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On a mixed problem for a coupled nonlinear system *

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(Dedicated to professor Luiz A. Medeiros for his 70th birthday)

Abstract

In this article we prove the existence and uniqueness of solutions to the mixed problem associated with the nonlinear system

$$u_{tt} - M(\int_\Omega |\nabla u|^2 dx)\Delta u + |u|^\rho u + \theta = f$$

$$\theta_t - \Delta\theta + u_t = g$$

where M is a positive real function, and f and g are known real functions.

1 Introduction

Let Ω be an open and bounded subset of $\mathbb{R}^m$, with smooth boundary Γ. Let Q be the cylinder $Q = \Omega\times]0,T[$ and $\sum$ its lateral boundary. Let us denote the usual norm in $H_0^m(\Omega)$ by $\|\cdot\|$ and the usual norm in $L^2(\Omega)$ by $|\cdot|$, where $H_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$, and $H^m(\Omega)$ is the standard Sobolev space.

We shall consider the nonlinear system

$$u_{tt} - M(\textstyle\int_\Omega |\nabla u|^2 dx)\Delta u + |u|^\rho u + \theta = f \quad \text{in} \quad Q \tag{1}$$

$$\theta_t \ - \ \Delta\theta + u_t \ = g \ \text{ in } \ Q \tag{2}$$

$$u = \theta = 0 \ \text{ on } \textstyle\sum \tag{3}$$

$$u(0) = u_{0;} \ u'(0) = u_1; \ \theta(0) = \theta_0 \tag{4}$$

When $M(s)$ is a positive constant α and $\theta = 0$, the dynamical part of the above system is a nonlinear perturbation of the linear wave equation $u_{tt}-\alpha\Delta u = f$, (cf. Lions [6]). When $M(s) = m_0 + m_1 s$, with m_0 and m_1 positive constants and $\theta = 0$, Equation (1) is a nonlinear perturbation of the canonical Kirchhoff-Carrier's model which describes small vibrations of a stretched string when tension is assumed to have only a vertical component at each point of the string

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(cf. Pohozhaev [10], Arosio-Spagnolo [1]). For $\theta = 0$, Hosoya-Yamada [9], investigate the existence, uniqueness and regularity of solutions of (1.1).

In [7], L. A. Medeiros studies the equation (1) when $\theta = 0$ and the nonlinear perturbation is equal to u^2. Lastly, in [8] Maciel-Lima, studied the existence of a local weak solution of the mixed problem for the perturbed Kirchhoff-Carrier's equation

$$u'' - M(\int_\Omega |\nabla u|^2 dx)\Delta u + \lambda |u|^\rho u = f\,,$$

when $\lambda = -1$, $M : [0, \infty) \to [0, \infty)$ is a C^1 function such that $M(s) \geq m_0 > 0, \forall s \in \mathbb{R}$, where $\rho \in \mathbb{R}$ and satisfies $0 < \rho \leq 2/(n-4)$ if $n \geq 5$ or $\rho \geq 0$ if $n = 1, 2, 3,$ or 4. For other perturbations of Kirchhoff-Carrier's operator, among several works, we cite D'ancona-Spagnolo [3], and Bisognin [2].

In the present work we discuss the existence of a weak solution for the coupled nonlinear system (1)–(3) where we impose the appropriate assumptions on M, ρ,f and g. For the proof of existence, we employ the Galerkin's approximation method plus a compactness argument (see, e.g., Lions [5]).

2 Notation and main result

We make the following assumptions:

$$M \in C^1[0, \infty) \text{ and } M(s) \geq m_0 > 0 \text{ for } s \geq 0. \tag{A.1}$$

$$0 < \rho \leq \frac{2}{n-2} \text{ if } n \geq 5 \text{ and } 0 \leq \rho < \infty \text{ if } n = 1,\ 2,\ 3 \text{ or } 4 \tag{A.2}$$

$$f,\ g \in C^0(0, T; H_0^1(\Omega)) \tag{A.3}$$

The main result of the present work is given in the following theorem.

Theorem 1 *Assume (A.1)–(A.3). For*

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega),\ u_1 \in H_0^1(\Omega), \quad \textit{and} \quad \theta_0 \in H_0^1(\Omega)$$

there exist $T_0 \in \mathbb{R}$, $0 < T_0 < T$ such that (1)–(4) has a unique weak solution $\{u, \theta\}$ on $[0, T_0]$ satisfying (1) and (2) in the following sense:

$$\tfrac{d}{dt}(u'(t), w) + M(\textstyle\int_\Omega |\nabla u(t)| dx) a(u(t), w) + (|u(t)|^\rho u(t), w) + (\theta(t), w) = (f(t), w)$$
$$\tfrac{d}{dt}(\theta(t), w) + a(\theta(t), w) + (u'(t), w) = (g(t), w)$$

for all $w \in H_0^1(\Omega)$ in the sense of $D'(0, T)$.

$$u(0) = u_0,\ u'(0) = u_1,\ \theta'(0) = \theta_0$$

Proof of Theorem 1. Let $w_1, ..., w_m$ be the eigenfunctions of the Laplacian on Ω and let V_m be the space generated by the first m eigenfunctions. Now let us consider the approximated system

$$\begin{aligned}
&(u_m''(t), w_k) - M(\|u_m(t)\|^2)(\Delta u_m(t), w_k) \\
&+(|u_m(t)|^\rho u_m(t), w_k) + (\theta_m(t), w_k) = (f(t), w_k) && (5)\\
&(\theta_m'(t), w_k) - (\Delta\theta_m(t), w_k) + (u_m'(t), w_k) = (g(t), w_k) && (6)\\
&u_m(0) = u_{0m} \longrightarrow u_0 \text{ strongly in } H_0^1(\Omega)\cap H^2(\Omega) && (7)\\
&u_m'(0) = u_{1m} \longrightarrow u_1 \text{ strongly in } H_0^1(\Omega) && (8)\\
&\theta_m(0) = \theta_{0m} \longrightarrow \theta_0 \text{ strongly in } H_0^1(\Omega) && (9)
\end{aligned}$$

where $1 \le k \le m$. Then there exist functions c_{km} and d_{mk} such that

$$u_m(t) = \sum_{k=1}^m c_{km}(t)w_k \quad\text{and}\quad \theta_m(t) = \sum_{k=1}^m d_{km}(t)w_k$$

are the unique local solutions of the above system on some interval $[0, t_m[$, where $t_m \in [0, T[$.

The estimates that we obtain below will allow us to extend the solutions $\{u_m, \theta_m\}$ to the interval $[0, T[$.

Estimate (i). Multiply (5) by $c_{km}'(t)$ and multiply (6) by $d_{km}(t)$, then sum over k to obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{|u_m'(t)|^2 + \hat{M}(\|u_m(t)\|^2)\} + \frac{1}{p}\frac{d}{dt}\|u_m(t)\|_{L^p(\Omega)}^p && (10)\\
&= \quad -(\theta_m(t), u_m'(t)) + (f(t), u_m'(t)) && (11)
\end{aligned}$$

where $p = \rho + 2$.

$$\frac{1}{2}\frac{d}{dt}\{|\theta_m(t)|^2 + \|\theta_m(t)\|^2\} = -(u_m'(t), \theta_m(t)) + (g(t), \theta_m(t)) \qquad (12)$$

Define

$$E(u(t), \theta(t)) = \frac{1}{2}\{|u'(t)|^2 + |\theta(t)|^2 + \hat{M}(\|u(t)\|^2) + \|\theta(t)\|^2\} + \frac{1}{p}\|u(t)\|_{L^p(\Omega)}^p$$

where $\hat{M}(\lambda) = \int_0^\lambda M(s)ds$.

Sum (11) and (12). Using the inequality $ab \le \frac{1}{2}\left(a^2 + b^2\right)$ and the Poincaré inequality we integrate from 0 to $t \le t_m$ to obtain

$$\begin{aligned}
&\frac{1}{2}\{|u_m'(t)|^2 + |\theta_m(t)|^2 + m_0\|u_m(t)\|^2 + \|\theta_m(t)\|^2\} + \frac{1}{p}\|u_m(t)\|_{L^p(\Omega)}^p \\
&\le \quad E(u_m(t), \theta_m(t)) \\
&\le \quad E(u_{0m}, \theta_{0m}) + \frac{1}{2}\int_0^T \|f(s)\|^2 + \frac{1}{2}\int_0^T \|g(s)\|^2 ds \\
&\quad +\frac{3}{2}\int_0^t \|u_m'(s)\|^2 ds + \int_0^t |\theta_m(s)|^2 ds
\end{aligned}$$

From (7)–(9) and hypotheses (A.3), it follows from Gronwall's inequality that

$$\begin{aligned}|u_m'(t)|^2 + |\theta_m(t)|^2 + m_0\|u_m(t)\|^2 + \|\theta_m(t)\|^2 + \frac{1}{2}\|u_m(t)\|^p_{L^p}\\ \leq \{2E(u_0,\theta_0) + \int_0^T \|f(s)\|^2 ds + \int_0^T \|g(s)\|^2 ds\}e^T\end{aligned}$$

Then we extend the approximate solution $\{u_m(t), \theta_m(t)\}$ to the interval $[0, T[$ and we have the estimates

$$|u_m'(t)| \leq C_1, \quad \|u_m(t)\| \leq C_2, \quad \text{and} \quad \|\theta_m(t)\| \leq C_1 \tag{13}$$

where $C_1 = \{2E(u_0,\theta_0) + \int_0^T \|f(s)\|^2 ds + \int_0^T \|g(s)\|^2 ds\}e^T$ and $C_2 = C_1 m_0^{-1}$.

From now on we denote by C various positive constants independent of m and t in $[0, T[$.

Estimate (ii). Observe that the system (5), (6) is equivalent to

$$\begin{aligned}(u_m''(t), w) - M(\|u_m(t)\|^2)(\Delta u_m(t), w) + (|u_m(t)|^\rho u_m(t), w) + (\theta_m(t), w)\\ = (f(t), w)\end{aligned} \tag{14}$$

$$(\theta_m'(t), w) - (\Delta\theta_m(t), w) + (u_m'(t), w) = (g(t), w) \tag{15}$$

for all $w \in V_m$. Putting $w = -\Delta u_m'(t) \in V_m$ in (14) and $w = -\Delta\theta_m(t) \in V_m$ in (15) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\|u_m'(t)\|^2 + M(\|u_m(t)\|^2)|\Delta u_m(t)|^2\} \\ &= -(\nabla(|u_m(t)|^\rho u_m(t), \nabla u_m'(t)) + M'(\|u_m'(t)\|^2)(\nabla u_m(t), \nabla u_m'(t))|\Delta u_m(t)|^2 \\ &\quad -(\nabla u_m'(t), \nabla\theta_m(t)) + (\nabla f(t), \nabla u_m'(t))\end{aligned} \tag{16}$$

$$\frac{1}{2}\frac{d}{dt}\|\theta_m(t)\|^2 + |\Delta\theta_m(t)|^2 = -(\nabla u_m'(t), \nabla\theta_m(t)) + (\nabla g(t), \nabla\theta_m(t)) \tag{17}$$

Adding equations (16) and (17) we have:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\|u_m'(t)\|^2 + \|\theta_m(t)\|^2 + M(\|u_m(t)\|^2)|\Delta u_m(t)|^2\} + |\Delta\theta_m(t)|^2 \\ &= -(\nabla(|u_m(t)|^\rho u_m(t)), \nabla u_m'(t)) + M'(|u_m'(t)|^2)(\nabla u_m(t), \nabla u_m'(t))|\Delta u_m(t)|^2 \\ &\quad -2(\nabla u_m'(t), \nabla\theta_m(t)) + (\nabla f(t), \nabla u_m'(t)) + (\nabla g(t), \nabla\theta_m(t))\end{aligned} \tag{18}$$

We have that

$$\begin{aligned}|\,(\nabla(|u_m(t)|^\rho u_m(t)), \nabla u_m'(t))\,| \leq \\ (\rho+1)\textstyle\int_\Omega |u(t)|^\rho |\nabla u_m(t)\|\nabla u_m'(t)\,|dx(\rho+1)|u(t)|^\rho_{L^{\rho q}} \cdot |\nabla u_m(t)|_{L^r} \cdot \|u_m'(t)\,\|\end{aligned}$$

with $1/q + 1/r = 1/2$.

From hypotheses (A.2) we can take q and r such that

$$\frac{1}{q} \geq \frac{\rho(n-4)}{2n} \quad \text{and} \quad \frac{1}{r} \geq \frac{n-2}{2n}\,.$$

Sobolev's inequality gives

$$|\nabla u_m(t)|_{L^r} \leq C|u_m(t)|_{H^2} \quad \text{and} \quad |u_m(t)|_{L^{\rho q}} \leq C|u_m(t)|_{H^2}$$

and the regularity theory for elliptic equations ensures that

$$|u_m(t)|_{H^2} \leq C|\Delta u_m(t)|$$

(see, e. g., Friedman [4]).

Therefore,

$$|\,(\nabla(|u_m(t)|^\rho u_m(t)), \nabla u_m'(t))\,| \leq C|\Delta u_m(t)|^{\rho+1}\|u_m'(t)\| \tag{19}$$

The second, third, fourth, and fifth terms of the right side in (18) are bounded as follows

$$\left|M'(|u_m'(t)|^2)\,(\nabla u_m(t), \nabla u_m'(t))\,|\Delta u_m(t)|^2\right| \leq M_1C_2\|u_m'(t)\| \cdot |\Delta u_m(t)|^2$$

where $M_1 = \max\{|M'(s)|; 0 \leq s \leq C_2\}$,

$$\begin{aligned}
2|(\nabla u_m'(t)\,, \nabla\theta_m(t))| &\leq \|u_m'(t)\|^2 + \|\theta_m(t)\|^2 && (20)\\
|(\nabla f(t), \nabla u_m'(t))| &\leq \frac{1}{2}\|f(t)\|^2 + \frac{1}{2}\|u_m'(t)\|^2 && (21)\\
|(\nabla g(t), \nabla\theta_m(t))| &\leq \frac{1}{2}\|g(t)\|^2 + \frac{1}{2}\|\theta_m(t)\|^2 && (22)
\end{aligned}$$

Let us define the functional

$$F(u(t), \theta(t)) = \|u_m'(t)\|^2 + \|\theta_m(t)\|^2 + M(\|u(t)\|^2)|\Delta u(t)|^2 + |\Delta\theta(t)|^2\,.$$

Then by (13) we have

$$\begin{aligned}
&\|u_m'(t)\|^2 + \|\theta_m(t)\|^2 + m_0|\Delta u_m(t)|^2 + |\Delta\theta_m(t)|^2\\
&\quad\leq F(u_m(t), \theta_m(t))\\
&\quad\leq \|u_m'(t)\|^2 + \|\theta_m(t)\|^2 + M_2|\Delta u_m(t)|^2 + |\Delta\theta_m(t)|^2
\end{aligned} \tag{23}$$

where $M_2 = \max\{M(s); 0 \leq s \leq C_2^2\}$. Making use of inequalities (19)–(23) in (18) it follows that

$$\begin{aligned}
&\frac{d}{dt}F(u_m(t), \theta_m(t))\\
&\leq 2C|\Delta u_m(t)|^{\rho+1} \cdot \|u_m'(t)\| + 2M_1C_2\|u_m'(t))\| \cdot |\Delta u_m(t)|^2\\
&\quad + \|f(t)\|^2 + \|g(t)\|^2 + 3\|u_m'(t)\|^2 + 3\|\theta_m(t)\|^2
\end{aligned}$$

By (23) we have,

$$\begin{aligned}
&\frac{d}{dt}F(u_m(t),\theta_m(t)) \\
&\leq \quad C\left\{F(u_m(t),\theta_m(t))^{\frac{\rho+2}{2}} + F(u_m(t),\theta_m(t))^{\frac{3}{2}} + F(u_m(t),\theta_m(t))\right\} \\
&\qquad +\|f(t)\|^2 + \|g(t)\|^2
\end{aligned}$$

A simple computation shows that

$$\frac{d}{dt}F(u_m(t),\theta_m(t)) \leq C\{F(u_m(t),\theta_m(t))^\gamma + \|f(t)\|^2 + \|g(t)\|^2\},$$

with $\gamma = \max\{(\rho+2)/2,\ 3/2\}$. Here we need the following lemma which will be proved later.

Lemma 1 *Let μ a positive and differentiable function such that*

$$\mu'(t) \leq \theta(t) + \alpha\mu(t) + \beta\mu^\gamma(t) \tag{24}$$

where $\theta(t)$ is a positive function, $\theta \in L^1(0,T)$, α, β, and γ are positive constants, with $\gamma > 1$. Then there exists $T_0 \in \mathbb{R}$, where $0 < T_0 < T$, such that μ is bounded on $[0, T_0]$.

By Lemma 1, there exist $T_0 > 0$ such that

$$F(u_m(t),\theta_m(t)) \leq C \quad \text{for} \quad 0 \leq t \leq T_0$$

Hence, we have

$$\begin{aligned}
\|u'_m(t)\| &\leq C \qquad (25)\\
|\Delta u_m(t)| &\leq C \qquad (26)\\
|\Delta\theta_m(t)| &\leq C \qquad (27)\\
\|\theta_m(t)\| &\leq C \qquad (28)
\end{aligned}$$

for $0 \leq t \leq T_0$. Putting $w = \theta'_m(t)$ in (15) we have

$$\begin{aligned}
|\theta'_m(t)|^2 &\leq (|g(t)| + |\Delta\theta_m(t)| + |u'_m(t)|)\,|\theta'_m(t)| \\
|\theta'_m(t)| &\leq |g(t)| + |\Delta\theta_m(t)| + |u'_m(t)|
\end{aligned}$$

Now, using the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$, it follows from (25) and (27) that

$$|\theta'_m(t)| \leq C + |g(t)| \quad \text{or} \quad |\theta'_m(t)|^2 \leq C + 2|g(t)|^2\,.$$

Integrating from 0 to T_0, we have

$$\int_0^{T_0} |\theta'_m(t)|^2 dt \leq C \tag{29}$$

Estimate (iii). Putting $w = u''_m(t)$ in (14) we have

$$\begin{aligned}|u''_m(t)|^2 &= M(\|u_m(t)\|^2)(\Delta u_m(t), u''_m(t)) - (|u_m(t)|^\rho u_m(t), u''_m(t)) \\ &\quad -(\theta_m(t), u''_m(t)) + (f(t), u''_m(t))\end{aligned}$$

Then estimating we obtain

$$\begin{aligned}|u''_m(t)|^2 &\le M_2|\Delta u_m(t)|\, |u''_m(t)| + |u_m(t)|^{\rho+1}_{L^{2(\rho+1)}}|u''_m(t)| \\ &\quad +|\theta_m(t)|\, |u''_m(t)| + |f(t)|\, |u''_m(t)| \\ |u''_m(t)| &\le M_2|\Delta u_m(t)| + |u_m(t)|^{\rho+1}_{L^{2(\rho+1)}} + |\theta_m(t)| + |f(t)|\end{aligned}$$

By (A.3), it follows that $H^1_0(\Omega) \hookrightarrow L^{2(\rho+1)}$. Using (13), (25) and Sobolev's embedding theorem, from (26) we get

$$|u''_m(t)| \le C\,.$$

Passage to the limit

From estimates (13) and (25) we have that (u_m) and (θ_m) are bounded in $L^\infty(0,T_0;H^1_0(\Omega)\cap H^2(\Omega))$ and $L^\infty(0,T_0;H^1_0(\Omega))$, respectively. From (25) the sequence (u'_m) is bounded in $L^\infty(0,T_0;H^1_0(\Omega))$, and, by (2.35), the sequence (u''_m) is bounded in $L^\infty(0,T_0;L^2(\Omega))$. Because the embedding from $H^1_0(\Omega)\cap H^2(\Omega)$ into $H^1_0(\Omega)$ is compact we can extract a subsequence, again denoted by (u_m), such that:

$$u_m \longrightarrow u \;\text{ strongly in }\; L^2(0,T_0;H^1_0(\Omega))$$

Analogously, from (28), (29), the compact embedding $H^1_0(\Omega)$ into $L^2(\Omega)$, and the Aubin-Lions lemma (see, e.g., [5]) it follows that

$$\theta_m \longrightarrow \theta \;\text{ strongly in }\; L^2(0,T_0;L^2(\Omega))\,.$$

Then taking the limit in equations (5)–(6), when $m \longrightarrow \infty$, we have that $\{u\,,\theta\}$ is a weak solution of the system (1)–(4).

Proof of the Lemma 1. Multiply (24) by $e^{-\alpha t}$ to obtain

$$(\mu(t)e^{-\alpha t})' \le \theta(t) + \beta\mu^\gamma(t) \tag{30}$$

(Note that $e^{-\alpha t} \le 1$). Integrating (30) in $[0,t[\subset [0,T[$ we obtain

$$\mu(t) \le \left[\mu(0) + \int_0^T \theta(s)ds + \beta\int_0^t \mu^\gamma(s)ds\right] e^{\alpha T}$$

Letting

$$K_1 = \left[\mu(0) + \int_0^T \theta(s)ds\right] e^{\alpha T} \quad\text{and}\quad K_2 = \beta e^{\alpha T}$$

it follows that

$$\mu(t) \leq K_1 + K_2 \int_0^t \mu^\gamma(s)\,, ds\,. \tag{31}$$

If we denote by $z(t)$ the function $z(t) = \int_0^t \mu^\gamma(s)ds$, it follows that $z(0) = 0$ and $z'(t) = \mu^\gamma(t)$. Then,

$$\frac{z'(t)}{(K_1 + K_2 z(t))^\gamma} \leq 1$$

Choosing T_0 such that

$$K_1 + K_2 z(t) \leq K_3\,,$$

where

$$K_3 = \left\{ [\frac{K_1^{1-\gamma}}{K_2(\gamma-1)} - T_0]^{1/(\gamma-1)} \cdot [K_2(\gamma-1)]^{1/(\gamma-1)} \right\}^{-1}$$

Thus, from (31), we obtain $\mu(t) \leq K_3$, if $0 \leq t \leq T_0$. This concludes the proof of this Lemma.

3 Uniqueness

Let $[u, \theta]$ and $[\hat{u}, \hat{\theta}]$ be solutions of (1)–(4) under the conditions of Theorem 1. Let $w = u - \hat{u}$ and $v = \theta - \hat{\theta}$. Then $[w, v]$ satisfies

$$\begin{aligned} & \frac{d}{dt}(w', z) + M(\int_\Omega |\nabla u|^2 dx)(\nabla w, \nabla z) + (|u|^\rho u - |\hat{u}|^\rho \hat{u}, z) + (v, z)] \\ = \quad & M(\int_\Omega |\nabla \hat{u}|^2 dx)(\nabla \hat{u}, \nabla z) - M(\int_\Omega |\nabla u|^2 dx)(\nabla \hat{u}, \nabla z) \qquad (32) \\ & \frac{d}{dt}(v, z) + (\nabla v, \nabla z) + (w', z) = 0 \qquad (33) \\ & w(0) = 0, \quad w'(0) = 0 \quad \text{and} \quad v(0) = 0 \qquad (34) \end{aligned}$$

Taking $z = w'$ in (32) and $z = v$ in (33), we obtain

$$\begin{aligned} & \frac{d}{dt}|w'|^2 + M(\int_\Omega |\nabla u|^2 dx)\frac{d}{dt}\|w\|^2 + \int_\Omega (|u|^\rho u - |\hat{u}|^\rho \hat{u}) w' dx + (v, w') \\ = \quad & M(\int_\Omega |\nabla \hat{u}|^2 dx)(\nabla \hat{u}, \nabla w') - M(\int_\Omega |\nabla u|^2 dx)(\nabla \hat{u}, \nabla w') \qquad (35) \\ & \frac{d}{dt}|v|^2 + \|v\|^2 + (w', v) = 0 \qquad (36) \end{aligned}$$

in the $D'(0, T)$ sense. Adding (35) to (36) we have

$$\begin{aligned} & \frac{d}{dt}|w'|^2 + M(\int_\Omega |\nabla u|^2 dx)\frac{d}{dt}\|w\|^2 + \frac{d}{dt}|v|^2 + \|v\|^2 \\ = \quad & \int_\Omega (|\hat{u}|^\rho \hat{u} - |u|^\rho u) w' dx - 2(v, w') + M(\int_\Omega |\nabla \hat{u}|^2 dx)(\nabla \hat{u}, \nabla w') \end{aligned}$$

$$\begin{aligned}
&-M(\int_\Omega |\nabla u|^2 dx)(\nabla\hat u, \nabla w') \\
\le\quad & \left|\int_\Omega (|\hat u|^\rho \hat u - |u|^\rho u) w' dx\right| + 2|(v,w')| \\
&+\left|M(\int_\Omega |\nabla\hat u|^2 dx) - M(\int_\Omega |\nabla u|^2 dx)\right| |(\nabla\hat u, \nabla w')|
\end{aligned}$$

On the other hand, by Holder's inequality with $\frac{1}{q}+\frac{1}{n}+\frac{1}{2}=1$, we have

$$\begin{aligned}
\left|\int_\Omega (|\hat u|^\rho \hat u - |u|^\rho u) w' dx\right| \quad &\le \quad (\rho+1)\int_\Omega \sup(|u|^\rho, |\hat u|^\rho)|w|\,|w'|dx \\
&\le \quad C\left(\|\,|u|^\rho\|_{L^n(\Omega)} + \|\,|\hat u|^\rho\|_{L^n(\Omega)}\right)\,\|w\|_{L^q(\Omega)}|w'|_{L^2(\Omega)}
\end{aligned}$$

By condition (A.2), we have $\rho n \le q$ and from the immersion $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ with $1/q = 1/2 - 1/n$, we have

$$\left|\int_\Omega (|\hat u|^\rho \hat u - |u|^\rho u) w' dx\right| \le C(\|u\|^\rho + \|\hat u\|^\rho)\,\|w\|\,|w'|$$

and since $u,\, \hat u \in L^\infty(0,T;H_0^1(\Omega))$, we have

$$\begin{aligned}
\left|\int_\Omega (|\hat u|^\rho \hat u - |u|^\rho u) w' dx\right| \quad &\le \quad C\|w\|\,|w'| &(37)\\
2|(v,w')| \quad &\le \quad 2|v|\,|w'| &(38)
\end{aligned}$$

Observe that

$$\begin{aligned}
&\left|M(\int_\Omega |\nabla\hat u|^2 dx) - M(\int_\Omega |\nabla u|^2 dx)\right| |(\nabla\hat u, \nabla w')| \\
&\le \quad |M'(\xi)|\,\big||\nabla\hat u|^2 - |\nabla u|^2\big|\,|(-\Delta)\hat u|\,|w'|
\end{aligned}$$

where ξ is between $|\nabla\hat u|^2$ and $|\nabla u|^2$. Then we have

$$\begin{aligned}
&\left|M(\int_\Omega |\nabla\hat u|^2 dx) - M(\int_\Omega |\nabla u|^2 dx)\right| |(\nabla\hat u, \nabla w')| \\
&\le \quad C\,||\nabla\hat u| + |\nabla u||\,\;||\nabla\hat u| - |\nabla u||\,|(-\Delta)\hat u|\,|w'| &(39)\\
&\le \quad C\|\hat u - u\|\,|(-\Delta)\hat u|\,|w'| \\
&\le \quad C\|w\|\,|w'|
\end{aligned}$$

Substituting (37)–(39) in (35) and noting that

$$\begin{aligned}
&M(\int_\Omega |\nabla u|^2 dx)\frac{d}{dt}|\nabla w|^2 \\
=\quad & \frac{d}{dt}\left(M(\int_\Omega |\nabla u|^2 dx)|\nabla w|^2\right) - \left[\frac{d}{dt}M(\int_\Omega |\nabla u|^2 dx)\right]|\nabla w|^2
\end{aligned}$$

we obtain:

$$\begin{aligned}
&\frac{d}{dt}\left\{|w'|^2+|v|^2+M(\int_\Omega|\nabla u|^2dx)|\nabla w|^2\right\}+\|v\|^2\\
&\leq \quad |v|^2+C|w'|^2+C\|w\|^2+\left|\frac{d}{dt}M(\int_\Omega|\nabla u|^2dx)\right||\nabla w|^2 \qquad (40)\\
&\leq \quad C\left\{|v|^2+|w'|^2+\|w\|^2\right\}
\end{aligned}$$

Integrating (40) from 0 to $t\leq T_0$, we have

$$\begin{aligned}
&|w'(t)|^2+|v(t)|^2+m_0\|w(t)\|^2+\int_0^T\|v(s)\|^2ds\\
&\leq \quad C\int_0^t\left\{|v(s)|^2+|w'(s)|^2+\|w(s)\|^2\right\}ds
\end{aligned}$$

By Gronwall's Lemma it follows that

$$|v(s)|^2+|w'(s)|^2+\|w(s)\|^2\leq 0\,.$$

This implies that $v(t)=w(t)=0\ \forall t\in[0,T]$. Or $u(t)=\hat{u}(t)$ and $\theta(t)=\hat{\theta}(t)\ \forall t\in[0,T]$. This concludes the proof of uniqueness.

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