Electronic Journal of Differential Equations, Vol. **1997**(1997) No. 02, pp. 1–24.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp (login: ftp) 147.26.103.110 or 129.120.3.113

QUALITATIVE BEHAVIOR OF AXIAL-SYMMETRIC SOLUTIONS OF ELLIPTIC FREE BOUNDARY PROBLEMS

Andrew F. Acker & Kirk E. Lancaster

Abstract

A general free boundary problem in $\mathbb{R}^3$ is investigated for axial-symmetric solutions and qualitative geometric properties of the free boundary are compared to those of the fixed boundary for the axial and radial directions. Counterexamples obtained previously by the first author show that our results cannot hold in the same generality as those for similar free boundary problems in $\mathbb{R}^2$.

§0. INTRODUCTION

Let G be the quasilinear, elliptic, second-order partial differential operator on $\mathbb{R}^N$ given by

$$GU = \sum_{i,j=1}^{N} A_{ij}(X, DU(X))D_iD_jU + B(X, DU(X)), \quad X \in \mathcal{O}, \tag{1}$$

for $U \in C^2(\mathcal{O})$, where $\mathcal{O}$ is any open set in $\mathbb{R}^N$, $A_{ij} \in C^{1,\delta}(\mathbb{R}^N \times \mathbb{R}^N)$, $i,j = 1,\dots,N$, satisfies $\sum_{i,j=1}^{N} A_{ij}(X,P)\xi_i\xi_j > 0$ for $X, P \in \mathbb{R}^N$ and $\Xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N \backslash \{0\}$, and $B \in C^{2,\delta}(\mathbb{R}^N \times \mathbb{R}^N)$ for some $\delta \in (0,1)$. For $\mathcal{S}^*$ a closed hypersurface in $\mathbb{R}^N$ and $\mathcal{S}$ a closed hypersurface in $\mathbb{R}^N$ which surrounds $\mathcal{S}^*$, we denote by $\mathcal{O}(\mathcal{S}^*, \mathcal{S})$ the open region between $\mathcal{S}^*$ and $\mathcal{S}$. The purpose of this paper is to study the qualitative geometric properties of axial-symmetric solutions of the following "Bernouli" free boundary problem when $N = 3$.

N-dimensional free boundary problem. Given a closed hypersurface $\mathcal{S}^* \subset \mathbb{R}^N$ and a positive constant λ, find a closed C^1 hypersurface $\mathcal{S} = \mathcal{S}_\lambda \subset \mathbb{R}^N$ which surrounds $\mathcal{S}^*$ and $U = U_\lambda \in C^2(\mathcal{O}) \cap C^0(\overline{\mathcal{O}}) \cap C^1(\mathcal{O} \cup \mathcal{S})$ such that

$$GU = 0 \quad \text{in } \mathcal{O}, \tag{2a}$$

$$U = 1 \quad \text{on } \mathcal{S}^*, \tag{2b}$$

$$U = 0 \quad \text{on } \mathcal{S}, \tag{2c}$$

$$|DU| = \lambda \quad \text{on } \mathcal{S}, \tag{2d}$$

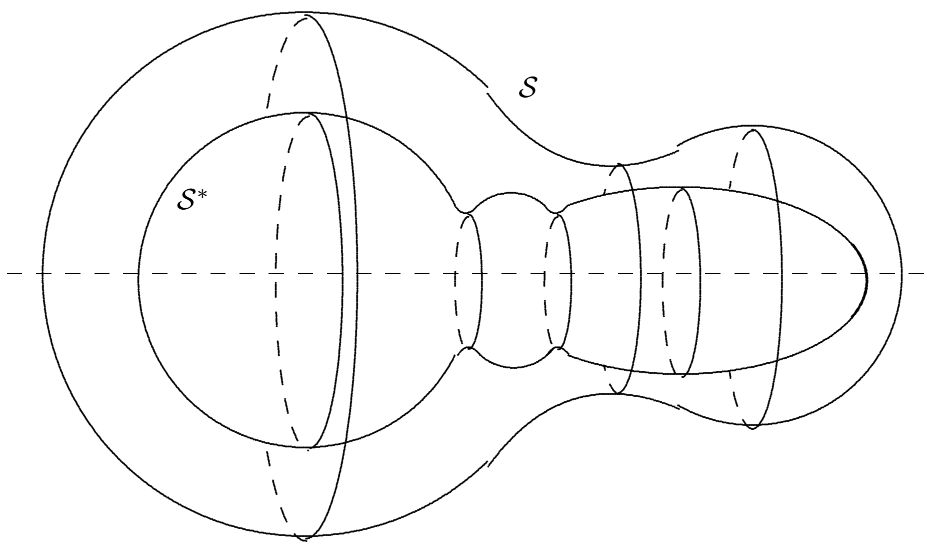
where $\mathcal{O} = \mathcal{O}(\mathcal{S}^*, \mathcal{S})$. We will call $\mathcal{S}^*$ the *fixed boundary* and $\mathcal{S}$ the *free boundary* of this problem.

1991 *Subject Classification:* Primary 35J65, 35R35; Secondary 35B99
Key words and phrases: Free boundary problem, curves of constant gradient direction.

Submitted: September 22, 1996. Published January 8, 1997.

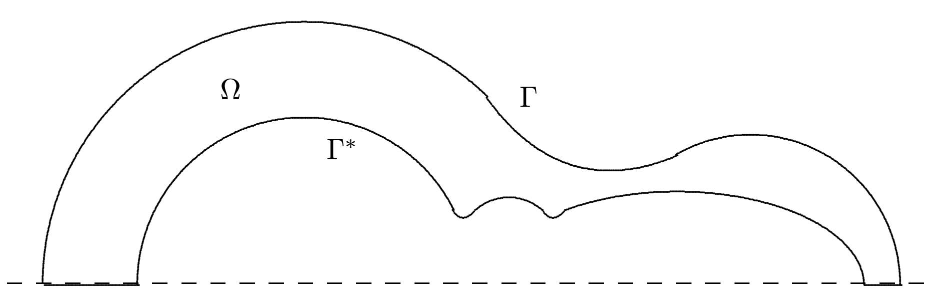
General existence results related to this free boundary problem were obtained by Alt, Caffarelli, and Friedman ([10]) using the method of variational inequalities which is discussed in greater generality in books by Friedman ([12]) and Kinderlehrer ([14]). However, their solutions might not be classical solutions and need not be doubly connected. When G is the Laplace operator and $\mathcal{S}^*$ is starlike relative to all points in a sufficiently small ball, the free boundary problem has a unique, starlike, classical solution $(\mathcal{S}_\lambda, U_\lambda)$ such that $\mathcal{S} = \mathcal{S}_\lambda$ is symmetric with respect to some line whenever $\mathcal{S}^*$ is symmetric with respect to that line ([9]). When G is the p-Laplace operator with $1 < p < \infty$, $\mathcal{S}^*$ is starlike relative to all points in a sufficiently small ball, and $(\mathcal{S}_\lambda, U_\lambda)$ is a classical solution, then it is unique and $\mathcal{S} = \mathcal{S}_\lambda$ is symmetric with respect to some line whenever $\mathcal{S}^*$ is symmetric with respect to that line ([9]).

In the axial-symmetric version of the three-dimensional free boundary problem, the given surface $\mathcal{S}^*$ and the operator G are symmetric with respect to a given axis, which may be taken to be the x_1-axis, and the free boundary $\mathcal{S}$ is also assumed to be symmetric with respect to this axis.



Clearly, this problem is actually two-dimensional, since the surfaces $\mathcal{S}$ and $\mathcal{S}^*$ are generated by the corresponding arcs

$$\Gamma = \{(x,y) : (x,y,0) \in \mathcal{S}, y > 0\} \tag{3a}$$
$$\Gamma^* = \{(x,y) : (x,y,0) \in \mathcal{S}^*, y > 0\}. \tag{3b}$$



Notice that the endpoints of Γ and Γ^* are the intersections of $\mathcal{S}$ and $\mathcal{S}^*$ respectively with the x_1-axis. Our conclusions regarding the qualitative geometric properties of $\mathcal{S}$ and $\mathcal{S}^*$ can then be expressed entirely in terms of the qualitative geometric properties of Γ and Γ^*.

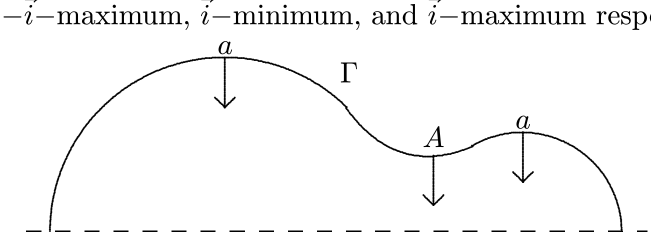
In order to discuss our results, let us adopt the notation of [2] and [4]. Thus Γ and Γ^* are oriented curves with initial and terminal points lying on the x_1-axis such that the x_1-coordinate of each initial point is smaller than the x_1-coordinate of the corresponding terminal point. We define $\vec{n}(x,y)$ to be the unit normal vector to $\Gamma \cup \Gamma^*$ at $(x,y) \in \Gamma \cup \Gamma^*$ which points to the right of the curve (with respect to the direction of the curve). Further, we have the following:

Definition. Given a unit vector $\vec{\nu}$, we call $(x_0,y_0) \in \Gamma$ a $\vec{\nu}$-minimum ($\vec{\nu}$-maximum) of Γ if $\vec{n}(x_0,y_0) = \vec{\nu}$ and (x_0,y_0) is a strict local minimum (maximum) relative to Γ of $f(x,y) = \vec{\nu} \cdot (x,y)$ (see, for example, Figures 2 and 3 in [4]).

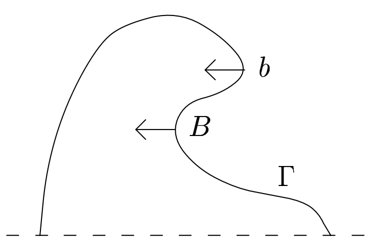
Definition. Given a unit vector $\vec{\nu}$, we call $(x_0,y_0) \in \Gamma^*$ a $\vec{\nu}$-minimum ($\vec{\nu}$-maximum) of Γ^* if $\vec{n}(x_0,y_0) = \vec{\nu}$ and either (x_0,y_0) is a strict local minimum (maximum) relative to Γ of $f(x,y) = \vec{\nu} \cdot (x,y)$ or there is a closed line segment $\gamma^* \subset \Gamma^*$ such that $(x_0,y_0) \in \gamma^*$ and $\vec{\nu} \cdot (x,y) > \ (<) \ \vec{\nu} \cdot (x_0,y_0)$ for $(x,y) \in \Gamma^* \backslash \gamma^*$ near γ^*. Here γ^* is considered as a single local extremum.

We may define $\vec{\nu}-$inflection points of Γ and Γ^* similarly (see [2], [16]). Notice that Lemma 2(b.) implies the definitions of $\vec{\nu}-$extrema are equivalent.

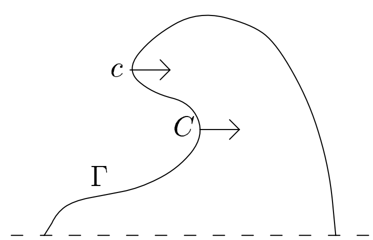
The following figures illustrate the definition of $\vec{\nu}-$extrema of Γ; the letters a, A, b, B, c, C represent points at which Γ has a $-\vec{j}-$minimum, $-\vec{j}-$maximum, $-\vec{i}-$minimum, $-\vec{i}-$maximum, $\vec{i}-$minimum, and $\vec{i}-$maximum respectively.



$-j$ extrema of Γ



$-i$ extrema of Γ



i extrema of Γ

Let us assume that Γ^* contains a finite number of maximal line segments (including isolated points) on which $\vec{n}(x,y) = \pm\vec{i}$ or $\vec{n}(x,y) = -\vec{j}$. Let equation (2a) be either Laplace's equation (i.e. (15)) or the minimal surface equation (i.e. (18)) in $\mathbb{R}^3$, $\mathcal{S}^*$ be a closed surface in $\mathbb{R}^3$, and $(\mathcal{S}, U)$ be a solution of the free boundary problem for some $\lambda > 0$. Suppose $\mathcal{O} = \mathcal{O}(\mathcal{S}^*, \mathcal{S})$ is rotationally symmetric with respect to the x_1-axis and set

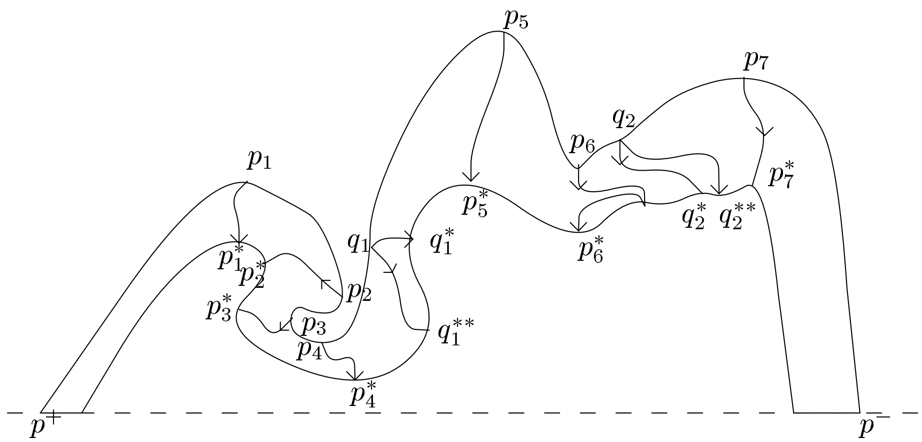
$$W = \{(x,y) \in \mathbb{R}^2 : (x,y,0) \in \mathcal{O}\}. \tag{4}$$

Let $\partial_i W$ and $\partial_o W$ denote the inner boundary and outer boundary of W respectively and let Γ and Γ^* be given by (3). Then our main results, which are given in §1, include the following as a special case:

Theorem 1. *Suppose there exists $u \in C^2(W \cup \partial_o W \cup \Gamma^*) \cap C^1(\overline{W})$ such that*

$$U(x_1, y\cos(\theta), y\sin(\theta)) = u(x_1, y) \tag{5}$$

for $(x_1, y) \in W$ and $\theta \in \mathbb{R}$. Suppose also that Γ^* and $\partial_o W$ are C^2 curves and W satisfies an interior sphere condition at each point of $\partial_i W$. Then Γ has no more $\vec{\nu}-$minima (maxima) than does Γ^* and each $\vec{\nu}-$minimum (maximum) of Γ can be joined to a (distinct) $\vec{\nu}-$minimum (maximum) of Γ^* by a curve along which ∇u has a constant direction, for each $\vec{\nu} = -\vec{\imath}, \vec{\imath}, -\vec{\jmath}$. In particular, if Γ^* is a graph over the $x-$axis, then Γ is also a graph over the $x-$axis.



The study of the relationship between $\vec{\nu}-$extrema of the free and fixed boundaries of solutions of the $N-$dimensional free boundary problem has previously been restricted to the case $N = 2$. In this case, we have a quasilinear elliptic partial differential operator Q on $\mathbb{R}^2$, a constant $\lambda > 0$, and a Jordan curve Γ^* in $\mathbb{R}^2$ and the free boundary problem consists of finding a Jordan curve Γ in $\mathbb{R}^2$ which surrounds Γ^* and a function $u \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma) \cap C^0(\overline{\Omega})$ such that

$$Qu = 0 \quad \text{in} \quad \Omega, \tag{6a}$$

$$u = 1 \quad \text{on} \quad \Gamma^*, \tag{6b}$$

$$u = 0, \ |\nabla u| = \lambda \quad \text{on} \quad \Gamma, \tag{6c}$$

where $\Omega = \mathcal{O}(\Gamma^*, \Gamma)$. The "geometric study" of this two-dimensional free boundary problem began with the consideration of the case in which Q is the Laplace operator. In this case, the principal model for later work was established by the first author in [1], [2], and [4], where a method of curves of constant gradient direction was developed and applied in an analysis of the number and ordering of the directional extrema and inflection points of the free boundary. At approximately the same time, curves of constant gradient direction were independently used to study ideal fluid flows by Friedman and Vogel ([11]). The use of curves of constant gradient direction was extended to solutions of the two-dimensional free boundary problem by Vogel ([18]) and the first author ([3]) when (6a) is Poisson's equation, by the authors when (6a) is the minimal surface equation ([7]) or the heat equation ([8]), and by the second author ([16]) when Q is any elliptic partial differential operator of the form

$$Qu \equiv au_{xx} + 2bu_{xy} + cu_{yy}, \tag{7}$$

where a, b, c depend on x, y, u_x, and u_y. The conclusion obtained (in the elliptic cases) is that if Γ and u constitute a solution of the free boundary problem, Ω is a C^2 domain, and $u \in C^2(\overline{\Omega})$, then each $\vec{\nu}-$extremum of the free boundary can be joined to a corresponding (distinct) $\vec{\nu}-$extremum of the fixed boundary by a curve (γ) along which ∇u remained parallel to $\vec{\nu}$ (i.e. $\nabla u(x, y) = |\nabla u(x, y)|\ \vec{\nu}$ for each (x, y) on γ) and, in particular, Γ has no more $\vec{\nu}-$minima ($\vec{\nu}-$maxima) than does Γ^*, for each $\vec{\nu}$. In addition, the number of $\vec{\nu}-$inflection points of Γ cannot exceed the number of $\vec{\nu}-$inflection points of Γ^*.

When the three-dimensional free boundary problem is symmetric with respect to the x_1-axis and $(\mathcal{S}, U)$ is an axial-symmetric solution, the function $u(x, y) = U(x, y, 0)$ is the solution of a related two-dimensional free boundary problem. In fact, we obtain immediately the following

Proposition. *Suppose $(\mathcal{S}, U)$ is a solution of the three-dimensional free boundary problem, $U \in C^1(\overline{\mathcal{O}})$, and there exists $u \in C^2(W) \cap C^1(\overline{W})$ which satisfies (5) for $(x_1, y) \in W$ and $\theta \in \mathbb{R}$, where $W = \{(x, y) : (x, y, 0) \in \mathcal{O}\}$. Let $x = x_1$ and define $\Omega = \{(x, y) \in W : y > 0\}$ and Q to be the quasilinear, elliptic operator given by*

$$Qu(x, y) = a(x, y, \nabla u)u_{xx} + 2b(x, y, \nabla u)u_{xy} + c(x, y, \nabla u)u_{yy} + d(x, y, \nabla u) \qquad (8)$$

for $u \in C^2(\Omega)$ and $(x, y) \in \Omega$, where $a(x, y, p, q) = A_{11}(x, y, 0, p, q, 0)$, $b(x, y, p, q) = A_{12}(x, y, 0, p, q, 0)$, $c(x, y, p, q) = A_{22}(x, y, 0, p, q, 0)$, and $d(x, y, p, q) = B(x, y, 0, p, q, 0) + \frac{q}{y}A_{33}(x, y, 0, p, q, 0)$. Then u is a solution of free boundary problem (6) when Q is given by (8).

It is natural to conjecture that the results obtained for the two-dimensional free boundary problem (6) with Q given by (7) apply to solutions of the $N-$dimensional free boundary problem for arbitrary $N \geq 3$. Such a generalization, if true, would be esthetically more satisfactory than the (3-dimensional) axial-symmetric results we obtain. However, this conjecture is incorrect, as the first author ([6]) has shown by means of a counterexample in which $N = 3$, $G = \triangle$ is the Laplace operator, $\lambda > 0$, the fixed boundary $\mathcal{S}^*$ has precisely one $\vec{\nu}-$minimum, and the free boundary $\mathcal{S} = \mathcal{S}_\lambda$ has two distinct $\vec{\nu}-$minima, for some direction $\vec{\nu}$.

The study of qualitative properties of axial-symmetric solutions in $\mathbb{R}^3$ is suggested by the facts that the properties in question seem to correspond to two-dimensional problems and axial-symmetric solutions of three-dimensional free boundary problems are of physical interest (e.g. [15]). The results in Theorems 1 and 3 about the directional extrema of Γ would be more appealing if they applied to arbitrary directions in $\mathbb{R}^2$. However, when such a problem is reduced to the two-dimensional free boundary problem (6), the differential operator (8) may contain a lower order term (i.e. d) which complicates the situation. The conjecture that the solution of (6) has the same qualitative properties with regard to arbitrary directions is false. The first author ([6]) has obtained a counterexample when $N = 3$, $G = \triangle$, and $\lambda > 0$ in which the generator Γ^* has only one $\vec{\nu}-$minimum while the free boundary $\Gamma = \Gamma_\lambda$ has two $\vec{\nu}-$minima, for some direction $\vec{\nu}$ (which is not an axial or radial direction). Thus, while our results seem somewhat restricted, the most natural and appealing generalizations are false.

The paper is organized as follows. In §1, we state our main results. In §2, we present some examples of free boundary problems in $\mathbb{R}^3$ to which our results apply.

The statements of our preliminary results, which consist of nine lemmas, are given in §3 and these lemmas are proven in §4; the statements are separated from their proofs in the hope of making the paper more readable. Our main results are proven in §5 and we include some concluding remarks in §6.

§1. MAIN RESULTS

Suppose $(\mathcal{S}, U)$ is a solution of the free boundary problem, $U \in C^1(\overline{\mathcal{O}})$, $\mathcal{O}$ is axial-symmetric, W is given by (4), and there exists $u \in C^2(W) \cap C^1(\overline{W})$ which satisfies (5). Let us write $x = x_1$. We set $\Gamma^* = \{(x,y) : (x,y,0) \in \mathcal{S}^*, y > 0\}$, $\Gamma = \{(x,y) : (x,y,0) \in \mathcal{S}, y > 0\}$, and $\Omega = \{(x,y) \in W : y > 0\}$. Define Q to be the quasilinear, elliptic operator given by (8). Then u is a solution of the free boundary problem (6). We will assume that linear functions of the form $U(x,y,z) = \alpha x + \beta$ are solutions of (2a); this is equivalent to assuming

$$B(x,y,0,p,0,0) = 0. \tag{9}$$

Let us define the ratio of the coefficient a of u_{xx} in Q to the lower order term d in Q to be

$$g(x,y,p,q) = \frac{d(x,y,p,q)}{a(x,y,p,q)} \equiv \frac{qA_{33}(x,y,0,p,q,0) + yB(x,y,0,p,q,0)}{yA_{11}(x,y,0,p,q,0)}.$$

Notice that $g(x,y,p,0) = 0$, and so $\frac{\partial g}{\partial y}(x,y,p,0) = 0$, for all $x \in \mathbb{R}, y > 0$.

Theorem 2. *Let us assume the three-dimensional free boundary problem (2) has a solution $(\mathcal{S}, U)$, U is in $C^2(\mathcal{O} \cup \mathcal{S}) \cap C^1(\overline{\mathcal{O}})$, the solution $(\mathcal{S}, U)$ is axial-symmetric, and condition (9) holds. Let $\partial_i W$ be the inner portion of the boundary of W and assume W satisfies an interior sphere condition at each point of $\partial_i W$. If we define Γ and Γ^* as above and if Γ^* is the graph of a C^1 function, then Γ is the graph of a C^2 function.*

If we are willing to assume that additional conditions are satisfied, we can obtain a result which is stronger than that of Theorem 2. Let us define the function

$$h(x,y,p,q) = \frac{d(x,y,p,q)}{c(x,y,p,q)} \equiv \frac{qA_{33}(x,y,0,p,q,0) + yB(x,y,0,p,q,0)}{yA_{22}(x,y,0,p,q,0)}. \tag{10a}$$

Let us assume that

$$\frac{\partial h}{\partial x}(x,y,0,q) = 0 \tag{10b}$$

and there is a C^1 function $\Phi(y,q)$ satisfying

$$\Phi(y,q) < 0, \tag{10c}$$

$$\frac{\partial \Phi}{\partial y}(y,q) < 0, \tag{10d}$$

$$\frac{\partial \Phi}{\partial q}(y,q) > 0, \tag{10e}$$

$$\frac{\partial \Phi}{\partial y}(y,q) = h(x,y,0,q)\frac{\partial \Phi}{\partial q}(y,q), \tag{10f}$$

$$\frac{q_1}{\Phi(y,q_1)} \le \frac{q_2}{\Phi(y,q_2)} \quad \text{when} \quad q_2 < q_1 < 0 \tag{10g}$$

for $x \in \mathbb{R}, y > 0$, and $q < 0$. If there exist C^0 functions $k : (-\infty, 0) \to (-\infty, 0)$ and $l : (0,\infty) \to (0,\infty)$ such that

$$k(q) \ge \frac{1}{q} \tag{11a}$$

and

$$\frac{d(x,y,0,q)}{c(x,y,0,q)} = \frac{l(y)}{k(q)}, \tag{11b}$$

for $x \in \mathbb{R}, y > 0, q < 0$, then $\Phi(y,q) = K(q)L(y)$ satisfies the conditions (10c)-(10g) above, where

$$K(q) = -\exp\left(\int_{-1}^{q} k(t)dt\right) \tag{11c}$$

and

$$L(y) = \exp\left(\int_{1}^{y} l(t)dt\right). \tag{11d}$$

Recall that we have oriented Γ (Γ^*) so that Ω lies locally to the right of Γ (left of Γ^*). Notice that the definition of the unit normal $\vec{n}$ on ∂W implies

$$\nabla u(x,y) = |\nabla u(x,y)|\ \vec{n}(x,y), \qquad (x,y) \in \Gamma \cup \Gamma^*. \tag{12}$$

We will assume that Γ^* contains a finite number of maximal line segments (including isolated points) on which $\vec{n}(x,y) = \pm\vec{i}$ or $\vec{n}(x,y) = -\vec{j}$.

Theorem 3. *Suppose W is an open, doubly connected region in the plane which is symmetric with respect to the $x-$axis and conditions (9) and (10) hold. Let $\partial_i W$ be the inner portion of the boundary of W and $\partial_o W$ be the outer portion. Let $\Gamma^* = \{(x,y) \in \partial_i W : y > 0\}$ and $\Gamma = \{(x,y) \in \partial_o W : y > 0\}$. Assume $\partial_o W$ and Γ^* are C^2 curves and that W satisfies an interior sphere condition at each point of $\partial_i W$. Let λ be a positive constant.*
Suppose there exists $u \in C^2(W \cup \partial_o W \cup \Gamma^*) \cap C^1(\overline{W})$ such that

$$\begin{aligned} Qu &= 0 && \text{in } \mathrm{W}, \\ u &= 1 && \text{on } \partial_{\mathrm{i}}\mathrm{W}, \\ u &= 0 && \text{on } \partial_{\mathrm{o}}\mathrm{W}, \\ |\nabla u| &= \lambda && \text{on } \partial_{\mathrm{o}}\mathrm{W}, \end{aligned} \tag{13}$$

and $u(x,-y) = u(x,y)$ for $(x,y) \in W$. Let E_1 be the set of $\pm\vec{i}-$extrema of Γ, E_2 be the set of $-\vec{j}-$extrema of Γ, and $E = E_1 \cup E_2$. Also let I_1 be the set of $\pm\vec{i}-$inflection points, I_2 be the set of $-\vec{j}-$inflection points, and $I = I_1 \cup I_2$.
Then every point $p \in E$ can be joined to a point $p^* \in \Gamma^*$ by a directed simple arc $\gamma_p \subset \overline{\Omega}$ (with $\gamma_p \cap \Omega$ piecewise C^1) and every point $q \in I$ can be joined to two

distinct points q^* and q^{**} by directed simple arcs σ_q and σ_{qq} in $\overline{\Omega}$ (with $\sigma_q \cap \Omega, \sigma_{qq} \cap \Omega$ piecewise C^1) such that:

(i) *If $p, q \in E$ and $p \neq q$, then $p^* \neq q^*$ and $\gamma_p \cap \gamma_q = \emptyset$. If $p \in E$ and $q \in I$, then p^*, q^*, q^{**} are distinct and the curves $\gamma_p, \sigma_q, \sigma_{qq}$ are disjoint. If $p, q \in I$, then p^*, p^{**}, q^*, q^{**} are distinct and the curves $\gamma_p, \gamma_{pp}, \sigma_q, \sigma_{qq}$ are disjoint.*

(ii) *If $p = (x, y) \in E$, $p^* = (x^*, y^*)$, and $(s, t) \in \gamma_p$, then $\nabla u(s, t)$ is parallel to $\vec{n}(x, y)$ and so $\vec{n}(x^*, y^*) = \vec{n}(x, y)$.*

(iii) *If $q = (x, y) \in I$ and $\vec{\nu} = \vec{n}(x, y)$, then $\nabla u(s, t)$ is parallel to $\vec{\nu}$ for $(s, t) \in \sigma_q \cup \sigma_{qq}$ and Γ^* has a $\vec{\nu}-$minimum at q^* and a $\vec{\nu}-$maximum at q^{**}.*

(iv) *If $(x, y) \in E$, $\vec{\nu} = \vec{n}(x, y)$, and (x, y) is a $\vec{\nu}-$minimum ($\vec{\nu}-$maximum) of Γ, then (x^*, y^*) is a $\vec{\nu}-$minimum ($\vec{\nu}-$maximum) of Γ^*.*

(v) *Suppose $p = (x, y) \in E_1$ and $\vec{\nu} = \vec{n}(x, y)$. If (x, y) is a $\vec{\nu}-$minimum of Γ, then u_x^2 is strictly increasing on γ_p, $(q - p) \cdot \vec{\nu} > 0$ for each point $q \in \gamma_p$ with $q \neq p$, $|\nabla u(p^*)| > \lambda$, and $0 < (p^* - p) \cdot \vec{\nu} < \frac{1}{\lambda}$.*

(vi) *Suppose $p = (x, y) \in E_1$ and $\vec{\nu} = \vec{n}(x, y)$. If (x, y) is a $\vec{\nu}-$maximum of Γ, then u_x^2 is strictly decreasing on γ_p, $(p^* - q) \cdot \vec{\nu} > 0$ for each point $q \in \gamma_p$ with $q \neq p^*$, $|\nabla u(p^*)| < \lambda$, and $(p^* - p) \cdot \vec{\nu} > \frac{1}{\lambda}$.*

(vii) *If $p = (x, y)$ is a $-\vec{j}-$minimum of Γ and $p^* = (x^*, y^*)$, then $\Phi(y, u_y)$ is strictly decreasing on γ_p, $\Phi(y, -\lambda)\big(v^p(x_1^*, y_1^*) - v^p(x, y)\big) < 1$, and $y > t$ for all points $q = (s, t) \in \gamma_p$ with $q \neq p$, where (x_1^*, y_1^*) is the first point of γ_p at which $y_1^* = y^*$.*

(viii) *If $p = (x, y)$ is a $-\vec{j}-$maximum of Γ and $p^* = (x^*, y^*)$, then $\Phi(y, u_y)$ is strictly increasing on γ_p, $\Phi(y, -\lambda)\big(v^p(x^*, y^*) - v^p(x, y)\big) > 1$, and $t > y^*$ for all points $q = (s, t) \in \gamma_p$ with $q \notin \Gamma^*$. Here*

$$v^p(s, t) = \int_{\gamma(s,t)} \frac{u_y}{\Phi(y, u_y)} dy, \quad (s, t) \in \gamma_p, \tag{14}$$

and $\gamma(s, t)$ is the portion of γ_p between (x_0, y_0) and (s, t).

Corollary. *Let $\Gamma^*, \lambda, \Gamma, \Omega$, and u be as in Theorem 3. Suppose Γ^* is the graph of a C^2 function $g^*(x)$, $\Gamma^* = \{(x, g^*(x))\}$. Then Γ is the graph of a C^2 function $g(x)$ and each point $(x, g(x))$ at which g has a relative maximum (minimum) corresponds to a distinct point $(x^*, g^*(x^*))$ at which g^* has a relative maximum (minimum).*

The proof of Theorem 2 will follow from Lemma 9, which does not depend on assumption (10). The proof of Theorem 3 will make use of nine preliminary lemmas, which constitute the bulk of the paper. Specifically, Lemmas 1, 4, 8, and 9 consider properties of the set $\{(x, y) \in \overline{\Omega} : u_y(x, y) = 0\}$, Lemmas 2, 3, 6, and 7 consider properties of the set $\{(x, y) \in \overline{\Omega} : u_x(x, y) = 0\}$, and Lemma 5 shows that the gradient of u does not vanish on $\overline{\Omega}$. Theorem 1 is a special case of Theorem 3.

§2. EXAMPLES

Laplace's Equation. Suppose G is the Laplacian, so that equation (2a) is

$$U_{x_1 x_1} + U_{x_2 x_2} + U_{x_3 x_3} = 0; \tag{15}$$

then equation (6a) becomes

$$u_{xx}+u_{yy}+\frac{1}{y}u_y=0 \tag{16}$$

and we observe that

$$k(q)=\frac{1}{q},\quad l(y)=\frac{1}{y},\quad \Phi(y,q)=yq. \tag{17}$$

Also $v(x,y)=\ln(y)$, the conclusions of Theorem 3 apply to solutions of (6), and the condition $\Phi(y_0,-\lambda)(v(x,y)-v(x_0,y_0))<(>)1$ becomes $y_0\exp(-(\lambda y_0)^{-1})<(>)y$.

Minimal Surface Equation. Suppose G is the minimal surface operator on $\mathbb{R}^3$, so that equation (2a) becomes

$$\left(1+|DU|^2\right)^{\frac{3}{2}}\operatorname{div}\left(\frac{DU}{\sqrt{1+|DU|^2}}\right)=0. \tag{18}$$

The conclusions of Theorem 3 apply to solutions of (6), since (6a) is

$$(1+u_y^2)u_{xx}-2u_xu_yu_{xy}+(1+u_x^2)u_{yy}+\frac{1}{y}(1+u_x^2+u_y^2)u_y=0, \tag{19}$$

and

$$k(q)=\frac{1}{q(1+q^2)},\quad l(y)=\frac{1}{y},\quad \Phi(y,q)=\frac{yq}{\sqrt{1+q^2}}. \tag{20}$$

A Contrived Equation. Suppose (2a) is

$$U_{x_1x_1}+U_{x_2x_2}+U_{x_3x_3}-\frac{x_2}{x_2^2+x_3^2}U_{x_2}-\frac{x_3}{x_2^2+x_3^2}U_{x_3}=0. \tag{21}$$

Then (6a) becomes

$$u_{xx}+u_{yy}=0 \tag{22}$$

and the results of [2] imply that the geometry of Γ is simpler than that of Γ^* with respect to all $\vec{\nu}-$extrema of Γ.

p-Laplace Equation. Suppose (2a) is

$$\operatorname{div}\left(|DU|^{\mathrm{p}-2}DU\right)=0 \tag{23}$$

for $\mathrm{p}>1$. Then (6a) becomes

$$\operatorname{div}\left(|\nabla u|^{\mathrm{p}-2}\nabla u\right)+\frac{1}{y}|\nabla u|^{\mathrm{p}-2}u_y=0 \tag{24}$$

and

$$k(q)=\frac{1}{q},\quad l(y)=\frac{1}{(\mathrm{p}-1)y},\quad \Phi(y,q)=y^{\mathrm{p}-1}q. \tag{25}$$

If $U:\overline{\mathcal{O}}\to\mathbb{R}$ is a C^2 solution of (2) with $|DU|>0$ on $\overline{\mathcal{O}}$, then the conclusions of Theorem 3 apply to this solution.

A Class of Operators. Suppose (2a) has the form $GU=\sum_{i,j=1}^{3}A_{ij}(X,DU(X))D_iD_jU$, where G is elliptic; hence $B\equiv 0$. Let $(\mathcal{S},U)$ be a solution of the Dirichlet problem (2) with $U\in C^2(\mathcal{O}\cup\mathcal{S})\cap C^1(\overline{\mathcal{O}})$. If U should happen to be axial-symmetric (with respect to the x_1-axis), the conclusions of Theorem 2 would apply to this solution. While our operator G above appears to be quite general, the assumption that U is axial-symmetric may impose some symmetry condition on G.

§3. PRELIMINARY RESULTS

In §3 and §4, we will suppose the assumptions given at the beginning of §1 hold. In particular, we assume u is given by (5) and conditions (9) and (10) hold. Notice, however, that Lemmas 1, 2, 4(a), 4(c), 5, 8, and 9 do not depend on condition (10).

Lemma 1. *Suppose $u \in C^2(\Omega)$ satisfies $Qu = 0$ in Ω. Define $T_0 = \{(x,y) \in \Omega : u_y(x,y) = 0\}$. Suppose $(x_0,y_0) \in T_0$, $|\nabla u(x_0,y_0)| \neq 0$, and $D^2u(x_0,y_0) \neq \vec{0}$. Then locally near (x_0,y_0), the set T_0 is a simple, C^1 curve σ which divides its complement into two connected components on which u_y has opposite signs. Further, u_x^2 is strictly increasing on σ if we choose the forward direction on σ such that $u_x u_y > 0$ locally to the right of σ (or $u_x u_y < 0$ locally to the left of σ).*

Lemma 2.
(a.) Let $\gamma = \{(x,y) \in \overline{\Omega} : u_x(x,y) = 0\}$ and $\Sigma = \{(x,y) \in \Omega \cup \Gamma : u_{xx}(x,y) = u_{xy}(x,y) = 0\}$. Then $\gamma \backslash \Sigma$ is dense in γ.

(b.) Γ does not contain any line segments.

Let us define

$$\phi(x,y) = \Phi(y, u_y(x,y)) \tag{26a}$$

and

$$\psi(x,y) = u_x(x,y). \tag{26b}$$

Lemma 3. *Suppose $u \in C^2(\Omega)$ satisfies $Qu = 0$ in Ω. Define $\Lambda_0 = \{(x,y) \in \Omega : u_x(x,y) = 0\}$. Suppose $(x_0,y_0) \in \Lambda_0$, $|\nabla u(x_0,y_0)| \neq 0$, and $|\nabla u_x(x_0,y_0)| \neq 0$. Then locally near (x_0,y_0), the set Λ_0 is a simple, C^1 curve γ which divides its complement into two connected components on which u_x has opposite signs. Further, ϕ is strictly decreasing on γ if we choose the forward direction on γ such that $u_x > 0$ locally to the right of γ (or $u_x < 0$ locally to the left of γ).*

Lemma 4.
(a.) Let $(x_0,y_0) \in W$ and suppose $u_y(x_0,y_0) = 0$. Define

$$v(x,y) = u(x_0,y_0) + u_x(x_0,y_0)(x - x_0). \tag{27}$$

Then there is an integer $n \geq 2$ such that the zeros of $u - v$ in a neighborhood of (x_0,y_0) lie on n C^1 curves $\delta_1, \ldots, \delta_n$ which divide a neighborhood of (x_0,y_0) into $2n$ disjoint open sectors such that $u - v$ has opposite signs on adjacent sectors and $|\nabla(u - v)| \neq 0$ in a deleted neighborhood of (x_0,y_0).
(b.) Let $(x_1,y_1) \in W$. Suppose $u_x(x_1,y_1) = 0$ and $|\nabla u_x(x_1,y_1)| = 0$. Then there is an integer $m \geq 2$ such that the zeros of u_x in a neighborhood of (x_1,y_1) lie on m C^1 curves $\gamma_1, \ldots, \gamma_m$ which divide a neighborhood of (x_1,y_1) into $2m$ disjoint open sectors such that u_x has opposite signs on adjacent sectors and $|\nabla u_x| \neq 0$ in a deleted neighborhood of (x_1,y_1).
(c.) Let $(x_2,y_2) \in W$. Suppose $u_y(x_2,y_2) = 0$ and $|\nabla u_y(x_2,y_2)| = 0$. Then there is an integer $m \geq 2$ such that the zeros of u_y in a neighborhood of (x_2,y_2) lie on m C^1 curves $\sigma_1, \ldots, \sigma_m$ which divide a neighborhood of (x_2,y_2) into $2m$ disjoint

open sectors such that u_y *has opposite signs on adjacent sectors and* $|\nabla u_y| \neq 0$ *in a deleted neighborhood of* (x_1, y_1).

Lemma 5. $|\nabla u| > 0$ *on* $\overline{\Omega}$.

Lemma 6
(a.) Suppose Γ **has a** $-\vec{j}-$**minimum at** $(x_0, y_0) \in \Gamma$**,** γ **is a directed curve in** $\overline{\Omega}$ **starting at** (x_0, y_0) **along which** $u_x = 0$ **and** ϕ **is strictly decreasing.** *Then* $y < y_0$ *for each point* (x, y) *of* γ *with* $(x, y) \neq (x_0, y_0)$.
(b.) Suppose Γ^* *has a* $-\vec{j}-$*maximum at* $(x^*, y^*) \in \Gamma^*$. *Let* γ *be a directed curve in* $\overline{\Omega}$ *along which* $u_x = 0$ *and* ϕ *is strictly increasing. Suppose* γ *terminates at* (x^*, y^*). *Then* $y \geq y^*$ *for each point* (x, y) *of* γ *and* $y > y^*$ *for each point* (x, y) *of* γ *with* $(x, y) \notin \Gamma^*$.

Lemma 7. *Suppose* Γ *has a* $-\vec{j}-$*maximum* ($-\vec{j}-$*minimum) at* $(x_0, y_0) \in \Gamma$. *Let* $\Lambda = \{(x, y) \in \overline{\Omega} : u_x(x, y) = 0\}$. *Then there exists a directed curve* γ *in* Λ *(with* $\gamma \cap \Omega$ *piecewise* C^1*) starting at* (x_0, y_0) *along which* ϕ *is strictly increasing (decreasing) and which is maximal in the sense that* $\gamma = \sigma$ *whenever* σ *is a directed curve in* Λ *starting at* (x_0, y_0) *along which* ϕ *is strictly increasing (decreasing) and* $\gamma \subset \sigma$. *Further, if* γ *is any such curve, then:*

(a.) γ *does not intersect itself and has no terminal or accumulation points in* Ω.

(b.) γ *does not intersect the* $x-$*axis and intersects* Γ^* *only at points of* Λ.

(c.) If Γ *has* $-\vec{j}-$*minimum at* (x_0, y_0), *then* γ *does not return to* Γ *after leaving* (x_0, y_0) *and terminates at a point* $(x_1, y_1) \in \Gamma^*$ *at which* Γ^* *has a* $-\vec{j}-$*minimum. Further,* $y_1 < y_0$ *and* $\Phi(y_0, -\lambda)\big(v(x_1^*, y_1^*) - v(x_0, y_0)\big) < 1$, *where* (x_1^*, y_1^*) *is the first point of* γ *at which* $y_1^* = y_1$,

$$v(x, y) = \int_{\gamma(x,y)} \frac{u_y}{\Phi(y, u_y)} dy \tag{28}$$

and $\gamma(x, y)$ *is the portion of* γ *between* (x_0, y_0) *and* (x, y).

(d.) If Γ *has* $-\vec{j}-$*maximum at* (x_0, y_0) *and if* γ *does not return to* Γ *after leaving* (x_0, y_0), *then* γ *terminates at a point* $(x_1, y_1) \in \Gamma^*$ *at which* Γ^* *has a* $-\vec{j}-$*maximum. Further,* $y_1 < y_0$ *and* $\Phi(y_0, -\lambda)\big(v(x_1, y_1) - v(x_0, y_0)\big) > 1$, *where* $v(x, y)$ *is defined as in (c.).*

Lemma 8.

(a.) Suppose σ *is a directed curve in* $\overline{\Omega}$ *starting at* $(x_0, y_0) \in \Gamma$ *along which* $u_y = 0$, u_x^2 *is strictly increasing, and* $u_x > 0$ ($u_x < 0$). *For each point* (x, y) *of* σ *with* $(x, y) \neq (x_0, y_0)$, *we have* $x_0 < x$ ($x_0 > x$).

(b.) Suppose σ *is a directed curve in* $\overline{\Omega}$ *along which* $u_y = 0$, u_x^2 *is strictly decreasing, and* $u_x > 0$ ($u_x < 0$). *Suppose* σ *terminates at a point* $(x^*, y^*) \in \Gamma^*$. *For each point* (x, y) *of* σ *with* $(x, y) \neq (x_0, y_0)$, *we have* $x^* < x$ ($x^* > x$).

Lemma 9. *Suppose* T *has a* $\pm\vec{i}-$*maximum* ($\pm\vec{i}-$*minimum) at* $(x_0, y_0) \in \Gamma$. *Let* $\Sigma = \{(x, y) \in \overline{\Omega} : u_y(x, y) = 0\}$. *Then there exists a directed curve* σ *in* T *(with* $\sigma \cap \Omega$ *piecewise* C^1*) starting at* (x_0, y_0) *along which* u_x^2 *is strictly decreasing (increasing)* *and which is maximal in the sense that $\sigma = \sigma_0$ whenever σ_0 is a directed curve in Σ starting at (x_0, y_0) along which u_x^2 is strictly decreasing (increasing) and $\sigma \subset \sigma_0$. Further, if σ is any such curve, then:*

(a.) σ does not intersect itself and has no terminal or accumulation points in Ω.

(b.) σ does not intersect the $x-$axis and intersects Γ^* only at points of T.

(c.) Let $\vec{\nu} = \pm\vec{i}$. If Γ has $\vec{\nu}-$minimum at $p_0 = (x_0, y_0)$, then σ does not return to Γ after leaving (x_0, y_0) and terminates at a point $p^* = (x^*, y^*) \in \Gamma^*$ at which Γ^* has a $\vec{\nu}-$minimum. Also, $(p^* - p_0) \cdot \vec{\nu} < \frac{1}{\lambda}$ and $(q - p_0) \cdot \vec{\nu} > 0$ for each $q = (x, y) \in \sigma$ with $q \neq p_0$.

(d.) Let $\vec{\nu} = \pm\vec{i}$. If Γ has $\vec{\nu}-$maximum at $p_0 = (x_0, y_0)$, then σ does not return to Γ after leaving (x_0, y_0) and terminates at a point $p^* = (x^*, y^*) \in \Gamma^*$ at which Γ^* has a $\vec{\nu}-$maximum. Also, $(p^* - p_0) \cdot \vec{\nu} > \frac{1}{\lambda}$ and $(p^* - q) \cdot \vec{\nu} > 0$ for each $q = (x, y) \in \sigma$ with $q \neq p^*$.

§4. PROOFS OF LEMMAS

Proof of Lemma 1. If we set $r = u_{xx}, s = u_{xy}, t = u_{yy}$, we see that $0 = r(ar + 2bs + ct + d)$ and so $rt - s^2 = -\frac{dr}{c} - \frac{1}{c}(ar^2 + 2brs + cs^2)$. Similarly, we see that $rt - s^2 = -\frac{dt}{a} - \frac{1}{a}(as^2 + 2bst + ct^2)$. Recall that $d(x, y, p, q) = B(x, y, 0, p, q, 0) + \frac{q}{y} A_{33}(x, y, 0, p, q, 0)$ and $B(x, y, 0, p, 0, 0) = 0$, so $d = 0$ on T_0. Since Q is elliptic, $a\xi_1^2 + 2b\xi_1\xi_2 + c\xi_2^2 > 0$ if and only if $\vec{\xi} = (\xi_1, \xi_2) \neq 0$. Thus

$$u_{xx}u_{yy} - u_{xy}^2 \leq 0 \quad \text{on} \quad T_0. \tag{29}$$

Also, on T_0, $rt - s^2 = 0$ iff $r = s = t = 0$. Since $D^2u(x_0, y_0) \neq 0$, $rt - s^2 < 0$ near (x_0, y_0) on T_0. Then $|\nabla u_y(x_0, y_0)| > 0$ and so the first part follows from the implicit function theorem. The monotonicity of u_x^2 follows from Lemma 1 of [16] with the choice $\alpha = 0$ or $\alpha = \pi$. Q.E.D.

Proof of Lemma 2. (a.) Notice that $\Sigma \neq \overline{\Omega}$ since u cannot be a linear function. Let $\text{int}(\gamma)$ be the interior of γ in $\mathbb{R}^2$. If $\text{int}(\gamma) \neq \emptyset$, then the proof of Theorem 8.19 of [13] implies $u_x \equiv 0$, which is a contradiction. Suppose $\gamma \backslash \Sigma$ is not dense in γ. Then there exists a connected set $K \subset \gamma$ which is relatively open in γ such that $\overline{K} \subset \gamma \cap \Sigma$. Choose a point $(x_1, y_1) \in \Omega \backslash \gamma$ such that $\text{dist}((x_1, y_1), K) < \text{dist}((x_1, y_1), \partial\Omega \cup \gamma \backslash K)$, which is possible since γ is a closed set. Let $r = \text{dist}((x_1, y_1), K) > 0$ and let $B = B((x_1, y_1), r)$. Then $\partial B \cap K \neq \emptyset$ and $B \cap \gamma = \emptyset$. Let $(x_2, y_2) \in \partial B \cap K$. Then $u_x > 0$ or $u_x < 0$ in B and $u_x(x_2, y_2) = 0$; the Hopf boundary point lemma ([13]) implies $\frac{\partial}{\partial \vec{\eta}}(u_x) \neq 0$ at (x_2, y_2), where $\vec{\eta}$ is a unit normal direction to ∂B at (x_2, y_2). This contradicts the fact that $u_{xx} = u_{xy} = 0$ on K.

(b.) Suppose first that γ is a line segment parallel to the $x-$axis (i.e. a horizontal line segment). Then $u_x = u_{xx} = 0$ on λ. Also u_y is constant $(= \pm\lambda)$ on γ and so $u_{xy} = 0$ on γ, in contradiction to (a.). If Γ contains a line segment σ, we may rotate Ω so that σ is horizontal, thereby possibly changing Q, and apply the argument above to obtain a contradiction. Q.E.D.

Proof of Lemma 3. The first part follows from the implicit function theorem.

Notice that $\nabla\psi = (u_{xx}, u_{xy})$ and

$$\nabla\phi = \Big(\frac{\partial\Phi(y,u_y)}{\partial q}u_{xy}, \frac{\partial\Phi(y,u_y)}{\partial q}u_{yy} + \frac{\partial\Phi(y,u_y)}{\partial y}\Big). \tag{30}$$

From the proof of Lemma 1, we see that

$$u_{xx}u_{yy} - u_{xy}^2 \le \frac{-d}{c}u_{xx} \tag{31}$$

with equality only when $u_{xx} = u_{xy} = 0$. Now $(\nabla\psi)^{\perp} \equiv (-u_{xy}, u_{xx})$ is a forward tangent vector to γ and

$$\begin{aligned} \nabla\phi\cdot(\nabla\psi)^{\perp} &= \frac{\partial\Phi}{\partial q}(u_{xx}u_{yy} - u_{xy}^2) + \frac{\partial\Phi}{\partial y}u_{xx} \\ &< (\frac{\partial\Phi}{\partial y} - \frac{d}{c}\frac{\partial\Phi}{\partial q})u_{xx} \\ &= 0 \end{aligned} \tag{32}$$

since $\frac{\partial\Phi}{\partial q} > 0$ and $c(x,y,0,q)\Phi_y(y,q) = d(x,y,0,q)\Phi_q(y,q)(y,q)$. Q.E.D.

Proof of Lemma 4. Let us define the operator R by

$$Rw = a^0w_{xx} + 2b^0w_{xy} + c^0w_{yy} + d, \tag{33}$$

where $a^0(x,y) = a(x,y,u_x(x,y),u_y(x,y))$, $b^0(x,y) = b(x,y,u_x(x,y),u_y(x,y))$, and $c^0(x,y) = c(x,y,u_x(x,y),u_y(x,y))$. Then $R(u-v) = 0$ and $|\nabla(u-v)| = 0$ at (x_0,y_0). Since $d(x,y,0,0) = 0$, (a.) follows from the Proposition in [17].

Since $h_x(x,y,0,q) = 0$ and $h \in C^{1,\delta}(\mathbb{R}^4)$, $h_x(x,y,p,q) = ph_1(x,y,p,q)$ for some function $h_1 \in C^{0,\delta}(\mathbb{R}^4)$. Also, since $u_x(x_1,y_1) = 0$ and $u \in C^{2,\delta}(\Omega)$, we see that

$$u_x(x,y) = \mu(x,y)u_{xx}(x,y) + \chi(x,y)u_{xy}(x,y) \tag{34}$$

for some functions $\mu,\chi \in C^{0,\delta}$ in some neighborhood of (x_1,y_1) with $\mu(x_1,y_1) = \chi(x_1,y_1) = 0$. Now define

$$\begin{aligned} e^0(x,y,p,q) = c^0(x,y)\Big(\Big(\frac{a^0}{c^0}\Big)_x + h_p(x,y,u_x,u_y) + \mu(x,y)h_1(x,y,u_x,u_y)\Big)p \\ + c^0(x,y)\Big(\Big(\frac{2b^0}{c^0}\Big)_x + h_q(x,y,u_x,u_y) + \chi(x,y)h_1(x,y,u_x,u_y)\Big)q. \end{aligned} \tag{35}$$

Notice that $e^0(x,y,0,0) = 0$. Let L be the linear operator given by

$$Lw = a^0w_{xx} + 2b^0w_{xy} + c^0w_{yy} + e^0, \tag{36}$$

where $e^0 = e^0(x,y,w_x,w_y)$. As in [13], 13.2, we see that $L(u_x) = 0$ and part (b.) follows from the Proposition in [17]. The proof of (c.) follows in a manner similar to that of (b.).

Q.E.D.

Proof of Lemma 5. Let us first observe that $0 < u(x,y) < 1$ for each $(x,y) \in \Omega$. To see this, suppose that (x_0, y_0) is an interior point of Ω with $u(x_0,y_0) \geq u(x,y)$ for all $(x,y) \in \Omega$. Notice that $v \equiv u(x_0,y_0)$ is a solution of (6a); let us set $w = v - u$. Then $w \geq 0$ on $\overline{\Omega}$ and $Rw = 0$, where R is given in (33). Since $d(x,y,0,0) = 0$, we may write this equation as a linear equation in terms of w_{xx}, w_{xy}, w_{yy}, w_x, and w_y. Since R is uniformly elliptic, the strong maximum principle ([13], Lemma 3.5) implies w, and so u, is constant. This is a contradiction and therefore $u(x,y) < 1$ for each $(x,y) \in \Omega$; the proof that $0 < u(x,y)$ for each $(x,y) \in \Omega$ is similar.

Let us next observe that $|\nabla u| > 0$ on Γ^*. To see this, let $(x_0,y_0) \in \Gamma^*$ and define $w = 1 - u$. Then $Rw = 0$ as above, $w \geq 0$ on $\overline{\Omega}$, and $w(x_0,y_0) = 0$. The Hopf boundary point lemma then implies $|\nabla u(x_0,y_0)| > 0$ and so our observation holds. Thus $|\nabla u| > 0$ on ∂W. Suppose $|\nabla u(x_0,y_0)| = 0$ for some $(x_0,y_0) \in W$ and set $z_0 = u(x_0,y_0)$. Then $0 < z_0 < 1$ and so the horizontal plane $z = z_0$ does not intersect either of the curves $\{(x,y,0) : (x,y) \in \Gamma\}$ or $\{(x,y,1) : (x,y) \in \Gamma^*\}$. From Lemma 4, we see that for some integer $m \geq 2$ there exist m distinct curves $\sigma_1, \ldots, \sigma_m$ which meet at (x_0,y_0) and divide a neighborhood of (x_0,y_0) into $2m$ open "sectors" $\omega_1, \omega_2, \ldots, \omega_{2m}$ such that $u < z_0$ on $\omega_1 \cup \omega_3 \cup \ldots \cup \omega_{2m-1}$ and $u > z_0$ on $\omega_2 \cup \omega_4 \cup \ldots \cup \omega_{2m}$. The Jordan curve theorem and the fact that W is an annular domain implies that there is a component ω of $\{(x,y) \in W : u(x,y) \neq z_0\}$ whose closure does not intersect ∂W and the maximum principle implies $u \equiv z_0$ in ω, in contradiction to the fact that $u \neq z_0$ in ω. Thus $|\nabla u| > 0$ in W. Q.E.D.

Proof of Lemma 6. (a.) Let γ be directed curve in $\overline{\Omega}$ starting at $(x_0,y_0) \in \Gamma$ along which $u_x = 0$ and ϕ is strictly decreasing. From Lemma 2 (b.) and the fact that Γ has a $-\vec{j}$–minimum at (x_0,y_0), we see that Γ is "strictly concave down" near (x_0,y_0). Assume the claim is false and γ ends at a point $(x_1,y_1) \in \overline{\Omega}$ with $y_1 = y_0$ such that $y < y_0$ if $(x,y) \in \gamma$ is not an endpoint of γ. Since $u_y(x_0,y_0) = -\lambda$ and $u_y(x,0) = 0$ for $(x,0) \in \overline{\Omega}$, Lemma 5 implies $u_y < 0$ on γ and $\gamma \cap \{y = 0\} = \emptyset$. ¿From Lemma 4, we see that $F = \{(x,y) \in \gamma : u_{xx}(x,y) = u_{xy}(x,y) = 0\}$ is finite in every compact subset of Ω. Since ϕ is strictly monotonic, γ cannot intersect itself, and so we see that $\gamma \cap \Omega$ is a piecewise C^1 curve(s). We may write $\gamma = \{(x(t),y(t)) : 0 \leq t \leq 1\}$, with $(x(0),y(0)) = (x_0,y_0)$ and $(x(1),y(1)) = (x_1,y_1)$, such that $x(\cdot), y(\cdot) \in C^0([0,1]) \cap C^1([0,1] \backslash D_1)$ and $x'(t) = y'(t) = 0$ if $t \in D_2$, where $D_1 = \{t \in [0,1] : (x(t),y(t)) \in \partial\Omega \cap F\}$ and $D_2 = \{t \in [0,1] : (x(t),y(t)) \in \Omega \cap F\}$; notice that D_1 is a discrete subset of $[0,1] \backslash D_1$. Then $u_x(x(t),y(t)) = 0$ for $0 \leq t \leq 1$ and $y(t) < y_0$ for $0 < t < 1$. The maximum principle, the Hopf boundary point lemma, and the facts that the monotonicity of ϕ prevents γ from returning to Γ (since γ lies below $y = y_0$) and Γ^* has only a finite number of horizontal segments implies D_1 is a finite (possibly empty) set.

Let us define

$$v(t) = \int_0^t \frac{u_y(x(\tau),y(\tau))y'(\tau)}{\phi(x(\tau),y(\tau))} d\tau = \int_0^t \frac{\big(u(x(\tau),y(\tau))\big)'}{\phi(x(\tau),y(\tau))} d\tau \qquad (37)$$

for $0 \leq t \leq 1$. We claim that $v(\cdot)$ is well-defined and finite and that $v(t) < v(1)$ for

$0 < t < 1$. Let us assume this for a moment and write $\vec{\gamma}(t) = (x(t), y(t))$. Then

$$\begin{aligned}
0 &\le u(x_1, y_1) - u(x_0, y_0) \\
&= \int_0^1 \big(u_x(\vec{\gamma}(t))x'(t) + u_y(\vec{\gamma}(t))y'(t)\big)dt \\
&= \int_0^1 u_y(\vec{\gamma}(t))y'(t)dt \\
&= \int_0^1 \phi(\vec{\gamma}(t))v'(t)dt \\
&= v(t)\phi(\vec{\gamma}(t))|_0^1 - \int_0^1 v(t)(\phi(\vec{\gamma}(t)))'dt \\
&< v(t)\phi(\vec{\gamma}(t))|_0^1 - v(1)\int_0^1 (\phi(\vec{\gamma}(t)))'dt \\
&= \Big(v(t)\phi(\vec{\gamma}(t)) - v(1)\phi(\vec{\gamma}(t))\Big)|_0^1 \\
&= \phi(x_0, y_0)(v(1) - v(0)) \\
&\le 0,
\end{aligned} \tag{38}$$

since $(\phi(\vec{\gamma}(t)))' \le 0$ for $t \in [0,1]\backslash D_1$ and $\phi < 0$. This contradiction implies that conclusion (a.) of the lemma is correct.

Notice that $u_y(x,0) = 0$ for $(x,0) \in \overline{\Omega}$; Lemma 5 implies γ is bounded away from $y = 0$, $C_1 = \inf_\gamma \frac{u}{\phi} > -\infty$, and $C_2 = \sup_\gamma \frac{u}{\phi^2} < \infty$. Suppose $(s,t) \subset [0,1]\backslash D_1$. Integration by parts yields

$$\int_s^t \frac{(u(\vec{\gamma}(\tau)))'}{\phi(\vec{\gamma}(\tau))}d\tau = \frac{u(\vec{\gamma}(\tau))}{\phi(\vec{\gamma}(\tau))}|_s^t + \int_s^t \frac{u(\vec{\gamma}(\tau))}{\phi^2(\vec{\gamma}(\tau))}\big(\phi(\vec{\gamma}(\tau))\big)'d\tau. \tag{39}$$

Since $(\phi(\vec{\gamma}(\tau)))' \le 0$ and $u \ge 0$, we observe that the second integral exists and equals $-\infty$ or a finite nonpositive number. Using the fact that $0 \le \frac{u}{\phi^2} \le C_2$ on γ, we obtain

$$\frac{u(\vec{\gamma}(\tau))}{\phi(\vec{\gamma}(\tau))}|_s^t \ge \int_s^t \frac{\big(u(\vec{\gamma}(\tau))\big)'}{\phi(\vec{\gamma}(\tau))}d\tau \ge \frac{u(\vec{\gamma}(\tau))}{\phi(\vec{\gamma}(\tau))}|_s^t + C_2\phi(\vec{\gamma}(\tau))|_s^t \ge C_1 + C_2\phi(\vec{\gamma}(\tau))|_0^1 \tag{40}$$

and therefore $\int_s^t \frac{\left(u(\vec{\gamma}(\tau))\right)'}{\phi(\vec{\gamma}(\tau))}d\tau$ is a well-defined, finite number. Suppose $D_1 = \{t_1, t_2, \ldots, t_n\}$ with $0 \le t_1 < t_2 < \ldots < t_n \le 1$ and set $t_0 = 0$ and $t_{n+1} = 1$. If $t \in (0,1]$, then $t \in (t_m, t_{m+1}]$ for some $m \in \{0, \ldots, n\}$ and

$$v(t) = \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{\big(u(\vec{\gamma}(\tau))\big)'}{\phi(\vec{\gamma}(\tau))}d\tau + \int_{t_m}^t \frac{\big(u(\vec{\gamma}(\tau))\big)'}{\phi(\vec{\gamma}(\tau))}d\tau \tag{41}$$

is a well-defined, finite number. Notice that $v \in C^0([0,1])$.

Let us consider our claim that $v(t) \le v(1)$. Notice that if $s > t$ and $y(s) = y(t)$, the facts that $\phi(\vec{\gamma}(t))$ is decreasing in t and $\frac{\partial \Phi}{\partial q}(y,q) > 0$ imply that $u_y(\vec{\gamma}(t)) >$

$u_y(\vec{\gamma}(s))$. Suppose first that $0 \le t_1 < t_2 \le t_3 < t_4 \le 1$ such that $y'(\cdot) < 0$ on $(t_1,t_2)\backslash D_1$, $y'(\cdot) > 0$ on $(t_3,t_4)\backslash D_1$, $y(t_1) = y(t_4)$, and $y(t_2) = y(t_3)$. For each $t \in (t_1,t_2)$, there is exactly one solution in (t_3,t_4) to the equation $y(\cdot) = y(t)$; let us denote this value by $s(t)$. Set $s(t_1) = t_4$ and $s(t_2) = t_3$. Then $y(t) = y(s(t))$ for $t \in [t_1,t_2]$. Notice that $y'(t) = y'(s(t))s'(t)$ and $s(t) > t$ for each $t \in (t_1,t_2)\backslash D_1$. Then

$$\begin{aligned} \int_{t_1}^{t_2} v'(t)dt &= \int_{t_1}^{t_2} \frac{u_y(\vec{\gamma}(t))y'(t)}{\phi(\vec{\gamma}(t))}dt \\ &\le \int_{t_1}^{t_2} \frac{u_y(\vec{\gamma}(s(t)))y'(s(t))s'(t)}{\phi(\vec{\gamma}(s(t)))}dt \qquad \text{by (10g)} \\ &= \int_{t_4}^{t_3} \frac{u_y(\vec{\gamma}(s))y'(s)}{\phi(\vec{\gamma}(s))}ds \\ &= -\int_{t_3}^{t_4} v'(s)ds \end{aligned} \tag{42}$$

and so $v(t_4) - v(t_1) + v(t_2) - v(t_1) = \int_{t_1}^{t_2} v'(t)dt + \int_{t_3}^{t_4} v'(t)dt \ge 0$.

Recall that $y(0) = y(1)$, $y(t) < y(1)$ for $t \in (0,1)$, and $\operatorname{sgn}(v'(t)) = \operatorname{sgn}(y'(t))$ for $t \in [0,1]\backslash D_1$. Let $H = \{t \in [0,1]\backslash D_1 : y'(t) = 0\}$ and H_0 be the interior of H. Then v is constant on the closure $\overline{H_0}$ of H_0. We may write

$$[0,1]\backslash H_0 = \bigcup_n \big([a_n,b_n] \cup [c_n,d_n]\big), \tag{43}$$

where $(a_1,b_1),(c_1,d_1),(a_2,b_2),(c_2,d_2),\dots$ are disjoint intervals, $0 \le a_n < b_n \le c_n < d_n \le 1$, $y(a_n) = y(d_n)$, $y(b_n) = c_n$, $y' < 0$ on $(a_n,b_n)\backslash D_1$, and $y' > 0$ on $(c_n,d_n)\backslash D_1$. Since

$$v(d_n) - v(c_n) + v(b_n) - v(a_n) \ge 0, \tag{44}$$

we see that

$$v(1) - v(0) = \sum_n \big(v(d_n) - v(c_n) + v(b_n) - v(a_n)\big) \ge 0. \tag{45}$$

(If $D_1 \ne \emptyset$, then we do not know if $v \in L^1([0,1])$ and so we may need to modify our argument slightly. Lemma 5 implies that for some $\epsilon_0 > 0$, $\{(x,y) \in \overline{\Omega} : \operatorname{dist}((x,y),\gamma) \le \epsilon_0\} \cap \{(x,y) \in \overline{\Omega} : y = 0 \ \text{ or } \ u_y(x,y) = 0\} = \emptyset$. If I is a compact subset of $[0,1]\backslash D_1$, then $v \in L^1(I)$. For $\epsilon \in (0,\epsilon_0)$, let $I_\epsilon = \cup_{j=1}^{n+1}[t_{j-1}^+, t_j^-]$ be a compact subset of $[0,1]\backslash D_1$ such that $\sum_{j=1}^{n+1}(t_j^- - t_{j-1}^+) > 1-\epsilon$, $t_j^- < t_j < t_j^+$, $j = 1,\dots,n$, and $|\vec{\gamma}(t_j^+) - \vec{\gamma}(t_j^-)| < \epsilon$, $j = 0,\dots,n+1$, where $t_0^- = 0$ and $t_{n+1}^+ = 1$. Let $\gamma_\epsilon = \{\vec{\gamma}_\epsilon(t) : t \in [0,1]\}$ be a piecewise C^1 curve in $\overline{\Omega}$ from (x_0,y_0) to (x_1,y_1) such that $\vec{\gamma}_\epsilon(t) = \vec{\gamma}(t)$ for $t \in I_\epsilon \cup D_1, \gamma_\epsilon(\cdot)$ is monotonic on $[t_j^-, t_j]$ and on $[t_j, t_j^+]$, for $j = 1,\dots,n+1$, and $|\vec{\gamma}_\epsilon(t) - \vec{\gamma}(t)| < \epsilon$ for $t \in [0,1]$, where $\vec{\gamma}_\epsilon(t) = (x_\epsilon(t), y_\epsilon(t))$. Let us define $l(t) = \frac{u_y(\vec{\gamma}(t))}{\phi(\vec{\gamma}(t))}$ if $t \in I_\epsilon$, $l(t) = C_3$ if $t \notin I_\epsilon$ with $y_\epsilon'(t) > 0$ and $l(t) = 0$ if $t \notin I_\epsilon$ with $y_\epsilon'(t) \le 0$, where $C_3 = \sup\{\frac{u_y(x,y)}{\phi(x,y)} : \operatorname{dist}((\mathrm{x},\mathrm{y}),\gamma) \le \epsilon\}$. Using an argument similar to that for (45), we see that

$$\int_0^1 l(t)y_\epsilon'(t)\ dt \ge 0, \tag{46}$$

since $l\ y'_\epsilon \in L^1([0,1])$, and so

$$\int_{I_\epsilon} \frac{u_y(\vec{\gamma}(t))}{\phi(\vec{\gamma}(t))} y'(t)\ dt \geq -(n+1)C_3\epsilon. \tag{47}$$

Letting $\epsilon \to 0$, we obtain $v(1) - v(0) \geq 0$.)

Now suppose $t \in (0,1)$; then $y(t) < y(1)$. Let $t_1 = \sup\{s \in [t,1] : y(\tau) \leq y(t)$ for all $\tau \in [t,s]\}$. Using essentially the same argument as above, we see that $v(t) \leq v(t_1)$. Set $t_2 = \sup\{s \in [t_1,1] : y(\cdot)$ is (weakly) increasing on $[t_1,s]\}$; then $t_1 < t_2$ and $v(t_1) < v(t_2)$ since $v(\cdot)$ is increasing on $[t_1,t_2]$ and $v'(s) > 0$ when $y'(s) > 0$. Now setting $t_3 = \sup\{s \in [t_2,1] : y(\tau) \leq y(t_2)$ for all $\tau \in [t_2,s]\}$, obtaining $v(t_2) \leq v(t_3)$, and continuing to argue in this manner, we see that $v(t) < v(1)$.

(b.) Notice that if $s > t$ and $y(s) = y(t)$, the facts that $\phi(\vec{\gamma}(t))$ is increasing in t and $\frac{\partial\Phi}{\partial q}(y,q) > 0$ imply that $u_y(\vec{\gamma}(t)) < u_y(\vec{\gamma}(s))$ and so

$$\frac{u_y(\vec{\gamma}(t))}{\phi(\vec{\gamma}(t))} \geq \frac{u_y(\vec{\gamma}(s))}{\phi(\vec{\gamma}(s))}. \tag{48}$$

Suppose $\gamma = \{(x(t),y(t)) : 0 \leq t \leq 1\}$ is a directed curve from a point $(x(0),y(0)) = (x_0,y_0)$ to $(x(1),y(1)) = (x^*,y^*)$ such that $y_0 = y^*$, $y(t) > y^*$ for $0 < t < t_1$, and $y(t) = y^*$ for $t_1 \leq t \leq 1$, for some $t_1 \in (0,1]$. If we define $v(t)$ by (37), $v(t) \geq v(1)$ for $0 \leq t \leq 1$ and $v(t) > v(1)$ for $0 < t < t_1$. Then (38) still holds, since $(\phi(\vec{\gamma}(t)))' \geq 0$, and this contradiction implies that a curve γ as above cannot exist and so (b.) holds. Q.E.D.

Proof of Lemma 7. Since (x_0,y_0) is a $-\vec{j}-$extrema, Lemma 2(b.) implies $u_x < 0$ on one side of (x_0,y_0) on Γ and $u_x > 0$ on the other side. Setting $\Pi = \{(x,y) \in \overline{\Omega} : u_x(x,y) > 0\}$ and letting γ_0 be a component of $\Omega \cap \partial M$ with $(x_0,y_0) \in \gamma$, where γ is the closure of γ_0, we see that there is at least one directed curve γ from (x_0,y_0) into Ω along which $u_x = 0$.

Suppose (x_0,y_0) is a $-\vec{j}-$minimum of Γ; then $u_x > 0$ locally to the left of (x_0,y_0) on Γ (i.e. preceeding (x_0,y_0) on Γ) and $u_x < 0$ locally to the right (i.e. following (x_0,y_0) on Γ). Let F be as in the proof of Lemma 6 (i.e. $F = \{u_{xx} = u_{xy} = 0\} \cap \Lambda$) and let γ_1 be the component of $\overline{\partial\Pi \cap \Omega \backslash F}$ which contains (x_0,y_0), where Π is the component of $\{(x,y) \in \Omega : u_x(x,y) > 0\}$ whose closure contains a portion of Γ immediately preceeding (x_0,y_0). ¿From Lemma 3, we see that ϕ is strictly decreasing on γ_1 and so γ_1 cannot return to (x_0,y_0). Now define γ to contain γ_1 and to be maximal with respect to forward continuation in $\Lambda\backslash\gamma_1$ under the condition that ϕ remain strictly decreasing. ¿From Lemmas 3 and 4(b.), we see that $\gamma \cap \Omega$ is piecewise C^1. If (x_0,y_0) is a $-\vec{j}-$maximum, then the existence of a curve γ with the required properties follows in a similar manner.

Let γ be any maximal curve along which $u_x = 0$ and ϕ is strictly increasing (decreasing). ¿From the monotonicity of ϕ, we observe that γ does not intersect itself. Suppose γ terminates at a point $(x_2,y_2) \in \Omega$. If $|\nabla u_x| \neq 0$ at (x_2,y_2), then Lemma 3 implies that γ can be extended beyond (x_2,y_2) with $u_x < 0$ $(u_x > 0)$ locally to the right of γ and ϕ is strictly increasing (decreasing) along γ, in violation of the maximal property of γ. Suppose $|\nabla u_x| = 0$ at (x_2,y_2). For some $\epsilon > 0$ and integer $m \geq 2$, Lemma 4 implies $N = \{(x,y) : |(x-x_2, y-y_2)| < \epsilon\}$ is contained in Ω and the set $\{(x,y) : u_x(x,y) < 0\} \cap N$ ($\{(x,y) : u_x(x,y) > 0\} \cap N$) contains m components; denote by V the component whose closure contains an interval of γ. Then we may extend γ by adding the component of $\partial V \cap N$ which includes (x_2, y_2). Lemma 3 implies ϕ is strictly increasing (decreasing) on the extension of γ, in contradiction to the maximal property of γ. Thus γ cannot terminate at a point of Ω. The fact that γ has no accumulation points follows from the characterization of the points of γ given by Lemmas 3 and 4; thus (a.) holds. The monotonicity of ϕ implies γ does not intersect itself. Also, since $u \in C^1(\overline{\Omega})$, $\gamma \cap \Gamma^* \subset \Lambda$. Therefore (b.) holds.

Let (x_0, y_0) be a $-\vec{j}-$minimum of Γ. Suppose γ intersects Γ at a point $(x_2, y_2) \neq (x_0, y_0)$. ¿From Lemma 6, we see that $y_2 \leq y_0$. Since $u_y < 0$ on γ, $u_y(x_0, y_0) = u_y(x_2, y_2) = -\lambda$. Now $\Phi(y_0, u_y(x_0, y_0)) > \Phi(y_2, u_y(x_2, y_2))$ and so $y_0 < y_2$, since $\frac{\partial \Phi}{\partial y} < 0$. This contradiction implies $\gamma \cap \Gamma = \{(x_0, y_0)\}$. Since γ cannot terminate at a point of $\Omega \cup \Gamma$, it must do so at a point (x_1, y_1) of Γ^*.

We observe that (x_1, y_1) need not be the first point of intersection of γ and Γ^*. We will show that if (x_2, y_2) is a point of $\gamma \cap \Gamma^*$ at which Γ^* has a $-\vec{j}-$maximum or a $-\vec{j}-$inflection point, then γ continues past (x_2, y_2). This means (x_1, y_1) must be a $-\vec{j}-$minimum of Γ^*. Suppose first that Γ^* has a $-\vec{j}-$maximum at (x_2, y_2). Then there is a (possibly degenerate) line segment $\sigma = \{(x, y_2) : \alpha \leq x \leq \beta\} \subset \Gamma^*$ with $\alpha \leq x_2 \leq \beta$ such that, for some $\epsilon > 0$, $y > y_2$ when $(x,y) \in \Gamma^*$ satisfies $\alpha - \epsilon < x < \alpha$ or $\beta < x < \beta + \epsilon$. Let $P = \{(x,y) \in \Omega : u_x(x,y) > 0\}$ and $L = \{(x,y) \in \Omega : u_x(x,y) < 0\}$. Then ∂P contains a neighborhood of Γ^* immediately following (β, y_2) and ∂L contains a neighborhood of Γ^* immediately preceeding (α, y_2). Since $u_x > 0$ locally to the right of γ and $u_x < 0$ locally to the left of γ at all but a finite number of points of γ, γ may be extended beyond (x_2, y_2) so that $u_x = 0$ and ϕ is strictly decreasing on γ; for example, if P_1 is the component of P which lies to the immediate right of γ near (x_2, y_2) and $\partial_1 P$ is the component of ∂P_1 which contains (x_2, y_2), then $\gamma \cup \partial_1 P$ is one of the two possible extensions. If Γ^* has a $-\vec{j}-$inflection point at (x_2, y_2), then there is a (possibly degenerate) line segment σ as above and either $u_x < 0$ locally preceeding (α, y_2) or $u_x > 0$ locally following (β, y_2) on Γ^*. Then γ may be extended past (x_2, y_2) either as illustrated above (if $u_x < 0$ locally preceeding (α, y_2)) or by replacing γ by $\gamma \cup \partial_1 L$, where L_1 is the component of L which lies immediately to the left of γ near (x_2, y_2) and $\partial_1 L$ is the component of ∂L_1 which contains (x_2, y_2). Since γ extends beyond $-\vec{j}-$maximum and $-\vec{j}-$inflection points of Γ^*, it must terminate at a $-\vec{j}-$minimum of Γ^*. Further, if Γ has a $-\vec{j}-$maximum at (x_0, y_0) and γ does not return to Γ, then a similar argument shows γ must terminate at a point $(x_1, y_1) \in \Gamma^*$ at which Γ^* has a $-\vec{j}-$maximum.

Let us consider the last part of (c.). Let $\gamma_0 = \{(x(t), y(t)) : 0 \leq t \leq 1\}$ be the portion of γ from $\vec{\gamma}(0) = (x_0, y_0)$ to $\vec{\gamma}(1) = (x_1^*, y_1^*)$. Notice that $v(t) = v(\vec{\gamma}(t))$ is given by (36) and $v(t) > v(1)$ for $0 \le t < 1$. Then

$$\begin{aligned} 1 &\ge u(x_1^*, y_1^*) - u(x_0, y_0) \\ &= v(t)\phi(\vec{\gamma}(t))|_0^1 - \int_0^1 v(t)\big(\phi(\vec{\gamma}(t))\big)' dt \\ &> \big(v(t)\phi(\vec{\gamma}(t)) - v(1)\phi(\vec{\gamma}(t))\big)|_0^1 \\ &= \phi(x_0, y_0)(v(1) - v(0)). \end{aligned} \tag{49}$$

Together with Lemma 6(a.), this implies (c.). To see that the last part of (d.) is valid, notice that $v(t) < v(1)$ for $0 \le t < 1$ and so

$$1 = u(x_1, y_1) - u(x_0, y_0) < \phi(x_0, y_0)(v(1) - v(0)). \tag{50}$$

The lemma then follows. Q.E.D.

Proof of Lemma 8. The proof of this lemma is similar to that of Lemma 6. Suppose $\sigma_1 = \{(x(t), y(t)) : 0 \le t \le 1\} \subset \sigma$ is a directed curve in $\overline{\Omega}$ starting at $p_0 = (x_0, y_0)$ and ending at $p_1 = (x_1, y_1)$ along which $u_y = 0$ and u_x^2 is strictly increasing such that $x_1 = x_0$ and $x(t) > x_0$ ($x(t) < x_0$) when $0 < t < 1$. From Lemma 5, we see that $u_x > 0$ ($u_x < 0$) on σ. If we set $\vec{\gamma}(t) = (x(t), y(t))$, we see

$$\begin{aligned} 0 &\le u(p_1) - u(p_0) \\ &= x(t)u_x(\vec{\gamma}(t))|_0^1 - \int_0^1 x(t)d\big(u_x(\vec{\gamma}(t))\big) \\ &< \big(x(t)u_x(\vec{\gamma}(t)) - x(0)u_x(\vec{\gamma}(t))\big)|_0^1 \\ &= u_x(p_1)(x_1 - x_0) = 0. \end{aligned} \tag{51}$$

This contradiction implies $x > x_0$ ($x < x_0$) for all $(x, y) \in \sigma$ with $(x, y) \ne (x_0, y_0)$ and so (a.) follows.

Suppose now that $\sigma_1 = \{(x(t), y(t)) : 0 \le t \le 1\} \subset \sigma$ is a directed curve in $\overline{\Omega}$ starting at $p_1 = (x_1, y_1)$ and ending at $p^* = (x^*, y^*)$ along which $u_y = 0$ and u_x^2 is strictly decreasing such that $x_1 = x^*$ and $x(t) > x^*$ ($x(t) < x^*$) when $0 < t < 1$. From Lemma 5, we see that $u_x > 0$ ($u_x < 0$) on σ. Then $u(p^*) = 1$ and so

$$\begin{aligned} 0 &\le u(p^*) - u(p_1) \\ &< \big(x(t)u_x(\vec{\sigma}(t)) - x^* u_x(\vec{\sigma}(t))\big)|_0^1 \\ &= u_x(p_1)(x^* - x_1) = 0. \end{aligned} \tag{52}$$

This contradiction implies (b.) holds. Q.E.D.

Proof of Lemma 9. Using Lemmas 1 and 4 in a manner similar to the proof of Lemma 7, we see that there is a maximal directed curve σ along which $u_y = 0$ and u_x^2 is strictly monotonic as claimed. By Lemma 1, σ does not intersect itself. Let us suppose for a moment that σ does not intersect the $x-$axis. The monotonicity of u_x^2 implies σ does not return to Γ. The fact that σ has no terminal or accumulation points in Ω follows as in Lemma 6 from Lemmas 1 and 4. The fact that σ terminates at a point $p^* = (x^*, y^*)$ at which Γ^* has a $\pm\vec{i}-$extrema of the same type as that of Γ at (x_0, y_0) follows as in the proof of Lemma 7.

Let us consider the claim that σ does not intersect the $x-$axis. Let $V = \{(x,0) \in \overline{\Omega}\}$ and $p^\pm = (x^\pm, 0) \in \partial_o W$ with $x^+ < x^-$. Denote by V^+ the component of V which contains p^+ and by V^- the component of V which contains p^-. Let $p^0 = (x^0, y^0)$ be the first $-\vec{j}-$minimum of Γ and denote by Γ^+ the portion of Γ between p^+ and p^0. Notice that $u_x > 0$ on Γ^+. From Lemma 7, we see that there is a curve γ^0 starting at p^0 and ending at a point p^1 of Γ^* such that $u_x = 0$ on γ^0 and $\Phi(y, u_y)$ is strictly decreasing along γ^0. ¿From Lemma 5, we see that any curve in $\overline{\Omega}$ along which $u_y = 0$ cannot intersect γ^0. Suppose σ intersects V^+. Then $p_0 = (x_0, y_0) \in \Gamma^+$, since σ cannot cross γ^0, and so p_0 is a $\vec{i}-$extrema of Γ. Let $p_1 = (x_1, 0)$ be the first point at which σ intersects V^+ and denote by σ_1 the portion of σ between p_0 and p_1. Let $W = \{(x,y) \in \Omega : u_y(x,y) \neq 0\}$ and let W_0 be any component of W contained in the open set bounded by σ_1, the portion of Γ between p^+ and p_0, and the portion of V^+ between p^+ and p_1. Let us suppose that $u_y > 0$ in W_0. If we orient ∂W_0 so W_0 lies to the right, then u_x^2 is strictly increasing on $\partial W_0 \backslash \Gamma$. Since $u_x = \lambda$ at p^+ and at each point of Γ^+ at which $u_y = 0$, we have a contradiction. Therefore σ does not intersect V^+; similar reasoning implies σ does not intersect V^-.

The proof of the lemma will be complete once we have shown that $(p^* - p_0) \cdot \vec{\nu} < 1$ if p_0 is a $\vec{\nu}-$minimum of Γ and $(p^* - p_0) \cdot \vec{\nu} > 1$ if p_0 is a $\vec{\nu}-$maximum of Γ, since Lemma 8 implies $(q - p_0) \cdot \vec{\nu} > 0$ for $q \in \sigma$, $q \neq p_0$ when p_0 is a $\vec{\nu}-$minimum and $(p^* - q) \cdot \vec{\nu} > 0$ for $q \in \sigma$, $q \neq p^*$ when p_0 is a $\vec{\nu}-$maximum.

Suppose first that p_0 is a $\vec{\nu}-$minimum and let $p_1 = (x_1, y_1)$ be the first point on σ at which $x_1 = x^*$. Let σ_1 be the portion of σ between p_0 and p_1 and write $\sigma_1 = \{(x(t), y(t)) : 0 \leq t \leq 1\}$. Then $x(t) > x_0$ if $\vec{\nu} = \vec{i}$ and $x(t) < x_0$ if $\vec{\nu} = -\vec{i}$ for each $t > 0$. If we write $\vec{\sigma}(t) = (x(t), y(t))$, we have

$$\begin{aligned} 1 = u(p_1) - u(p_0) &= \int_0^1 u_x(\vec{\sigma}(t)) x'(t)\, dt \\ &= x(t) u_x(\vec{\sigma}(t))|_0^1 - \int_0^1 x(t)\, d\big(u_x(\vec{\sigma}(t))\big) \\ &> x(t) u_x(\vec{\sigma}(t)) - x^* \big(u_x(p_1) - u_x(p_0)\big)|_0^1 \\ &= u_x(p_0)(x^* - x_0) = \lambda (p^* - p_0) \cdot \vec{\nu}. \end{aligned} \tag{53}$$

Suppose next that p_0 is a $\vec{\nu}-$maximum. Let us write $\sigma = \{(x(t), y(t)) : 0 \leq t \leq 1\}$. Then $x(t) < x^*$ if $\vec{\nu} = \vec{i}$ and $x(t) > x^*$ if $\vec{\nu} = -\vec{i}$ for each $t > 0$. We have

$$\begin{aligned} 1 &= u(p_1) - u(p_0) = x(t) u_x(\vec{\sigma}(t))|_0^1 - \int_0^1 x(t)\, d\big(u_x(\vec{\sigma}(t))\big) \\ &< x(t) u_x(\vec{\sigma}(t)) - x^* \big(u_x(p_1) - u_x(p_0)\big)|_0^1 \\ &= u_x(p_0)(x^* - x_0) = \lambda (p^* - p_0) \cdot \vec{\nu}. \end{aligned} \tag{54}$$

This completes the proof of Lemma 9. Q.E.D.

§5. PROOFS OF THE MAIN RESULTS

Proof of Theorem 3: We will begin by considering those claims in the theorem which refer only to points $p \in E$; these include (ii), (iv), (v), (vi), (vii), (viii) as well as a portion of (i). Notice that if $p \in E_1$ or p is a $-\vec{j}-$minimum of Γ, then there is a directed curve $\gamma = \gamma_p$ starting at p and ending at a point $p^* \in \Gamma^*$ with the properties given in Lemmas 7 and 9. Further, if $p \in E_2$ and Γ has a $-\vec{j}-$maximum at p, then there is a maximal directed curve γ_p starting at p and either intersecting Γ after leaving p or ending at a point p^* at which Γ^* has a $-\vec{j}-$maximum; this curve has the properties given in Lemma 7. Now notice that Lemma 5 implies $\gamma_p \cap \gamma_q = \emptyset$ whenever $p \in E_1$ and $q \in E_2$. Suppose $p = (x, y) \in E_2$ and Γ has a $-\vec{j}-$maximum at p. If p_1 is the member of E immediately preceeding p, then either $p_1 \in E_1$ or Γ has a $-\vec{j}-$minimum at p_1. Similarly, if p_2 is the member of E immediately following p, then either $p_2 \in E_1$ or Γ has a $-\vec{j}-$minimum at p_2. If $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$, then $y_1 > y$ and $y_2 > y$. Suppose $\gamma = \gamma_p$ returns to Γ at a point $q = (s, t) \in \Gamma$. Since $\phi(q) > \phi(p)$, $u_y(q) = u_y(p) = -\lambda$, and $\frac{\partial \Phi}{\partial y} < 0$, we see that $\Phi(t, -\lambda) > \Phi(y, -\lambda)$ and so $t < y$. Since Γ has a $-\vec{j}-$maximum at p, each point of Γ between p_1 and p_2 has a $y-$coordinate larger than y and therefore $q \neq p_1$, $q \neq p_2$, and q does not lie between p_1 and p_2 on Γ. Thus, the only way $\gamma = \gamma_p$ can return to Γ is if it crosses either $\gamma_1 = \gamma_{p_1}$ or $\gamma_2 = \gamma_{p_2}$. Let us assume this and let q be the first point at which γ intersects one of the curves γ_1 or γ_2; we may assume γ intersects γ_1 at q. Then p_1 cannot be in E_1 and so Γ has a $-\vec{j}-$minimum at p_1. Since $\phi(p_1) > \phi(q) > \phi(p)$ and $u_y(p_1) = u_y(p) = -\lambda$, we see that $\Phi(y_1, -\lambda) > \Phi(y, -\lambda)$ and so $y_1 < y$. This contradicts the fact that $y_1 > y$ and implies γ and γ_1 do not intersect. Therefore γ ends at a $-\vec{j}-$maximum of Γ^*. The fact that γ lies between γ_1 and γ_2 and does not intersect either of them implies that γ_1 and γ_2 cannot intersect. Similarly, if p_3 is a $-\vec{j}-$maximum of Γ and $p_3 \neq p$, then γ_{p_3} and γ cannot intersect. Using the monotonicity of u_x^2 or ϕ on γ_p and Lemma 5 as above, we see that if $p, q \in E$ with $p \neq q$, then $\gamma_p \cap \gamma_q = \emptyset$. Using Lemmas 7 and 9, we see that the claims involving points $p \in E$ follow.

We will next consider those claims which refer only to points $q \in I$; these include (iii) and a portion of (i). Let $q \in I_1$; then $u_y(q) = 0$ and $u_y(p) \geq (\leq) 0$ for $p \in \Gamma$ near q. Suppose, for a moment, that $u_y > (<) 0$ in $\Omega \cap V$, where V is a deleted neighborhood of q. Since $|\nabla u| = \lambda$, Lemma 2(b.) implies $u_y > (<) 0$ on $\Gamma \cap V$. Thus u_y has a local minimum (maximum) at q. As in the proof of Lemma 4, we see that u_y is the solution of the linear equation

$$Mw = a^0 w_{xx} + b^0 w_{xy} + c^0 w_{yy} + f^0, \tag{55}$$

where a^0, b^0, c^0 are as in the proof of Lemma 4, $f^0 = f^0(x, y, w_x, w_y)$, and f^0 is defined in a similar manner to the definition of e^0 in (35). Since $u \in C^2(\overline{\Omega \cap V})$, M is uniformly elliptic near q. The Hopf boundary point lemma then implies $u_{xy}(q) = (u_y)_x(q) \neq 0$. Now u_x^2 equals λ^2 at q, $u_x^2 < \lambda^2$ on $\Gamma \cap V$, and $\vec{j}$ is a tangent vector to Γ at q, so $u_{xy}(q) = (u_x)_y(q) = 0$, a contradiction. Therefore u_y changes signs in $\Omega \cap V$ for every neighborhood V of q.

Let us assume q is a $-\vec{i}-$inflection point (so $u_x(q) < 0$) and $u_y \leq 0$ on Γ near q. Let Π be a component of $\{(x, y) \in \Omega : u_y(x, y) > 0\}$ whose closure contains q and let σ_1 and σ_2 be distinct directed curves which are the closures of components of $\partial\Pi \cap \Omega \backslash F$ and each begin at q. In fact, we may assume that, in some neighborhood of q, $u_y < 0$ between σ_1 and Γ_q^+ and $u_y > 0$ between σ_1 and σ_2, where Γ_q^+ is the portion of Γ following q. Thus σ_1 is the curve in T_0 adjacent to Γ_q^+ and σ_2 is the curve adjacent to σ_1. Now let σ_1 (σ_2) represent a maximal extension of σ_1 (σ_2) with respect to forward continuation under the conditions that $u_y = 0$ and u_x^2 be strictly decreasing (increasing) on σ_1 (σ_2). As in the proof of Lemma 9, we see that σ_1 terminates at a point $q^* \in \Gamma^*$ at which Γ^* has a $-\vec{i}-$maximum, σ_2 terminates at a point $q^{**} \in \Gamma^*$ at which Γ^* has a $-\vec{i}-$minimum, and q^* follows q^{**} on Γ^*.

Suppose now that q is any element of I_1 and $\vec{\nu} = \vec{n}(q)$. In a similar manner to that above, we see there are two directed simple curves σ_1 and σ_2 and two points q^* and q^{**} on Γ^* such that Γ^* has a $\vec{\nu}-$minimum at one of these points and has a $\vec{\nu}-$maximum at the other. Using the monotonicity of u_x^2, it is easy to see that the points $p_1^*, p_2^*, \ldots, p_k^*$, $q_1^*, q_1^{**}, \ldots, q_l^*, q_l^{**}$ are all distinct $\pm\vec{i}-$extrema of Γ^* if $p_1, p_2, \ldots, p_k \in E_1$ and $q_1, \ldots, q_l \in I_1$.

Suppose $q = (x_0, y_0) \in I_2$. As above, the Hopf boundary point lemma implies there are two curves σ_1 and σ_2 starting at q along which $u_x = 0$ such that ϕ is strictly increasing on σ_1 and strictly decreasing on σ_2. Let σ_1 and σ_2 denote maximal extensions (with respect to forward continuation under the conditions $u_x = 0$ and ϕ be strictly monotonic) of σ_1 and σ_2. Once we know that σ_1 and σ_2 do not return to Γ after leaving q, the remainder of the proof follows as for the case of $\pm\vec{i}-$inflection points. Suppose $u_x(q) = 0$ and $u_x \leq 0$ on Γ near q. We may assume that, in a neighborhood of q, $u_x < 0$ between σ_1 and Γ_q^+ and $u_x > 0$ between σ_1 and σ_2. Then ϕ is decreasing on σ_1 and increasing on σ_2; hence σ_1 can only return to Γ at a point above the line $y = y_0$ and σ_2 can return to Γ at a point below $y = y_0$. Since σ_1 lies to the right of σ_2, σ_1 and σ_2 cannot intersect any of the curves γ_p for $p \in E$, $\Gamma_q^+ \cap V$ lies below the line $y = y_0$, and $\Gamma_q^- \cap V$ lies above $y = y_0$, where Γ_q^+ (Γ_q^-) is the portion of Γ following (preceeding) q and V is some neighborhood of q, we see that σ_2 and σ_2 cannot return to Γ. The case when $u_x(q) = 0$ and $u_x \geq 0$ on Γ near q is similar.

Finally, we will consider the proof of the remaining portion of (i) concerning points $p \in E$ and $q \in I$. We claim that if $p, q \in E \cup I$ and $p \neq q$, then γ_p (or σ_p) and γ_q (or σ_q) are disjoint. If p is a $\vec{\nu}$ extreme or inflection point, q is a $\vec{\mu}$ extreme or inflection point, and the appropriate curves starting at p and q intersect, then Lemma 5 implies $\vec{\nu} = \vec{\mu}$. To illustrate that two such curves cannot meet, suppose p is a $\vec{i}-$minimum, q is the next point (i.e. following on Γ), and q is a $\vec{i}-$inflection point. Then $u_y < 0$ between p and q on Γ. If σ_q is the first curve (i.e. adjacent to Γ_q^-) leaving q along which $u_y = 0$, then u_x^2 is strictly increasing along γ_p and strictly decreasing along σ_q. Since $u_x(p) = u_x(q) = \lambda$, we see that $\gamma_p \cap \sigma_q = \emptyset$. The general case is similar and so the theorem follows. Q.E.D.

Proof of Theorem 2: If $\vec{\nu} = \pm\vec{i}$, $q \in \Gamma$, and Γ has a $\vec{\nu}-$extrema or $\vec{\nu}-$inflection point at q, then, as in Lemma 9, there is a curve γ in $\overline{\Omega}$ starting at q along which $u_y = 0$ and ending at a point $q^* \in \Gamma^*$. Since there are no points on Γ^* at which $u_y = 0$, we see that Γ has no $\vec{\nu}$ extreme or inflection points; the theorem then follows. Q.E.D.

In fact, if we do not assume conditions (10b)-(10g) hold but otherwise assume the hypotheses of Theorem 3, then we claim that the conclusions of the theorem which concern $\pm\vec{i}$ extreme and inflection points of Γ continue to hold. To see this, recall that Lemmas 1,2,4(a.),4(c.),5,8, and 9 do not depend on condition (10). The proof of our claim will follow from the proof of the theorem once we know that none of the curves γ_p or σ_q can intersect the $x-$axis. If p is a $\vec{i}-$extrema or $\vec{i}-$inflection point of Γ, then γ_p cannot intersect V^+ as in the proof of Lemma 9 and cannot intersect V^- since $u_x > 0$ on γ_p and, from Lemma 5, $u_x < 0$ on V^-. Similarly, if p is a $-\vec{i}$ extreme or inflection point, then γ_p cannot intersect the $x-$axis.

§6. CONCLUDING REMARKS

The results of this paper were obtained by the first author ([5]) in the case where G is the Laplacian and $\mathcal{O}$ has analytic boundary and were extended to the general case of smooth domains and equations satisfying conditions (9) and (10) by the second author. We regard this work as one extension to three dimensions of the qualitative theory for two-dimensional free boundary problems given in [1]-[4], [7], [8], [11], [16], and [18]. Other authors might consider additional free boundary problems in $\mathbb{R}^N$ $(N \geq 3)$ which reduce to two-dimensional free boundary problems, determine conditions on the partial differential operator and/or the boundary conditions which allow one to compare the $\vec{\nu}-$extrema of the free boundary with the $\vec{\nu}-$extrema of the fixed boundary, and so obtain other extensions of the two-dimensional qualitative theory.

It would be interesting to determine genuine three-dimensional (or $N-$dimensional) generalizations of the qualitative geometric theory. While it seems unlikely that a relationship between $\vec{\nu}-$extrema of the fixed and free boundaries exists in some generic sense (e.g. for almost all operators G and almost all fixed boundaries $\mathcal{S}^*$), perhaps other kinds of geometric information, such as sectional or mean curvature, of the free and fixed boundaries can be compared. Unfortunately, we have no idea at the moment of an appropriate genuine higher dimensional generalization of this work.

REFERENCES

[1] A. Acker: On the Geometric Form of Free Boundaries Satisfying a Bernoulli Condition, Math. Meth. Appl. Sci. 6 (1984), 449-456.

[2] A. Acker: On the Geometric Form of Free Boundaries Satisfying a Bernoulli Condition.II., Math. Meth. Appl. Sci. 8 (1986), 387-404.

[3] A. Acker: On the Geometric Form of Solutions of a Free Boundary Problem involving Galvanization, Math. Meth. Appl. Sci. 9 (1987), 99-104.

[4] A. Acker: On the Geometric Form of Bernoulli Configurations, Math. Meth. Appl. Sci. 10 (1988), 1-14.

[5] A. Acker: On the geometric form of axially-symmetric free boundaries satisfying a Bernoulli condition. (unpublished 27-page manuscript, 1986)

[6] A. Acker: Theorems and Counterexamples on the Geometry of Solutions to Bernoulli Free Boundary Problems, Advances in Geometric Analysis and Continuum Mechanics, Boston: International Press, 1995.

[7] A. Acker and K. Lancaster: The Geometry of Curves of Constant Contact Angle for Doubly-Connected Minimal Surfaces, Comm. PDE 14 (1989), 375-390.

[8] A. Acker and K. Lancaster: Existence and Geometry of a Free Boundary Problem for the Heat Equation, Pac. J. Math. 148 (1991), 207-224.

[9] A. Acker and R. Meyer: A Free Boundary Problem for the p-Laplacian: Uniqueness, Convexity, and Successive Approximation of Solutions, Electron. J. Diff. Equ. 8 (1995).

[10] Alt, Caffarelli, and Friedman: A Free Boundary Problem for Quasi-linear Elliptic Equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 11 (1984), 1-44.

[11] A. Friedman and T. Vogel: Cavitational Flow in a Channel with Oscillatory Wall, Nonlinear Analysis TMA 7 (1983), 1175-1192.

[12] A. Friedman: Variational Principles and Free-Boundary Problems, New York: Wiley, 1982.

[13] D. Gilbarg and N. Trudinger: Elliptic Partial Differential Equations of Second Order, second edition, Berlin: Springer, 1983.

[14] D. Kinderlehrer and G. Stampacchia: An Introduction to Variational Inequalities and their Applications, New York: Academic Press, 1980.

[15] A. Lacey and M. Shillor: Electrochemical and Electro-Discharge Machining with a Threshold Current, IMA J. Appl. Math. 39 (1987), 121-142.

[16] K. Lancaster: Qualitative Behavior of Solutions of Elliptic Free Boundary Problems, Pac. J. Math 154 (1992), 297-316.

[17] K. Lancaster: The Relationship between the Boundary Behavior of and the Comparison Principles satisfied by "Approximate Solutions" of Elliptic Dirichlet Problems, Comparison Methods and Stability Theory, 237-259, Lecture Notes in Pure and Appl. Math., 162, New York: Dekker, 1994.

[18] T. Vogel: A Free Boundary Problem arising from a Galvanizing Process, SIAM J. Math. Mech. Anal. 16 (1985), 970-979.

Andrew Acker & Kirk Lancaster
Department of Mathematics and Statistics
Wichita State University
Wichita, Kansas 67260-0033
E-mail address: acker@twsuvm.uc.twsu.edu
E-mail address: lancaste@twsuvm.uc.twsu.edu