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Solutions to perturbed eigenvalue problems of the p-Laplacian in $\mathbb{R}^N$ *

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Abstract

Using a variational approach, we investigate the existence of solutions for non-autonomous perturbations of the $p-$Laplacian eigenvalue problem

$$-\Delta_p u = f(x,u) \quad \text{in} \quad \mathbb{R}^N .$$

Under the assumptions that the primitive $F(x,u)$ of $f(x,u)$ interacts only with the first eigenvalue, we look for solutions in the space $D^{1,p}(\mathbb{R}^N)$. Furthermore, we assume a condition that measures how different the behavior of the function $F(x,u)$ is from that of the $p-$power of u.

1 Introduction

In this paper we study the existence of solutions for non-autonomous perturbations of the p-Laplacian eigenvalue problem

$$-\Delta_p u \equiv -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(x,u) \quad \text{in} \quad \mathbb{R}^N , \tag{1}$$

where $u \in D^{1,p}(\mathbb{R}^N)$. We assume that the primitive $F(x,u)$ of the nonlinearity $f(x,u)$ interacts only with the first eigenvalue of some p-Laplacian eigenvalue problems with weights naturally associated with $F(x,u)$. We also assume that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the growth condition,

$$(f) \qquad |f\,(x,u)\,| \le a(x)|u|^r + b(x)|u|^s, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}$$

where a, b are continuous functions, $a \in L^\infty(\mathbb{R}^N) \cap L^{r_0}(\mathbb{R}^N)$ and $b \in L^\infty(\mathbb{R}^N) \cap L^{s_0}(\mathbb{R}^N)$ with $0 \le r \le p-1 \le s < p^* - 1$, $r_0 = Np/[Np-(r+1)(N-p)]$, and $s_0 = Np/[Np-(s+1)(N-p)]$. In this context $1 < p < N$, and p^* denotes the critical Sobolev exponent, $p^* = Np/(N-p)$.

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We are interested in finding weak solutions of (1) in the framework of the reflexive Banach space $D^{1,p}(\mathbb{R}^N)$, defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{D^{1,p}} = \|\nabla u\|_{L^p}$. By the Sobolev inequality, $\|u\|_{L^{p^*}} \le C_0\|\nabla u\|_{L^p}$ for all $u \in D^{1,p}(\mathbb{R}^N)$, we see that $D^{1,p}(\mathbb{R}^N)$ can be embedded continuously in $L^{p^*}(\mathbb{R}^N)$ and that

$$D^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in (L^p(\mathbb{R}^N))^N\}\,.$$

For more details about this space, see e.g. [3]. As we shall see, under assumption (f), the functional

$$I\,(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx, \quad F(x,u) = \int_0^u f(x,s)\,ds,$$

is a C^1 weakly lower semi-continuous functional defined on $D^{1,p}(\mathbb{R}^N)$, and

$$I'\,(u)\,\varphi = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla\varphi dx - \int_{\mathbb{R}^N} f(x,u)\varphi dx, \quad \forall\varphi \in D^{1,p}(\mathbb{R}^N).$$

Notice that critical points of the functional I are precisely the weak solutions of (1). To get critical points of I we apply two "minimax" methods: a mountain pass type argument and a minimization technique. To do this, we explore the interaction of the potential $F(x,u)$ with the first eigenvalue, in combination with the following hypotheses which are related with a compactness condition of Palais-Smale type.

(F_1) There exists a measurable function $a \in L^\infty(\mathbb{R}^N)$ and a constant μ, $0 < \mu < p^*$ such that

$$f(x,u)u - pF(x,u) \ge a(x)|u|^\mu > 0, \quad \forall(x,u) \in \mathbb{R}^N \times (\mathbb{R} - \{0\}),$$

(F_2) For some $p < q < p^*$, $\limsup\limits_{|u|\to+\infty} \frac{F(x,u)}{a(x)|u|^q} \le M < +\infty$ uniformly $x \in \mathbb{R}^N$

Let $\lambda_1(m)$ denote the first eigenvalue of the weighted nonlinear eigenvalue problem in $\mathbb{R}^N$,

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u,$$

where $m \in L^{N/p}(\mathbb{R}^N)$ is a weight function which is positive on a subset of positive measure. It is convenient to recall the variational characterization

$$\lambda_1(m) = \inf\{\int_{\mathbb{R}^N} |\nabla u|^p dx / \int_{\mathbb{R}^N} m|u|^p dx : u \in D^{1,p}(\mathbb{R}^N) - \{0\}\}\,, \tag{2}$$

and that $\lambda_1(m)$ is a positive real number, since by the Holder and Sobolev inequalities

$$\int_{\mathbb{R}^N} m|u|^p dx \le \|m\|_{L^{N/p}}\|u\|^p_{L^{p^*}} \le C\|m\|_{L^{N/p}}\|u\|^p_{D^{1,p}}.$$

Thus,

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^p dx}{\int_{\mathbb{R}^N} m|u|^p dx} \geq \frac{1}{C\|m\|_{L^{N/p}}} > 0,$$

for all $u \in D^{1,p}(\mathbb{R}^N) - \{0\}$. For more details about this eigenvalue problem, see e. g. [1].

Now, we are ready to present the main results of this article.

Theorem 1 *Assume that (f), (F_1), and (F_2) are satisfied. Furthermore, suppose*

(F_3) There exist a function $\alpha \in L^\infty(\mathbb{R}^N) \cap L^{\frac{N}{p}}(\mathbb{R}^N)$ and a positive Conant δ, such that

$$F(x,u) \leq \frac{1}{p}\alpha(x)|u|^p \quad \forall x \in \mathbb{R}^N, \ \forall |u| \leq \delta,$$

where either $\alpha \leq 0$ or $\lambda_1(\alpha) > 1$,

(F_4) There exist a function $\omega \in L^\infty(\mathbb{R}^N) \cap L^{\frac{N}{p}}(\mathbb{R}^N)$ and a positive constant R, such that

$$F(x,u) \geq \frac{1}{p}\omega(x)|u|^p \quad \forall x \in \mathbb{R}^N, \ \forall |u| \geq R,$$

where $\omega > 0$ on a subset of positive measure and $\lambda_1(\omega) < 1$.

Then, problem (1) has a nontrivial solution, provided that $0 < N(q-p)/p < \mu < q < p^*$.

Next, we consider the case when the potential $F(x,u)$ approaches $\omega(x)|u|^p$ as $|u| \to \infty$, and therefore, it interacts with the first eigenvalue of the problem $-\Delta_p u = \lambda\omega(x)|u|^{p-2}u$. In fact, we assume that $F(x,u)$ interacts with the first eigenvalue $\lambda_1(\omega)$, but its behavior is different from that of a $p-$power of u. As we shall see, these facts lead to a standard situation for a global minimum.

Theorem 2 *Assume that (f) and (F_1) hold. Furthermore, suppose*

(F_5) There is a function $\omega \in L^\infty(\mathbb{R}^N) \cap L^{\frac{N}{p}}(\mathbb{R}^N)$ such that

$$\limsup_{|u|\to+\infty} \frac{pF(x,u)}{|u|^p} \leq \omega(x) \quad uniformly\ x \in \mathbb{R}^N,$$

where either ω is ≤ 0 or $\lambda_1(\omega) \geq 1$.

Then, problem (1) has a solution which is a global minimum of the associated functional, provided that $0 < \mu < p$.

Remark 1. If in Theorem 1, assumption (F_1) is replaced by its "dual",

$$(\widetilde{F}_1) \qquad f(x,u)u - pF(x,u) \leq -a(x)|u|^{\mu} < 0, \quad \forall (x,u) \in \mathbb{R}^N \times (\mathbb{R} - \{0\}),$$

it is not difficult to see that similar result holds. Hypotheses of this kind are a measure of how different the behavior of F is from that of a $p-$power of u. It is worth to observe that in many cases condition (F_1) is implied by the following Ambrosetti-Rabinowitz condition,

(AR) $0 < \theta F(x,u) \leq uf(x,u)$, for all $(x,u) \in \mathbb{R}^N \times (\mathbb{R} - \{0\})$ and some $\theta > p$.

Indeed, by (AR), $f(x,u)u - pF(x,u) \geq (\theta - p)F(x,u)$ and, moreover, $F(x,u) \geq \min\{F(x,1), F(x,-1)\}|u|^{\theta} > 0, \quad \forall x \in \mathbb{R}^N$ and $|u| \geq 1$. Thus, (F_1) holds if we assume also that there exist $\mu \leq \theta$ and a measurable function $a \in L^{\infty}(\mathbb{R}^N)$ such that $F(x,u) \geq a(x)|u|^{\mu} > 0$, for all $x \in \mathbb{R}^N$ and $|u| \leq 1$. On the other hand, Example 1 shows a function which satisfies condition (F_1), but does not satisfy (AR). Requirements similar to $(F_1)-(F_2)$ were considered by Costa and Miyagaki in [7] (see also [8, 9]) where a is constant. They obtained a nontrivial solution for autonomous perturbations of the p-Laplacian on a unbounded cylindrical domain, $\Omega = \Omega_0 \times \mathbb{R}^{N-K}$, with Ω_0 bounded domain of $\mathbb{R}^K$. Their solution, in the space $W_0^{1,p}(\Omega)$, is obtained using the mountain pass argument combined with the concentration-compactness principle of Lions.

In order to get the geometry of the mountain-pass, the number

$$\lambda_1 = \inf\{\int_{\Omega} |\nabla u|^p dx / \int_{\Omega} |u|^p dx : u \in W_0^{1,p}(\Omega) - \{0\}\},$$

has been explored, and it is known ([5, 14]) to be equal to the first eigenvalue of $-\Delta_p$ acting on $W_0^{1,p}(\Omega_0)$. In fact, the following condition for crossing the first eigenvalue has been used,

$$\limsup_{u \to 0} pF(u)/|u|^p \leq \xi < \lambda_1 < \eta \leq \liminf_{|u| \to \infty} pF(u)/|u|^p .$$

For the non-autonomous perturbations of the p-Laplacian studied here, the crossing of the first eigenvalue is expressed by requirements such as (F_3) and (F_4). These conditions give the geometric shape required by the Mountain Pass Theorem. Condition (F_3) together with the growth condition (f) lead to the fact that the origin is a local minimum of the associated functional, while assumption (F_4) implies that the functional is not bounded below. The interaction of the potential $F(x,u)$ with the first eigenvalue in elliptic eigenvalue problems with weights in bounded domains have been considered by de Figueiredo and Massabó in [11].

Remark 2. The same procedures as in Theorem 1, along with obvious modifications, show the existence of a nontrivial solution to the non-autonomous perturbation of the p-Laplacian studied by Yu in [18]:

$$l(u) \equiv -\operatorname{div}(\big(a(x)|\nabla u|^{p-2}\nabla u + b(x)|u|^{p-2}u\big) = f(x,u) \text{ in } \Omega \text{ and } u|_{\partial\Omega} = 0,$$

where Ω is a $C^{1,\delta}$ $(0<\delta<1)$ exterior domain in $\mathbb{R}^N$, $0 < a_0 \leq a(x) \in L^\infty(\Omega) \cap C^\delta(\overline{\Omega})$, and $0 \leq b(x) \in L^\infty(\Omega) \cap C(\Omega)$. Following this approach we complement Theorem 1 in Yu, which corresponds to the to "super-linear case". In [18], a solution was obtained by working in the framework of a reflexive Banach space E, defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_E^p = \int_\Omega [|\nabla u|^p + w(x)|u|^p]dx$, where $w(x) = \max\{b(x), (1+|x|)^{-p}\}$. From the Hardy-type inequality, $\int_\Omega (1+|x|)^{-p}|u|^p dx \leq \int_\Omega [|\nabla u|^p\, dx$ for all $u \in C_0^\infty(\Omega)$ it follows that the norm induced by the operator l, $\|u\|_l^p = \int_\Omega [|a(x)\nabla u|^p + b(x)|u|^p]dx$, is equivalent to the norm in E. Therefore, in order to establish conditions of type (F_3) and (F_4) we must explore "crossing conditions" involving the number

$$\lambda_1(m) = \inf\{\|u\|_l^p / \int_\Omega m(x)|u|^p dx : u \in E - \{0\}\},$$

which is positive in view of the Holder and Sobolev inequalities.

Remark 3. Problems involving unbounded domains have been studied recently by several authors, among others Berestycki and Lions [4], Strauss [17], and Drábek [12] (see also references therein).

2 Preliminary Results

Throughout this work C will denote a generic positive constant. Results similar to the one in the next lemma have appeared in [18], and we sketch its proof here for the benefit of the reader.

Lemma 1 *Suppose that f is continuous and satisfies (f). Then the functional $I : D^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ given by*

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \int_{\mathbb{R}^N} F(x,u)dx,$$

is well defined and of class C^1 with

$$I'(u)\,\varphi = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi dx - \int_{\mathbb{R}^N} f(x,u)\varphi dx, \quad \forall \varphi \in D^{1,p}(\mathbb{R}^N). \tag{3}$$

Moreover, the functional

$$K(u) = \int_{\mathbb{R}^N} F(x, u(x))\, dx$$

is weakly continuous and K' is compact.

Proof. As a consequence of assumption (f), with the aid of the Holder and Sobolev inequalities, we see that I and $I'(u)$ are well defined on $D^{1,p}(\mathbb{R}^N)$.

The idea in the rest of the proof is to use results similar to the ones known on the case of bounded domains. We also use that the restriction operator is continuous, that $\|a\|_{L^{r_0}(\mathbb{R}^N - B(0,r_n))} \to 0$, and that $\|b\|_{L^{s_0}(\mathbb{R}^N - B(0,r_n))} \to 0$ as $n \to \infty$, for any (r_n) increasing and unbounded sequence of positive real numbers.

Let $B(0,r_n)$ denote the ball of radius r_n centered at the origin of $\mathbb{R}^N$, and let $K_n : D^{1,p}(B(0,r_n)) \to \mathbb{R}$ denote the functional given by

$$K_n(u) = \int_{B(0,r_n)} F(x,u)dx.$$

In view of (f), it is well known that $K_n \in C^1\left(D^{1,p}\left(B(0,r_n)\right)\right)$ and that

$$K_n'(u)(v) = \int_{B(0,r_n)} f(x,u)vdx, \quad \forall v \in D^{1,p}\left(B(0,r_n)\right).$$

Moreover, K_n is weakly continuous and $K_n^{'}(u)$ is compact (cf. [10, 15]).

Next we prove that K is weakly continuous. Let $u_k \rightharpoonup u$ weakly in $D^{1,p}(\mathbb{R}^N)$ as $k \to \infty$. Using hypothesis (f) and the Holder and Sobolev inequalities, we obtain

$$\begin{aligned} |K(u_k) - K(u)| \quad \leq \quad & |K_n(u_k) - K_n(u)| \\ & + C\{\|a\|_{L^{r_0}(\mathbb{R}^N - B(0,r_n))}(\|u_k\|^{r+1}_{D^{1,p}} + \|u\|^{r+1}_{D^{1,p}}) \\ & + \|b\|_{L^{s_0}(\mathbb{R}^N - B(0,r_n))}(\|u_k\|^{s+1}_{D^{1,p}} + \|u\|^{s+1}_{D^{1,p}})\}\,. \end{aligned}$$

Given $\epsilon > 0$, we choose r_n sufficiently large such that

$$\max\{\|a\|_{L^{r_0}(\mathbb{R}^N - B(0,r_n))}, \|b\|_{L^{s_0}(\mathbb{R}^N - B(0,r_n))}\} < \epsilon/2C\,. \tag{4}$$

On the other hand, since the restriction operator $u \longmapsto u|_{B(0,r_n)}$ is continuous from $D^{1,p}(\mathbb{R}^N)$ into $D^{1,p}(B(0,r_n)$ and K_n is weakly continuous, up to a subsequence, we have

$$|K_n(u_k) - K_n(u)| < \epsilon/2\,. \tag{5}$$

Combining (4) and (5) we conclude that K is weakly continuous. To prove that $I \in C^1\left(D^{1,p}(\mathbb{R}^N)\right)$, we must show that $K \in C^1\left(D^{1,p}(\mathbb{R}^N)\right)$ and that

$$K'(u)v = \int_{\mathbb{R}^N} f(u,x)vdx,$$

since the first term in I is C^1 and its Fréchet derivative is the first term in (3). For fixed $u \in D^{1,p}(\mathbb{R}^N)$ and given $\epsilon > 0$, we must show that there exists $\delta = \delta(u,\epsilon)$ such that

$$|\int_{\mathbb{R}^N} F\left(x, u+v\right) dx - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} f(u,x)vdx| \leq \epsilon\|v\|_{D^{1,p}},$$

for all $v \in D^{1,p}(\mathbb{R}^N)$ with $\|v\|_{D^{1,p}} \le \delta$. Notice that if $\|v\|_{D^{1,p}} \le 1$,

$$\begin{aligned} &|\int_{\mathbb{R}^N} F\left(x, u+v\right)\, dx - \int_{\mathbb{R}^N} F(x,u) dx - \int_{\mathbb{R}^N} f(u,x) v\, dx| \\ &\quad\le \quad |K_n(u+v) - K_n(v) - K_n'(u)v| \\ &\qquad\quad +C\|v\|_{D^{1,p}}\{\|a\|_{L^{r_0}(\mathbb{R}^N - B(0,r_n))}(\|u_k\|^r_{D^{1,p}} + \|u\|^r_{D^{1,p}}) \\ &\qquad\quad +\|b\|_{L^{s_0}(\mathbb{R}^N - B(0,r_n))}(\|u_k\|^s_{D^{1,p}} + \|u\|^s_{D^{1,p}})\}\,. \end{aligned}$$

Since $a \in L^{r_0}(\mathbb{R}^N)$ and $b \in L^{s_0}(\mathbb{R}^N)$, as before, we can choose r_n sufficiently large such that $\max\{\|a\|_{L^{r_0}(\mathbb{R}^N - B(0,r_n))}, \|b\|_{L^{s_0}(\mathbb{R}^N - B(0,r_n))}\} < \epsilon/2C$. Using that $K_n \in C^1\left(D^{1,p}\left(B(0,r_n)\right)\right)$, there exists $\delta = \delta(u,\epsilon)$ such that, for all $v \in D^{1,p}(\mathbb{R}^N)$ with $\|v\|_{D^{1,p}} \le \delta$,

$$|K_n(u+v) - K_n(v) - K_n'(u)v| \le \epsilon \|v\|_{D^{1,p}},$$

and the proof that I is Fréchet differentiable is complete. To prove that K' is continuous we use the estimate

$$\begin{aligned} |K'\left(u_k\right) - K'\left(u\right)| \quad \le \quad & |K_n'\left(u_k\right) - K_n'\left(u\right)| \\ & +C\|a\|_{L^{r_0}(\mathbb{R}^N - B(0,r_n))}(\|u_k\|^r_{D^{1,p}} + \|u\|^r_{D^{1,p}}) \\ & +C\|b\|_{L^{s_0}(\mathbb{R}^N - B(0,r_n))}(\|u_k\|^s_{D^{1,p}} + \|u\|^s_{D^{1,p}}) \end{aligned}$$

together with the facts that K_n' is continuous, and that $\|a\|_{L^{r_0}(\mathbb{R}^N - B(0,r_n))}$ and $\|b\|_{L^{s_0}(\mathbb{R}^N - B(0,r_n))}$ converge to zero as $n \to \infty$.

Finally, to prove compactness, we use the diagonal method. Let $u_k \rightharpoonup u$ weakly in $D^{1,p}(\mathbb{R}^N)$ as $k \to \infty$. Since $a \in L^{r_0}(\mathbb{R}^N)$ and $b \in L^{s_0}(\mathbb{R}^N)$ we can choose an increasing and unbounded sequence of positive real numbers (r_n) such that

$$\max\{\|a\|_{L^{r_0}(\mathbb{R}^N - B(0,r_n))}, \|b\|_{L^{s_0}(\mathbb{R}^N - B(0,r_n))}\} \le 1/2n.$$

On the other hand, for each natural number n, we have a subsequence (u_{kn}), of (u_k), such that

$$\|K_n'(u_{kn}) - K_n'(u)\| \le 1/2n,$$

since K_n' is compact. Thus, combining these two estimates,

$$\|K'(u_{kn}) - K'(u)\| \le C/n.$$

Therefore, the diagonal subsequence (u_{kk}) leads to $K'(u_{kk}) \to K'(u)$. □

In the next lemmas we prove that the functional I satisfies a compactness condition of the Palais-Smale type which was introduced by Cerami in [6].

Definition Let $J : E \to \mathbb{R}$ be a C^1 functional. We say that J satisfies condition (C) if every sequence (u_n) in E such that

$$\begin{array}{ll} (i) & J(u_n) \to c \\ (ii) & (1 + \|u_n\|_{D^{1,p}})\|J'(u_n)\|_{(D^{1,p})^*} \to 0, \end{array} \tag{6}$$

possesses a convergent subsequence.

Using this compactness condition, Bartolo, Benci and Fortunato in [2] obtained rather general minimax results.

Lemma 2 (Compactness Condition I) *Assume that (f), (F_1), (F_2), and (F_3) are satisfied with $0 < N(q-p)/p < \mu < q < p^*$. Then, the functional I satisfies the compactness condition (C).*

Proof. Let $(u_n) \subset D^{1,p}(\mathbb{R}^N)$ be a sequence satisfying (6). From (F_1) together with (6), we have

$$\begin{aligned} C &\geq pI(u_n) - I'(u_n)u_n \\ &= \int_{\mathbb{R}^N} [f(x,u_n)u_n - pF(x,u_n)]dx \\ &\geq \int_{\mathbb{R}^N} a(x)|u_n|^\mu dx, \end{aligned}$$

for some positive constant C and for sufficiently large n. Thus, (u_n) is a bounded sequence in the Lebesgue space

$$L_a^\mu(\mathbb{R}^N) = \{u : \int_{\mathbb{R}^N} a(x)|u_n|^\mu dx < \infty\}.$$

From [16], we have the interpolation inequality

$$\|u\|_{L_a^q} \leq \|u\|_{L_a^\mu}^{1-t} \|u\|_{L_a^{p^*}}^t, \quad \forall u \in L_a^\mu(\mathbb{R}^N) \cap L_a^{p^*}(\mathbb{R}^N),$$

where $0 < \mu < q < p^*$ and $t \in (0,1)$ are such that $1/q = (1-t)/\mu + t/p^*$. The previous two inequalities and $a \in L^\infty(\mathbb{R}^N)$ imply

$$\|u_n\|_{L_a^q} \leq C\|u_n\|_{L^{p^*}}^t. \tag{7}$$

Using (F_2) and (F_3) we obtain

$$F(x,u) \leq \frac{\alpha(x)}{p}|u|^p + Ca(x)|u|^q, \ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus, from (i) in (6), we achieve

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_n|^p dx &= pI(u_n) + p\int_{\mathbb{R}^N} F(x,u_n)\,dx \\ &\leq C + \int_{\mathbb{R}^N} \alpha(x)|u_n|^p dx + C\int_{\mathbb{R}^N} a(x)|u_n|^q\,dx\,, \end{aligned}$$

which, in view of the variational characterization of the first eigenvalue $\lambda_1(\alpha)$ and (7), guarantees that

$$\begin{aligned}\|u_n\|^p_{D^{1,p}} &\leq C + \lambda_1^{-1}(\alpha)\|u_n\|^p_{D^{1,p}} + C\|u_n\|^{tq}_{L^{p^*}(\mathbb{R}^N)}\\ &\leq C + \lambda_1^{-1}(\alpha)\|u_n\|^p_{D^{1,p}} + C\|u_n\|^{tq}_{D^{1,p}}\,,\end{aligned}$$

where, in the last estimate we used the Sobolev inequality. Thus,

$$(1-\lambda_1^{-1}(\alpha))\|u_n\|^p_{D^{1,p}} \leq C(1+\|u_n\|^{tq}_{D^{1,p}}).$$

Therefore, due to $\lambda_1(\alpha) > 1$, the sequence (u_n) is bounded, since $\mu > \frac{N}{p}(q-p)$ is equivalent to $tq < p$.

Finally, by considering the functional $J : D^{1,p}(\mathbb{R}^N) \to \mathbb{R}$; $J(u) = \frac{1}{p}\|u_n\|^p_{D^{1,p}}$, whose derivative, $J' : D^{1,p}(\mathbb{R}^N) \to J'(D^{1,p}(\mathbb{R}^N))$, $J'(u)\varphi = \int |\nabla u|^{p-2}\nabla u \nabla \varphi$, is an isomorphism, we have

$$(J')^{-1}I'(u) = u - (J')^{-1}K'(u).$$

This fact together with $\|I'(u_n)\|_{(D^{1,p})^*} \to 0$ and the compactness of K', established in Lemma 1, imply that (u_n) has a convergent subsequence. □

Lemma 3 (Compactness Condition - II) *Assume that* (f), (F_1), *and* (F_5) *hold. Then the functional* I *satisfies the compactness condition* (C), *provided that* $0 < \mu < p$.

Proof. Let (u_n) be a sequence, in $D^{1,p}(\mathbb{R}^N)$, satisfying

$$\begin{array}{ll}(i) & I(u_n) \to c\\ (ii) & (1+\|u_n\|_{D^{1,p}})\|I'(u_n)\|_{(D^{1,p})^*} \to 0.\end{array} \tag{8}$$

Due to the same argument used in the proof of the previous lemma, we need only to prove that the sequence (u_n), is bounded. The proof will be carried out by contradiction: suppose, up to a subsequence, that $\|u_n\|_{D^{1,p}} \to \infty$. Proceeding as in the proof of Lemma 2, condition (F_1) together with (8) lead to

$$\|u_n\|_{L^\mu_a} \leq C. \tag{9}$$

Note that, by assumption (F_1),

$$\frac{d}{ds}\left(\frac{F(x,s)}{s^p}\right) = \frac{f(x,s)s - pF(x,s)}{s^{p+1}} \geq a(x)s^{\mu-p-1},$$

for all $x \in \mathbb{R}^N$ and $s > 0$. Now, integrating the above inequality over an interval $[u, U] \subset (0, +\infty)$, we obtain

$$\frac{F(x,U)}{U^p} - \frac{F(x,u)}{u^p} \geq \frac{a(x)}{\mu-p}\left(\frac{1}{U^{p-\mu}} - \frac{1}{u^{p-\mu}}\right).$$

From the hypothesis $\mu < p$, passing to the lim sup in the last inequality, as $U \to +\infty$, and using condition (F_5), we obtain

$$F(x,u) \le \frac{1}{p}\omega(x)u^p - \frac{a(x)}{p-\mu}u^\mu, \quad \forall (x,u) \in \mathbb{R}^N \times (0,+\infty).$$

Similarly, we show that

$$F(x,u) \le \frac{1}{p}\omega(x)|u|^p - \frac{a(x)}{p-\mu}|u|^\mu, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{10}$$

Let $v_n = u_n/\|u_n\|_{D^{1,p}}$. By passing to subsequences, we can find that v_n converges weakly in $D^{1,p}(\mathbb{R}^N)$ and almost everywhere to a function $v_0 \in D^{1,p}(\mathbb{R}^N)$. Therefore,

$$\int_{\mathbb{R}^N} \omega(x)|v_n|^p\,dx \to \int_{\mathbb{R}^N} \omega(x)|v_0|^p\,dx\,, \tag{11}$$

because, by lemma 1, the functional $K(u) \doteq \int_{\mathbb{R}^N} \omega(x)|u|^p dx$ is weakly continuous. Dividing (8.i) by $\|u_n\|^p_{D^{1,p}}$ and using estimate (10), we have

$$\frac{1}{p}\int_{\mathbb{R}^N} |\nabla v_n|^p dx - \frac{1}{p}\int_{\mathbb{R}^N} \omega(x)|v_n|^p dx + \frac{\|u_n\|^\mu}{(p-\mu)\|u_n\|_{D^{1,p}}} \le \frac{C}{\|u_n\|^p_{D^{1,p}}}.$$

In view of $\|u_n\|_{D^{1,p}} \to \infty$, (9), and (11), when passing to the limit we obtain

$$1 \le \int_{\mathbb{R}^N} \omega(x)|v_0|^p\,;$$

therefore, v_0 is not identically zero. On the other hand, from (9), we have

$$\int_{\mathbb{R}^N} a(x)|v_n|^\mu dx \le \frac{C}{\|u_n\|^\mu_{D^{1,p}}}.$$

Which by Fatou's Lemma, implies

$$\int_{\mathbb{R}^N} a(x)|v_0|^\mu dx \le 0$$

and $v_0 = 0$, almost everywhere, which is a contradiction. □

Proof of Theorem 1

To obtain a nontrivial critical point of the functional I, will apply the following version of the Mountain-Pass Theorem, with condition (C) instead of the usual Palais-Smale compactness condition.

Theorem 3 *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying condition (C). Suppose that $I(0) = 0$ and for some $\sigma, \rho > 0$ and $e \in E$, with $\|e\| > \rho$, one has $\sigma \le \inf_{\|u\|=\rho} I(u)$ and $I(e) < 0$. Then I has a critical value $c \ge \sigma$, characterized by*

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le \tau \le 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) = e\}$.

Remark 4. It is not difficult to see that the same proof of the standard Mountain-Pass Theorem (cf. [15]) applies to the present context; since the deformation theorem, Theorem 1.3 in [2], is obtained with condition (C) in a Banach space framework. It is worth to observe also that this version of the Mountain-Pass Theorem can be obtained as a consequence of Lemma 5 in [13].

The proof of Theorem 1 follows from Lemma 2, where the compactness condition is proved, and the next lemma, where the geometric conditions are checked.

Lemma 4 (Mountain-Pass Geometry) *Suppose (f), (F_3), and (F_4) hold. Then there exist positive constants σ and ρ such that $I(u) \ge \sigma$ if $\|u\|_{D^{1,p}} = \rho$. Moreover, there exists $\varphi \in D^{1,p}(\mathbb{R}^N)$ such that $I(t\varphi) \to -\infty$ as $t \to \infty$.*

Proof. Using the growth condition (f) and (F_3), there exists a positive constant C_δ such that

$$F(x,u) \le \frac{\alpha(x)|u|^p}{p} + C_\delta |u|^{p^*}.$$

Hence

$$\begin{aligned} I(u) &\ge \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p}\int_{\mathbb{R}^N} \alpha(x)|u|^p dx - C_\delta \int_{\mathbb{R}^N} |u|^{p^*} dx \\ &\ge \frac{1}{p}(1 - \lambda_1^{-1}(\alpha))\|u\|_{D^{1,p}}^p - \widehat{C}_\delta \|u\|_{D^{1,p}}^{p^*}, \end{aligned}$$

for some positive constant $\widehat{C}_\delta$, where in the last inequality we have used the variational characterization of the first eigenvalue $\lambda_1(\alpha)$ and the Sobolev inequality. Since $\lambda_1(\alpha) > 1$, we can fix positive constants σ and ρ such that $I(u) \ge \sigma$ if $\|u\|_{D^{1,p}} = \rho$.

Let us prove the second assertion. Consider $\varepsilon > 0$ such that $\lambda_1(\omega) + \varepsilon < 1$, and choose $\varphi \in C_0^\infty(\mathbb{R}^N) - \{0\}$ satisfying

$$\int_{\mathbb{R}^N} |\nabla \varphi|^p dx \le (\lambda_1(\omega) + \varepsilon) \int_{\mathbb{R}^N} \omega |\varphi|^p dx.$$

From (F_4), we have

$$\int_{\mathbb{R}^N} F(x, t\varphi)\, dx = \int_{\{x : |t\varphi(x)| \ge R\} \cap \operatorname{supp} \varphi} F(x, t\varphi)\, dx$$

$$
\begin{aligned}
&+ \int_{\{x:|t\varphi(x)|\leq R\}\cap \operatorname{supp}\varphi} F(x,t\varphi)\,dx \\
&\geq \frac{|t|^p}{p}\int_{\{x:|t\varphi(x)|\geq R\}\cap \operatorname{supp}\varphi} \omega(x)|\varphi|^p\,dx - C\,,
\end{aligned}
$$

for some positive constant C. Thus,

$$
\begin{aligned}
I(t\varphi) &\leq \frac{|t|^p}{p}\left[\int_{\mathbb{R}^N}|\nabla\varphi|^p\,dx - \int_{\{x:|t\varphi(x)|\geq R\}\cap \operatorname{supp}\varphi}\omega|\varphi|^p\,dx\right] + C \\
&= \frac{|t|^p}{p}\Big[\int_{\mathbb{R}^N}|\nabla\varphi|^p dx - \int_{\mathbb{R}^N}\omega|\varphi|^p\,dx \\
&+ \int_{\{x:|t\varphi(x)|<R\}\cap \operatorname{supp}\varphi}\omega|\varphi|^p dx\Big]
\end{aligned}
$$

Since, by the Lebesgue convergence Theorem,

$$\lim_{|t|\to\infty}\int_{\{x:|t\varphi(x)|<R\}\cap \operatorname{supp}\varphi}\omega|\varphi|^p dx = 0,$$

there exists $M>0$, such that

$$\int_{\{x:|t\varphi(x)|<R\}\cap \operatorname{supp}\varphi}\omega|\varphi|^p dx \leq -\frac{(\lambda_1(\omega)+\varepsilon-1)}{2}\int_{\mathbb{R}^N}\omega|\varphi|^p dx,\ \forall|t|\geq M.$$

This fact together with our choice of φ imply that

$$I(tu) \leq \frac{|t|^p}{p}\frac{(\lambda_1(\omega)+\varepsilon-1)}{2}\int_{\mathbb{R}^N}\omega|\varphi|^p dx + C, \quad \forall|t|\geq M.$$

Thus, we have $I(tu)<0$, for sufficiently large $|t|$. □

Proof of Theorem 2

We shall obtain here the critical point by minimization. We have proved in Lemma 3 that assumptions (F_1) and (F_5) imply

$$F(x,u) \leq \frac{1}{p}\omega(x)|u|^p - \frac{a(x)}{p-\mu}|u|^\mu, \quad \forall(x,u)\in\mathbb{R}^N\times\mathbb{R}.$$

Combining this estimate with the variational characterization of the first eigenvalue $\lambda_1(\alpha)$, yields

$$
\begin{aligned}
I(u) &\geq \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p\,dx - \frac{1}{p}\int_{\mathbb{R}^N}\omega(x)|u|^p\,dx + \frac{1}{p-\mu}\int_{\mathbb{R}^N}a(x)|u|^\mu\,dx \\
&\geq \frac{1}{p}(1-\lambda_1^{-1}(\omega))\int_{\mathbb{R}^N}|\nabla u|^p\,dx + \frac{1}{p-\mu}\int_{\mathbb{R}^N}a(x)|u|^\mu\,dx.
\end{aligned}
$$

Thus, the functional I is bounded below, since $\lambda_1(\omega) \geq 1$. Therefore, in view of Lemma 3 in [13] and Lemma 3, I has a critical point u which is its global minimum. $\square$

Examples

Example 1. Let $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$F(x,u) = \begin{cases} \frac{1}{p}\omega(x)|u|^p \ln |u| & if \quad u \neq 0, \\ 0 & if \quad u = 0, \end{cases}$$

where ω is a positive measurable function in $L^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$ such that $\lambda_1(\omega) < 1$. Notice that, for all $x \in \mathbb{R}^N$ and $u \neq 0$,

$$u\frac{\partial F}{\partial u}(x,u) - \mu F(x,u) = \frac{(p-\mu)}{p}\omega(x)|u|^p \ln |u| + \frac{1}{p}\omega(x)|u|^p.$$

Thus, condition (F_1) is satisfied with $\mu = p$, since

$$u\frac{\partial F}{\partial u}(x,u) - pF(x,u) = \frac{1}{p}\omega\,(x)\,|u|^p > 0, \quad \forall x \in \mathbb{R}^N, \forall u \neq 0.$$

It is not difficult to see that the remaining hypotheses of Theorem 1 are satisfied. On the other hand, if $\mu > p$,

$$u\frac{\partial F}{\partial u}(x,u) - \mu F(x,u) = [(p-\mu)\ln |u| + 1]\frac{1}{p}\omega(x)|u|^p < 0$$

for all $x \in \mathbb{R}^N$ and $|u|$ sufficiently large. Consequently, Ambrosetti-Rabinowitz condition (AR) does not hold.

Example 2. Let $\psi : [1,+\infty) \to [0,+\infty)$ be a continuous nontrivial function satisfying $\int_1^{+\infty} \psi(t)dt < \infty$, and let $H(u) = d + \int_1^u [\psi(t) + t^{\mu-p-1}]\, dt$, $u \geq 1$, where d is such that $\lim_{u\to+\infty} H(u) = 1$. Let $F(x,u) = \omega(x)u^p H(u)/p$ for $x \in \mathbb{R}^N$, $u \geq 1$, where ω is a positive measurable function in $L^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$, such that $\lambda_1(\omega) \geq 1$. Notice that $\lim_{u\to+\infty} pF(x,u)/u^p = \omega(x)$ and that

$$u\frac{\partial F}{\partial u}(x,u) - pF(x,u) = \frac{1}{p}\omega(x)u^{p+1}H'(u) \geq \frac{1}{p}\omega(x)u^\mu, \quad x \in \mathbb{R}^N,\ u \geq 1.$$

It is not difficult to see that F can be extended as a function $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that the conditions of Theorem 2 are satisfied for all $u \in \mathbb{R}$ (cf. [9]).

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