$, $\omega(0,\,x) = x$.

The next notions were introduced by Guan and Sawyer in [Gu-S2].

Definition 1. The vector field $\ell(x)$ satisfies condition $\mathcal{A}_p^{\mp}$ on S if for each $y \in S$ there exist constants $r > 0$, $R^- < 0 < R^+$ such that $\gamma(\omega(R^-,\,x)) \ne 0$, $\gamma(\omega(R^+,\,x)) \ne 0$ for all $x \in S$, $|x-y| < r$ and both of the following conditions hold:

$$\left[\frac{1}{\int_{t_1}^{t_2}\gamma(\omega(t,\,x))\,dt}\int_{t_1}^{t_2}\gamma(\omega(t,\,x))^{\frac{p}{p-1}}\,dt\right]^{p-1} \le C\frac{1}{t_3-t_2}\int_{t_2}^{t_3}\gamma(\omega(t,\,x))\,dt$$

for all $x \in S$, $|x-y| < r$ and all $0 < t_1 < t_2 < t_3 < R^+$ with $\int_{t_1}^{t_2}\gamma(\omega(t,\,x))\,dt = \int_{t_2}^{t_3}\gamma(\omega(t,\,x))\,dt$, and also

$$\left[\frac{1}{\int_{t_2}^{t_3}|\gamma(\omega(t,\,x))|\,dt}\int_{t_2}^{t_3}|\gamma(\omega(t,\,x))|^{\frac{p}{p-1}}\,dt\right]^{p-1} \le C\frac{1}{t_2-t_1}\int_{t_1}^{t_2}|\gamma(\omega(t,\,x))|\,dt$$

for all $x \in S$, $|x-y| < r$ and all $R^- < t_1 < t_2 < t_3 < 0$ with $\int_{t_1}^{t_2}|\gamma(\omega(t,\,x))|\,dt = \int_{t_2}^{t_3}|\gamma(\omega(t,\,x))|\,dt$.

Definition 2. The vector field $\ell(x)$ satisfies the condition $\mathcal{T}_\theta$ if

$$t_2 - t_1 \le C\left(\int_{t_1}^{t_2}|\gamma(\omega(t,\,x))|\,dt\right)^{\theta}$$

for all $t_1 < t_2$ and $x \in \partial\Omega$.

Our final assumption concerns the behavior of $\ell(x)$ on $\partial\Omega$:

$$\begin{cases}\text{The vector field } \ell(x) \text{ satisfies conditions } \mathcal{A}_q^{\mp} \text{ and } \mathcal{T}_\theta \\ \text{for some } q > n \text{ and } \theta \in [0,\,1), \quad \theta \ne \alpha.\end{cases} \tag{1.6}$$

We are in a position now to state the main result of the paper.

Theorem 1.1. *Suppose assumptions* (1.2) – (1.6) *to be fulfilled.*

Then the degenerate oblique derivative problem (1.1) *admits a unique classical* $C^{2+\alpha}(\overline{\Omega})$ *solution for each* $\varphi \in C^{2+\alpha-\theta}(\partial\Omega)$ *and* $\psi \in C^{2+\alpha}(S)$.

Remark 1.2. 1. The requirements in (1.3) on $b(x,\,z,\,p)$ to be differentiable with respect to z and p may be replaced by its Lipschitz continuity in z and p.

2. The quadratic growth assumption (1.5) includes for example the natural conditions in studying regular oblique derivative problems for fully nonlinear elliptic operators (cf. [L-T]), as well as the structure conditions on $b(x,\,z,\,p)$ imposed in [P-Pa].

3. Conditions $\mathcal{A}_p^{\mp}$ and $\mathcal{T}_\theta$ correspond to the requirement of "finite type" vector field ℓ in the C^∞ case (cf. [Gu], [P-Pa], [Gu-S1]). In fact, supposing $\partial\Omega \in C^\infty$, $\ell \in C^\infty$, we say that the field $\ell(x)$ is of finite type if there exists an integer k, such that

$$\sum_{i=1}^{k} \left| \frac{\partial^i}{\partial t^i} \gamma(\omega(t,\, x)) \Big|_{t=0} \right| > 0 \quad \text{for all } x \in \partial\Omega.$$

(Indeed, the number k is exactly the order of contact between $\ell(x)$ and $\partial\Omega$.) Now, if ℓ is of type k, then [Tr, Lemma C.1] implies condition $\mathcal{T}_\theta$ with $\theta = 1/(k+1)$. Moreover, it follows from [Gu-S1] that the $\mathcal{A}_p^{\mp}$ condition is satisfied for all p in the range $(1,\, \infty)$.

4. Careful analysis on the condition $\mathcal{T}_\theta$ shows that, if it is satisfied by a field $\ell(x)$ which becomes tangential to $\partial\Omega$ then the exponent θ is necessary strictly less than one.

2. Some preliminaries

For the sake of completeness we will sketch in this section some of the results proved by Guan and Sawyer in [Gu-S2].

Define the linear uniformly elliptic operator

$$\mathcal{L} \equiv a^{ij}(x) D_{ij} + b^i(x) D_i + c(x)$$

with $C^\alpha(\overline{\Omega})$ coefficients $(0 < \alpha < 1)$ and assume $\ell(x)$ to be an emergent type vector field as in the preceding section, with (1.2) and (1.3) being fulfilled.

Let us consider the linear tangential oblique derivative problem

$$\begin{cases} \mathcal{L}u \equiv a^{ij}(x) D_{ij}u + b^i(x) D_i u + c(x) u = f(x) & \text{in } \Omega, \\ \partial u/\partial\ell = g(x) \quad \text{on } \partial\Omega, \qquad u = h(x) & \text{on } S. \end{cases} \tag{2.1}$$

The following result is a special case of [Gu-S2, Theorem 10] that concerns the properties of the problem (2.1) in Hölder spaces.

Lemma 2.1. *Let the field ℓ satisfy condition $\mathcal{T}_\theta$ for some $\theta \geq 0$, and $c(x) \leq 0$.*

Then for each $(f,\, g,\, h) \in C^\alpha(\overline{\Omega}) \times C^{2+\alpha-\theta}(\partial\Omega) \times C^{2+\alpha}(S)$ there exists a unique solution $u \in C^{2+\alpha}(\overline{\Omega})$ of the problem (2.1). Moreover, if $u \in C^{2+\alpha'}(\overline{\Omega})$ $(0 < \alpha' < \alpha)$ satisfies (2.1) with f, g and h as above, then $u \in C^{2+\alpha}(\overline{\Omega})$ and there is a constant C (independent of u) such that

$$\|u\|_{C^{2+\alpha}(\overline{\Omega})} \leq C\Big(\|f\|_{C^\alpha(\overline{\Omega})} + \|g\|_{C^{2+\alpha-\theta}(\partial\Omega)} + \|h\|_{C^{2+\alpha}(S)} + \|u\|_{C^0(\overline{\Omega})} \Big). \tag{2.2}$$

To summarize the corresponding results in the Sobolev functional scale, denote by $H_p^s(\Omega)$ and $B^{s,\,p}(\Omega)$ the Sobolev and Besov L^p-spaces, respectively ([Ad]).

Theorem 12 and Remark 3 of [Gu-S2] yield the following

Lemma 2.2. *Let the field $\ell(x)$ satisfy condition $\mathcal{T}_\theta$ on $\partial\Omega$ $(\theta \geq 0)$, condition $\mathcal{A}_p^{\mp}$ on S $(p > 1)$, and $c(x) \leq 0$.*

For each $(f,\, g,\, h) \in L^p(\Omega) \times B^{2-\theta-1/p,\,p}(\partial\Omega) \times B^{2-\theta/p-1/p,\,p}(S)$ there exists a unique solution $u \in H_p^2(\Omega)$ of the problem (2.1), and there is a constant C such that

$$\|u\|_{H_p^2(\Omega)} \leq C\Big(\|f\|_{L^p(\Omega)} + \|g\|_{B^{2-\theta-1/p,\,p}(\partial\Omega)} + \|h\|_{B^{2-\theta/p-1/p,\,p}(S)} + \|u\|_{L^p(\Omega)} \Big). \tag{2.3}$$

The remaining part of this section is devoted to comparison principles for linear and quasilinear elliptic operators.

Lemma 2.3. *Suppose conditions* (1.2) *and* $c(x) \leq 0$ *to be fulfilled and let* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *satisfy*

$$\begin{cases} \mathcal{L}u \equiv a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u \geq 0 \quad \text{in } \Omega, \\ \partial u/\partial \ell = 0 \quad \text{on } \partial\Omega, \qquad u \leq 0 \quad \text{on } S. \end{cases}$$

Then $u \leq 0$ *on* $\overline{\Omega}$.

Proof. We argue by contradiction. If $u(x)$ assumes positive values on $\overline{\Omega}$ then there exists $x_0 \in \overline{\Omega}$ such that $u(x_0) = \max_{\Omega} u > 0$ and the strong interior maximum principle asserts $x_0 \in \partial\Omega$. Further, $u \leq 0$ on S and it remains $x_0 \in \partial\Omega \setminus S$ which is impossible since $\partial u/\partial \ell = 0$ on $\partial\Omega \setminus S$ while the boundary maximum principle yields $|\partial u/\partial \ell| > 0$ at the point x_0 (ℓ is strictly transversal to $\partial\Omega$ on $\partial\Omega \setminus S$). □

Corollary 2.4. *Let* (1.2) *hold true and suppose the function* $b(x, z, p)$ *to be non-increasing in* z *for each* $(x, p) \in \Omega \times \mathbb{R}^n$ *and differentiable with respect to* p *in* $\Omega \times \mathbb{R} \times \mathbb{R}^n$. *Let* $u,\ v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *satisfy*

$$\begin{cases} a^{ij}(x)D_{ij}u + b(x, u, Du) \geq a^{ij}(x)D_{ij}v + b(x, v, Dv) \quad \text{in } \Omega, \\ \partial u/\partial \ell = \partial v/\partial \ell = 0 \quad \text{on } \partial\Omega, \qquad u \leq v \quad \text{on } S. \end{cases}$$

Then $u \leq v$ *on* $\overline{\Omega}$.

Proof. Defining $w = u - v$, we have

$$\mathcal{L}w \equiv a^{ij}(x)D_{ij}w + b^i(x)D_iw \geq 0 \quad \text{on } \{x \in \Omega \colon w(x) > 0\},$$

where

$$b^i(x) = \int_0^1 b_{p_i}(x, v(x), sDw(x) + Dv(x))\, ds.$$

Furthermore,

$$\partial w/\partial \ell = 0 \quad \text{on } \partial\Omega \qquad \text{and} \qquad w \leq 0 \quad \text{on } S.$$

Thus, the assertion of Corollary 2.4 follows by Lemma 2.3. □

3. A priori estimates

Theorem 1.1 will be proved with the aid of the Leray–Schauder fixed point theorem that reduces the classical solvability of (1.1) to the establishment of an a priori estimate in the Banach space $C^{1+\beta}(\overline{\Omega})$ ($\beta \in (0, 1)$ is a suitable number) for all solutions to a family of problems related to (1.1). This section deals with deriving of these estimates.

To making our exposition more clear, we shall start with the homogeneous case, i.e. we take $\varphi \equiv 0$, $\psi \equiv 0$ and consider the problem

$$\begin{cases} a^{ij}(x)D_{ij}u + b(x, u, Du) = 0 \quad \text{in } \Omega, \\ \partial u/\partial \ell = 0 \quad \text{on } \partial\Omega, \qquad u = 0 \quad \text{on } S \end{cases} \tag{3.1}$$

instead of (1.1).

3.1. A priori estimate for $\|u\|_{C^0(\overline{\Omega})}$.

Lemma 3.1. *Suppose the conditions* (1.2), (1.3) *and* (1.4) *to be fulfilled.*

Then
$$\|u\|_{C^0(\overline{\Omega})} \equiv \max_{\overline{\Omega}} |u(x)| \leq \frac{1}{b_0} \max_{\overline{\Omega}} |b(x,\,0,\,0)|$$
for each solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *of the problem* (3.1).

Proof. Choosing the positive constant M such that
$$M \geq \frac{1}{b_0} \max_{\overline{\Omega}} |b(x,\,0,\,0)|,$$
one has
$$\begin{aligned} a^{ij}(x)D_{ij}u + b(x,\,u,\,Du) &\geq -Mb_0 + \max_{\overline{\Omega}} |b(x,\,0,\,0)| \\ &\geq M\int_0^1 b_z(x,\,sM,\,0)\,ds + b(x,\,0,\,0) = b(x,\,M,\,0) \\ &= a^{ij}(x)D_{ij}(M) + b(x,\,M,\,DM) \quad \text{in } \Omega \end{aligned}$$
as consequence of (1.4). Moreover,
$$\partial u/\partial\ell = 0 = \partial M/\partial\ell \quad \text{on } \Omega, \qquad u = 0 < M \quad \text{on } S.$$
Therefore, the comparison principle (Corollary 2.4) implies $u(x) \leq M$ for all $x \in \overline{\Omega}$.

In the same fashion it can be proved $u(x) \geq -M$ $\forall x \in \overline{\Omega}$ that completes the proof. □

3.2. A priori estimate for $\|Du\|_{L^{2q}(\Omega)}$, $q > n$.

In view of the Morrey lemma ($H^2_q(\Omega) \subset C^{2-n/q}(\overline{\Omega})$, $q > n$), the a priori bound for the C^β-Hölder norm of the gradient Du with $\beta = 1 - n/q$ (and therefore, the solvability of (3.1)) is equivalent to an estimate of the $H^2_q(\Omega)$-norm of u. On the other hand, Lemma 2.2 (and especially (2.3)) reduces that bound to a uniform with respect to u estimate of $\|b(x,\,u,\,Du)\|_{L^q(\Omega)}$, which becomes equivalent to an a priori estimate of $\|Du\|_{L^{2q}(\Omega)}$ through the quadratic growth assumption (1.5) and Lemma 3.1. We shall employ a technique inspired by Amann–Crandall's approach (cf. [A-C]) in proving an $L^\infty(\Omega)$ gradient estimate for semilinear elliptic equations.

Lemma 3.2. *Let conditions* (1.2), (1.3), (1.5) *and* (1.6) *be satisfied.*

Then there exists a constant C depending on known quantities only and on $\|u\|_{C^0(\overline{\Omega})}$, *such that*
$$\|Du\|_{L^{2q}(\Omega)} \leq C \tag{3.2}$$
for each solution $u \in C^{2+\alpha}(\overline{\Omega})$ *of the problem* (3.1).

Proof. The function $u \in C^{2+\alpha}(\overline{\Omega})$ solves the equation
$$a^{ij}(x)D_{ij}u + B(x)|Du|^2 - u(x) = F(x) \quad \text{in } \Omega,$$
where
$$\begin{cases} B(x) = \dfrac{b(x,\,u(x),\,Du(x))}{1+|Du|^2} \in C^\alpha(\overline{\Omega}), \\[2ex] F(x) = -u(x) - \dfrac{b(x,\,u(x),\,Du(x))}{1+|Du|^2} \in C^\alpha(\overline{\Omega}). \end{cases} \tag{3.3}$$

For the fixed solution $u(x)$ we imbed (3.1) into the one-parameter family of tangential oblique derivative problems

$$\begin{cases} a^{ij}(x)D_{ij}u(\rho;\,x) + B(x)|Du(\rho;\,x)|^2 - u(\rho;\,x) = \rho F(x) \quad \text{in } \Omega, \\ \partial u(\rho;\,x)/\partial \ell = 0 \quad \text{on } \partial\Omega, \qquad u(\rho;\,x) = 0 \quad \text{on } S \end{cases} \tag{3.4}$$

with solutions $u(\rho;\,x) \in H^2_q(\Omega)$ ($\rho \in [0,\,1]$) if they do exist. Let us point out that $q > n$ and Sobolev's imbedding theorem ensure that the values of $u(\rho;\,x)$ and its derivatives on $\partial\Omega$ are well defined.

Indeed, $u(0;\,x) = 0$ and $u(1;\,x) \equiv u(x)$ is the fixed solution of (3.1). Our aim is to estimate $\|D_x u(\rho_2;\,x)\|_{L^{2q}(\Omega)}$ in terms of $\|D_x u(\rho_1;\,x)\|_{L^{2q}(\Omega)}$ when $\rho_2 - \rho_1 > 0$ is small enough. After that, having in addition the unique solvability of (3.4) in $H^2_q(\Omega)$ for each value $\rho \in [0,\,1]$, it will be easy to derive the desired estimate (3.2) by iteration of the $L^{2q}(\Omega)$-norms of $Du(\rho;\,x)$ for $\rho < 1$.

Step 1. To realize our program, we shall estimate at first the difference between two solutions of (3.4) in terms of the difference between the corresponding values of the parameter ρ. Let $u(\rho_1;\,x)$, $u(\rho_2;\,x) \in H^2_q(\Omega)$ solve (3.4) with $\rho_1 \le \rho_2$. Then

$$\|u(\rho_1;\,x) - u(\rho_2;\,x)\|_{C^0(\overline{\Omega})} \le (\rho_2 - \rho_1)\left[\mu\left(\|u\|_{C^0(\overline{\Omega})}\right) + \|u\|_{C^0(\overline{\Omega})}\right]. \tag{3.5}$$

To prove this, put $w(x) = u(\rho_1;\,x) - u(\rho_2;\,x)$ and observe that $w \in H^2_q(\Omega)$ solves the linearized problem

$$\begin{cases} a^{ij}(x)D_{ij}w + B^i(x)D_i w - w = (\rho_1 - \rho_2)F(x) \quad \text{in } \Omega, \\ \partial w/\partial \ell = 0 \quad \text{on } \partial\Omega, \qquad w = 0 \quad \text{on } S \end{cases} \tag{3.6}$$

with

$$B^i(x) = 2B(x)\int_0^1 \left(sD_i w + D_i u(\rho_2;\,x)\right)\,ds \in C^{\min(\alpha,\,1-n/q)}(\overline{\Omega}).$$

Now, the result of Lemma 3.1 can be applied to (3.6) whence

$$\|w\|_{C^0(\overline{\Omega})} \le (\rho_2 - \rho_1)\max_{\overline{\Omega}}|F(x)| \le (\rho_2 - \rho_1)\left[\mu\left(\|u\|_{C^0(\overline{\Omega})}\right) + \|u\|_{C^0(\overline{\Omega})}\right]$$

by means of (1.5). The only difference we have to point out is that the Aleksandrov–Pucci maximum principle ([G-T, Theorem 9.6]) is to be used ($w \in H^2_q(\Omega) \subset C^{2-n/q}(\overline{\Omega})$, $q > n$) instead of the strong interior maximum principle. The estimate (3.5) is proved.

Remark 3.3. Putting $\rho_1 = \rho_2$ in (3.5) we obtain uniqueness of solutions to (3.4) for each value of $\rho \in [0,\,1]$.

Step 2. Let $\rho_1 < \rho_2$ be two arbitrary numbers and suppose there exist solutions $u(\rho_1;\,x)$ and $u(\rho_2;\,x) \in H^2_q(\Omega)$ of (3.4). The difference $w(x) = u(\rho_1;\,x) - u(\rho_2;\,x) \in H^2_q(\Omega)$ solves

$$\begin{cases} a^{ij}(x)D_{ij}w = (\rho_1 - \rho_2)F(x) - B(x)\left(|Du(\rho_1;\,x)|^2 - |Du(\rho_2;\,x)|^2\right) + w \quad \text{a.e. } \Omega, \\ \partial w/\partial \ell = 0 \quad \text{on } \partial\Omega, \qquad w = 0 \quad \text{on } S, \end{cases}$$

and therefore Lemma 2.2 yields

$$\|w\|_{H^2_q(\Omega)} \le C\Bigg(\|w\|_{L^q(\Omega)} \\ + \Big\|(\rho_1-\rho_2)F(x) - B(x)\Big(|Du(\rho_1;\,x)|^2 - |Du(\rho_2;\,x)|^2\Big)\Big\|_{L^q(\Omega)}\Bigg).$$

The conditions (1.5), (3.3) and (3.5) lead to

$$\|w\|_{H^2_q(\Omega)} \le C\left(1 + \|Dw\|^2_{L^{2q}(\Omega)} + \|Du(\rho_1;\,\cdot)\|^2_{L^{2q}(\Omega)}\right) \tag{3.7}$$

with a new constant C that depends on $\|u\|_{C^0(\overline{\Omega})}$ in addition, but it is independent of $\rho_1-\rho_2$.

We utilize Gagliardo–Nirenberg's interpolation inequality (see [Ga], [N]) and the bound (3.5) in order to obtain

$$\begin{aligned}\|Dw\|^2_{L^{2q}(\Omega)} &\le C\|D^2w\|_{L^q(\Omega)}\|w\|_{L^\infty(\Omega)} \\ &\le C(\rho_2-\rho_1)\left[\mu\left(\|u\|_{C^0(\overline{\Omega})}\right) + \|u\|_{C^0(\overline{\Omega})}\right]\|D^2w\|_{L^q(\Omega)}.\end{aligned}$$

Making use of (3.7) one has

$$\|Dw\|^2_{L^{2q}(\Omega)} \le C\left(1 + (\rho_2-\rho_1)\|Dw\|^2_{L^{2q}(\Omega)} + \|Du(\rho_1;\,\cdot)\|^2_{L^{2q}(\Omega)}\right)$$

with a constant C independent of $\rho_1-\rho_2$.

Now, if $\rho_2-\rho_1\le\varepsilon$ where $C\varepsilon<1/2$, we have

$$\|Du(\rho_2;\,\cdot)\|^2_{L^{2q}(\Omega)} \le C_1 + C_2\|Du(\rho_1;\,\cdot)\|^2_{L^{2q}(\Omega)} \tag{3.8}$$

whenever $\rho_2-\rho_1\le\varepsilon$. In particular, taking $\rho_1=0$ and $\rho_2=\varepsilon$ above, the uniqueness result (Remark 3.3) implies

$$\|Du(\varepsilon;\,\cdot)\|^2_{L^{2q}(\Omega)} \le C_1 \tag{3.9}$$

whenever there exists a solution $u(\varepsilon;\,x)\in H^2_q(\Omega)$ of (3.4) with $\rho=\varepsilon$.

Step 3. The Leray–Schauder fixed point theorem ([G-T, Theorem 11.3]) will be used to prove solvability of the problem (3.4) for $\rho=\varepsilon$. For this goal, define the compact nonlinear operator

$$\mathcal{F}\colon H^1_{2q}(\Omega) \longrightarrow H^2_q(\Omega) \underset{\text{compactly}}{\hookrightarrow} H^1_{2q}(\Omega)$$

as follows: for each $v\in H^1_{2q}(\Omega)$ the image $\mathcal{F}v\in H^2_q(\Omega)$ is the unique solution of the *linear* oblique derivative problem:

$$\begin{cases} a^{ij}(x)D_{ij}(\mathcal{F}v) = \varepsilon F(x) - B(x)|Dv|^2 + v & \text{a.e. } \Omega,\\ \partial(\mathcal{F}v)/\partial\ell = 0 \quad \text{on } \partial\Omega, \qquad \mathcal{F}v = 0 & \text{on } S.\end{cases}$$

Indeed, this problem is uniquely solvable in $H^2_q(\Omega)$ in view of (3.3) and Lemma 2.2. Clearly, each fixed point of $\mathcal{F}$ will be a solution to (3.4) with $\rho=\varepsilon$. The estimate (2.3) shows that $\mathcal{F}$ is a continuous mapping from $H^1_{2q}(\Omega)$ into itself. Moreover, it follows by (3.9) an a priori estimate (uniformly with respect to σ and v) for each solution of the equation $v=\sigma\mathcal{F}v$, $\sigma\in[0,\,1]$, that is equivalent to the problem

$$\begin{cases} a^{ij}(x)D_{ij}v = \sigma\left(\varepsilon F(x) - B(x)|Dv|^2 + v\right) & \text{a.e. } \Omega,\\ \partial v/\partial\ell = 0 \quad \text{on } \partial\Omega, \qquad v = 0 & \text{on } S.\end{cases}$$

Hence, Leray–Schauder's theorem asserts existence of a fixed point of $\mathcal{F}$ that proves solvability in $H^2_q(\Omega)$ of the problem (3.4) with $\rho = \varepsilon$.

To complete the proof of Lemma 3.2, put $\rho_1 = k\varepsilon$ and $\rho_2 = (k+1)\varepsilon$ ($k = 1, 2, \dots$) in (3.8). Applying finitely many times the above procedure we get the desired estimate (3.2) for $u(x) \equiv u(1; x)$. $\square$

Corollary 3.4. *Let conditions* (1.2) – (1.6) *be fulfilled.*

Then there is the bound

$$\|u\|_{H^2_q(\Omega)} \le C$$

for each solution $u \in H^2_q(\Omega)$ *of the problem* (3.1).

Proof. It follows by the estimate (2.3) and Lemmas 3.1 and 3.2. $\square$

Corollary 3.5. *Assume conditions* (1.2) – (1.6) *to be satisfied.*

Then there exists a constant C *such that*

$$\|u\|_{C^{2-n/q}(\overline{\Omega})} \le C \tag{3.10}$$

for each solution $u \in C^{2+\alpha}(\overline{\Omega})$ *of the problem* (1.1) *with* $\varphi \in C^{2+\alpha-\theta}(\partial\Omega)$ *and* $\psi \in C^{2+\alpha}(S)$.

Proof. Taking into account the imbedding $H^2_q(\Omega) \subset C^{2-n/q}(\overline{\Omega})$ for $q > n$, the estimate (3.10) is an immediate consequence of Corollary 3.4 if $u \in C^{2+\alpha}(\overline{\Omega})$ solves the problem (3.1).

To handle with the non-homogeneous problem (1.1) we solve at first the linear problem

$$\begin{cases} a^{ij}(x)D_{ij}\delta = 0 \quad \text{in } \Omega, \\ \partial\delta/\partial\ell = \varphi \quad \text{on } \partial\Omega, \qquad \delta = \psi \quad \text{on } S. \end{cases}$$

Indeed, there exists a unique solution $\delta \in C^{2+\alpha}(\overline{\Omega})$ of that problem by virtue of Lemma 2.1.

Thus, if $u(x)$ solves (1.1) then the function $v = u - \delta$ is a solution of the homogeneous problem

$$\begin{cases} a^{ij}(x)D_{ij}v + b'(x, v, Dv) = 0 \quad \text{in } \Omega, \\ \partial v/\partial\ell = 0 \quad \text{on } \partial\Omega, \qquad v = 0 \quad \text{on } S, \end{cases}$$

where $b'(x, z, p) = b(x, z + \delta(x), p + D\delta(x))$ and conditions of the type (1.4) and (1.5) are fulfilled by $b'(x, z, p)$.

Since the bound (3.10) is satisfied by the function $v(x)$, it will be true for $u(x)$ also, with a new constant C depending on $\|\delta\|_{C^{2+\alpha}(\overline{\Omega})}$ in addition. $\square$

4. Proof of Theorem 1.1

The uniqueness assertion of Theorem 1.1 follows immediately by (1.4) and Corollary 2.4.

To prove existence, Leray–Schauder's fixed point theorem will be used again. Let us set $\beta = 1 - n/q$, and for $v \in C^{1+\beta}(\overline{\Omega})$ consider the linear tangential oblique derivative problem:

$$\begin{cases} a^{ij}(x)D_{ij}u + b(x, v, Dv) = 0 \quad \text{in } \Omega, \\ \partial u/\partial\ell = \varphi \quad \text{on } \partial\Omega, \qquad u = \psi \quad \text{on } S. \end{cases}$$

Since $b(x,\, v,\, Dv) \in C^{\alpha\beta}(\overline{\Omega})$ (cf. (1.3)), it follows by Lemma 2.1 that there exists a unique solution $u \in C^{2+\alpha\beta}(\overline{\Omega})$ of the above problem. This way, a nonlinear operator

$$\mathcal{F}\colon C^{1+\beta}(\overline{\Omega}) \longrightarrow C^{2+\alpha\beta}(\overline{\Omega})$$

is defined by the formula $\mathcal{F}v = u$. The mapping $\mathcal{F}$ is a continuous (in view of (2.2)) and compact ($C^{2+\alpha\beta}(\overline{\Omega}) \hookrightarrow C^{1+\beta}(\overline{\Omega})$ compactly) mapping acting from $C^{1+\beta}(\overline{\Omega})$ into itself. Moreover, the bound (3.10) provides an a priori estimate with a constant C, independent of u and $\sigma \in [0,\, 1]$, for each solution to the equation $u = \sigma\mathcal{F}u$ that is equivalent to the problem

$$\begin{cases} a^{ij}(x)D_{ij}u + \sigma b(x,\, u,\, Du) = 0 & \text{in } \Omega, \\ \partial u/\partial\ell = \sigma\varphi \quad \text{on } \partial\Omega, \qquad u = \sigma\psi & \text{on } S. \end{cases}$$

Therefore, the Leray–Schauder theorem ensures existence of a fixed point $u = \mathcal{F}u \in C^{2+\alpha\beta}(\overline{\Omega})$ that is a solution of (1.1). Finally, the assertion $u \in C^{2+\alpha}(\overline{\Omega})$ follows easily by Lemma 2.1 and by using standard bootstrapping arguments.

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Dian K. Palagachev, Department of Mathematics, Technological University of Sofia, 8 “Kl. Okhridski” blvd., 1756 Sofia, Bulgaria
E-mail address: `dian@bgcict.acad.bg`

Peter R. Popivanov, Institute of Mathematics, Bulgarian Academy of Sciences, “G. Bonchev” str., bl. 8, 1113 Sofia, Bulgaria
E-mail address: `popivano@bgearn.acad.bg`