Electronic Journal of Differential Equations, Vol. **1997**(1997), No. 18, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp (login: ftp) 147.26.103.110 or 129.120.3.113

Nonlinear weakly elliptic 2×2 systems of variational inequalities with unilateral obstacle constraints*

D.R. Adams & H.J. Nussenzveig Lopes†

Abstract

We study 2×2 systems of variational inequalities which are only weakly elliptic; in particular, these systems are not necessarily monotone. The prototype differential operator is the (vector-valued) p-Laplacian. We prove, under certain conditions, the existence of solutions to the unilateral obstacle problem. This work extends the results by the authors in [Annali di Mat. Pura ed Appl., **169**(1995), 183–201] to nonlinear operators.

In addition, we address the question of determining function spaces on which the p-Laplacian is a bounded nonlinear operator. This question arises naturally when studying existence for these systems.

Introduction

In [4] the authors studied the existence of solutions to a linear 2×2 system of variational inequalities with unilateral obstacle constraints. More precisely, we obtained existence results for the differential operator $\mathcal{L} = A\Delta - BI$, assuming only *weak ellipticity* (see [2]), which in this case reduces to A being invertible with additional sign restrictions on the entries of the constant matrices A^{-1} and $A^{-1}B$. In particular, these assumptions allowed for non-monotone systems.

The purpose of the present work is to extend the results of [4] to nonlinear operators. We observe that, while many of the arguments used in [4] can be carried through in an analogous manner, certain new and unexpected restrictions appear.

Let $\Psi = (\psi^1, \psi^2)$ be a smooth obstacle and set

$$\mathbb{K} = \{V \in (W_0^{1,p}(\Omega) \cap L^2(\Omega))^2 \,|\, V^i \geq \psi^i \text{ a.e. } \Omega, i = 1, 2\},$$

* *1991 Mathematics Subject Classifications:* 35J85, 35J45, 31C45.
Key words and phrases: p-Laplacian, obstacle problem,
non-monotone systems of variational inequalities.

Submitted July 28, 1997. Published October 31, 1997.
†Partially supported by CNPq grants 300158/93-9 and 451113/95-0,
and by FAEP grant 0212/95.

for some $1 < p < \infty$. We study the existence of a solution $U \equiv (u^1, u^2)^t \in \mathbb{K}$ to the system of variational inequalities:

$$\langle \mathcal{L}U,\, V - U\rangle \geq 0 \tag{1}$$

for all $V \in \mathbb{K}$. Here $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain, $\mathcal{L}$ is a possibly nonlinear differential operator and the brackets $\langle \cdot, \cdot \rangle$ denote duality pairings.

We consider differential operators of the form:

$$\mathcal{L}U \equiv A\,\mathrm{div}[\vec{F}(x, \nabla U)] - BU, \tag{2}$$

where $\mathrm{div}[\vec{F}(x,\nabla U)] = (\mathrm{div}\,\vec{F}_1(x,\nabla u^1), \mathrm{div}\,\vec{F}_2(x,\nabla u^2))^t$, and A and B are 2×2 constant, real matrices. We assume that each component operator $\mathrm{div}\,\vec{F}_i$ has the same structure as those considered by J. Heinonen, T. Kilpeläinen and O. Martio in [11]; the prototype operator we have in mind is the p-Laplacian Δ_p, for which $\vec{F}_i = |\nabla u^i|^{p-2}\nabla u^i$.

The study of variational inequalities with unilateral constraints has applications in modeling many problems in elasticity subject to obstacles. Applications include the study of vibrating systems such as the double-pendulum problem or double vibrating springs, which can be modeled by variational inequalities with differential operators in the form (2), see [15]. From a mathematical point-of-view, variational inequalities distinguish "ellipticity" of the associated differential operator. Consider, for example, the scalar obstacle problem: Find $u \in \mathbb{K}$ such that

$$\langle a\Delta u, v - u\rangle \geq 0 \quad \forall v \in \mathbb{K} = \{v \in W_0^{1,2}(\Omega) \mid v \geq \psi \text{ a.e. } \Omega\}\,, \tag{3}$$

where $a \neq 0$, $\psi \in C^\infty(\Omega)$, $\psi < 0$ near $\partial\Omega$ and $\max_\Omega \psi > 0$. It is easy to see that (3) has a solution if and only if $a < 0$. Hence the variational inequality distinguishes, at the level of existence of solutions, between $-\Delta$ and Δ, while the partial differential equation:

$$\begin{cases} a\Delta u = f, \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega, \end{cases}$$

does not.

As in [4], we assume that the system (1) is *weakly elliptic*, i.e. elliptic in the sense of Cauchy-Kowalewski (see [2]). For an operator of the form (2) this means we require A to be invertible. In particular A is allowed to have eigenvalues of opposite signs, something clearly not allowed for either rank-one convex or monotone systems. Even for linear systems very little is known unless strong ellipticity is assumed (see [12] and references therein). The analysis for systems is more difficult than for scalar problems mainly due to the absence of maximum principles (which were used in deriving the necessary and sufficient condition for the existence of a solution to (3)). In addition, it is much simpler to analyze the variational formulation of a scalar variational inequality than that of a system of variational inequalities, and thereby conclude or rule out existence. One of our main contributions in this paper is to show that there exists a solution to (1) provided we assume the same sign restrictions on the entries of A^{-1} and $A^{-1}B$ as in [4], but only for a restricted set of smooth obstacles and for $p > 2n/(n+2)$. (This is the content of Theorem 1.1).

The sign restrictions on the entries of A^{-1} and $A^{-1}B$ are the following:

a) The entries of A^{-1} are *negative* on the diagonal and *non-positive* off the diagonal;

b) The entries of $A^{-1}B$ are *non-negative* on the diagonal and *negative* off the diagonal; (4)

c) The smallest eigenvalue λ_0 of the symmetric part of $A^{-1}B$, denoted by $(A^{-1}B)_S$, is positive.

These conditions arise when we derive *a priori* estimates for an approximate problem. We use the penalty method, where the approximate problem consists of solving a penalized system. The restrictions on the entries of A^{-1} and $A^{-1}B$ imply that the penalized system can be written as a strongly elliptic system of the form:

$$-\text{div}\, \vec{F}(x, \nabla U_\varepsilon) = -A^{-1}BU_\varepsilon + F_\varepsilon, \tag{5}$$

where each component of F_ε is nonnegative and $A^{-1}B$ is an *M-matrix*. A matrix C is called an M-matrix if the diagonal entries of C are non-negative and the off-diagonal entries are non-positive (see [5] for the basic properties of M-matrices). We note that systems of this form, when $-\operatorname{div} \vec{F}(x, \nabla U) \equiv -\Delta$, are *cooperative* systems, for which a number of interesting properties are known (see [6] and references therein). In particular, these (linear) systems were studied in [6], where a necessary and sufficient condition for maximum principles was derived. (By a maximum principle we mean the property that the components of solutions of (5) which vanish on the boundary of a bounded set Ω are nonnegative whenever the components of F_ε are nonnegative.) The special properties of M-matrices played an important role in obtaining this result (see [6] for details). Finally, condition 4c) on the smallest eigenvalue of $A^{-1}B$ keeps the solutions of the penalized system away from possible eigenvectors, for which there can be no *a priori* estimates.

In the case of the p-Laplacian, the existence and regularity of solutions of $N \times N$ systems of variational inequalities has been established for *diagonal* systems with *natural growth* in [7, 8, 9]. A diagonal system is one in which the p-Laplacian of the i-th component of the solution appears only in the i-th inequality. This would correspond to A being diagonal for our $\mathcal{L}$. The condition of “natural growth” is that lower-order terms grow as $|\nabla U|^p$. In contrast, the operators we study are coupled in the highest-order terms, yet the lower-order terms are linear. We observe that it is possible to obtain at least the *a priori* estimates of Theorem 1.2 for more general systems for which the operator $\mathcal{L}$ has an additional, nonlinear, lower-order term $C(x, U)$. The nonlinear term must satisfy the following hypothesis: $C(x,U)$ grows at most linearly, $C(x,0)$ is essentially bounded, the derivatives of $C(x,U)$ with respect to both x and U, denoted $D_xC(x,U)$ and $D_UC(x,U)$, are globally bounded and their L^∞-norms are sufficiently small.

We will not carry out this analysis in the interest of clarity, however the treatment of this case is a simple adaptation of the proof of Theorem 1.2.

The proof of Theorem 1.1 follows the same pattern as that of Theorem 2.1 in [4]. We establish *a priori* estimates for the penalized system, and then pass to the limit using standard compactness arguments, recovering the principal problem and with it existence. The *a priori* estimates are derived analogously to Theorem 1.1 of [4], replacing the linearity of Δ with monotonicity properties of the component operators $\operatorname{div}\vec{F}_i$.

The result in Theorem 1.1 is valid only for nonlinearities such that $p > 2n/(n+2)$ and also only for a restricted set of smooth obstacles. This differs from Theorem 2.1 of [4]. The compactness argument we use requires that $W_0^{1,p}(\Omega)$ be compactly imbedded in $L^2(\Omega)$, which holds as long as $p > 2n/(n+2)$. (Observe that $2 > 2n/(n+2)$, hence this issue did not arise in [4].) We note that in the scalar case and for the operator $-\Delta_p u + \lambda u$, $\lambda > 0$, it is easy to obtain existence for all $p > 1$ by means of variational methods, since the functional $\int_\Omega (|\nabla u|^p/p + \lambda u^2/2)dx$ is weakly lower-semi-continuous over

$$\mathbb{K} = \{v \in W_0^{1,p}(\Omega) \cap L^2(\Omega) \,|\, v \geq \psi \text{ on } \Omega\} \neq \phi\,.$$

See [13] for details. These methods do not apply to the problem at hand because the operator $\mathcal{L}$ is not monotone.

Surprisingly, we must also impose restrictions on the set of admissible smooth obstacles. In order for the *a priori* estimates to be meaningful, the L^2–norms of $\operatorname{div}\vec{F}_i(\cdot,\nabla\psi^i)$ and the L^p–norms of $\nabla\operatorname{div}\vec{F}_i(\cdot,\nabla\psi^i)$ must be finite. This can be quite restrictive, as seen by considering the prototype operator Δ_p, $p \neq 2$, and the $C^\infty(\Omega)$-obstacle $\psi(x) = 1 - x_1^2$, $x = (x_1, x_2, ..., x_n)$, for which the conditions $\Delta_p\psi^i \in L^2(\Omega)$ and $\nabla\Delta_p\psi^i \in L^p(\Omega)$ may fail, depending on p. Hence the question: 'on which function spaces are the (nonlinear) operators $\operatorname{div}\vec{F}_i$ and $\nabla\operatorname{div}\vec{F}_i$ bounded?' arises naturally for this problem. Another important result in this work is a condition on p and q for the boundedness of Δ_p from $W^{3,q}_{\mathrm{loc}}$ to L^2_{loc}. We show: 1) a sufficient condition for $\Delta_p\psi \in L^q$ for $\psi \in C_c^2(\Omega) \cap C^3(\Omega)$ is that $p > \max\{3/2, 2 - 1/q\}$ (this condition is also necessary if $q = 2$), and 2) a sufficient condition for $\nabla\Delta_p\psi \in L^q$, $\psi \in C_c^2(\Omega) \cap C^3(\Omega)$, is $p > 3 - 1/q$.

The paper is divided into three sections. In Section 1, we prove existence of solutions to the unilateral obstacle problem (6). In Section 2, we investigate the restrictiveness of the conditions on the obstacle Ψ in the case of the p-Laplace operator. We employ the familiar interpolation theorem of E.M. Stein for linear analytic families of operators. We also discuss relaxations of the conditions on the obstacle, which still imply existence. In particular, if we use the concept of Choquet integrals with respect to variational capacity, then both conditions on

ψ^i, mentioned above, can be expressed more simply as

$$\int_\Omega (-\Delta_p \psi^i)_+^p \, dC_p < \infty,$$

which confines our attention to second order derivatives. In Section 3, we collect additional results. First we show that solutions to problem (6) are bounded (assuming $(-\operatorname{div} F_i(\cdot, \nabla\psi^i))_+ \in L^\infty(\Omega)$ for $i = 1, 2$). Then we analyze an example in which A has opposite-signed eigenvalues and $p > 2$. We prove that the components of any solution are comparable and non-negative. This is a maximum principle result, which holds for a small class of systems including this example, satisfying certain algebraic constraints. (The constraints are mutually contradictory if $1 < p \leq 2$.) These results complement those in [6], where the case $p = 2$ was treated.

1 Existence

The main result of this section is Theorem 1.1. The proof will be accomplished in several stages. First we derive *a priori* estimates for the solutions of the penalized system (9). Then we prove existence for the penalized system and *a priori* higher regularity estimates. Finally, we pass to the limit as the penalty parameter $\varepsilon \to 0$.

Let us begin by fixing notation. Throughout, Ω is a bounded, smooth domain in $\mathbb{R}^n$. We denote by $C_c^\infty(\Omega)$ the space of infinitely differentiable functions with compact support in Ω. We use standard notation for the Sobolev spaces $W_0^{1,p}(\Omega)$, $1 < p < \infty$, and their duals $W^{-1,p'}(\Omega)$, where $p' = p/(p-1)$. Let $A \in M_{2\times 2}(\mathbb{R})$ be an invertible matrix, and $B \in M_{2\times 2}(\mathbb{R})$. Consider the mappings $\vec{F}_i : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2$, and assume they satisfy the following structure conditions (as in [11]):

- (i) $x \to \vec{F}_i(x, \zeta)$ is measurable for all $\zeta \in \mathbb{R}^n$, $\zeta \to \vec{F}_i(x, \zeta)$ is continuous for a.e. $x \in \Omega$;
- (ii) (Growth) There exist constants $a_0 > 0$, $b_0 > 0$, such that: $|\vec{F}_i(x, \zeta)| \leq a_0 |\zeta|^{p-1}$, a.e. $x \in \Omega$, and $\vec{F}_i(x, \zeta) \cdot \zeta \geq b_0 |\zeta|^p$, a.e. $x \in \Omega$;
- (iii) (Monotonicity) $(\vec{F}_i(x, \zeta) - \vec{F}_i(x, \xi)) \cdot (\zeta - \xi) > 0$ for $\zeta \neq \xi$, $i = 1, 2$;
- (iv) (Homogeneity) $\vec{F}_i(x, \lambda\zeta) = |\lambda|^{p-2} \lambda \vec{F}_i(x, \zeta)$ for every $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

Let $\mathcal{L}$ be the differential operator given by

$$\mathcal{L}U \equiv A \begin{bmatrix} \operatorname{div} \vec{F}_1(x, \nabla u^1) \\ \operatorname{div} \vec{F}_2(x, \nabla u^2) \end{bmatrix} - B \begin{bmatrix} u^1 \\ u^2 \end{bmatrix},$$

where $U = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$. We seek a solution to the problem:

$$\text{Find } U \in \mathbb{K} \text{ such that } \langle \mathcal{L}U \,,\, V - U \rangle \geq 0 \,, \tag{6}$$

for all V in the admissible set

$$\mathbb{K} = \{V \in (W_0^{1,p}(\Omega) \cap L^2(\Omega))^2 \mid V^i \geq \psi^i \text{ a.e. } \Omega,\ i = 1,2\}\,.$$

The brackets $\langle\cdot,\cdot\rangle$ denote the obvious duality pairings. Throughout this paper we assume the obstacle $\Psi = (\psi^1, \psi^2) \in (C^3(\Omega))^2$ to be such that $\psi^i < 0$ near $\partial\Omega$ and $\max_\Omega \psi^i > 0$. Note that, in the case $p > 2n/(n+2)$, $\mathbb{K}$ above can be defined using only $W_0^{1,p}(\Omega) \subset L^2(\Omega)$.

We assume that the matrix A is invertible, and that the sign conditions (4) on the entries of A^{-1} and $A^{-1}B$ hold. We note that condition 4c) can be significantly weakened. Consider, for instance, matrices A and B satisfying conditions 4a)–b), which do not satisfy 4c), but such that $\det A^{-1}B > 0$. Now, multiply each column of A by positive numbers k_1 and k_2. Then the rows of A^{-1} are multiplied by $1/k_1$ and $1/k_2$, respectively, and the same happens with the rows of $A^{-1}B$. This does not alter the conditions 4a)–b). Additionally, it is easy to see that there are numbers $k_1 > 0$, and $k_2 > 0$ such that this procedure generates a matrix $\widetilde{A}$, which, together with the original B, satisfies all three conditions. Hence, by re-defining the mappings F_i, it is possible to relax 4c) to: $\det A^{-1}B > 0$.

It is also possible to relax 4c) for $p \geq 2$, allowing some $\lambda_0 < 0$, by refining the estimates below. We choose not to develop this here in the interest of clarity.

Throughout this section we assume the mappings F_i and the matrices A and B, are fixed and satisfy (i)–(iv) and 4a)–c), respectively. We now state the main result of this section.

Theorem 1.1 *Let $p > 2n/(n+2)$. Suppose the obstacle $\Psi = (\psi^1, \psi^2) \in (C^3(\Omega))^2$, with $\psi^i < 0$ near $\partial\Omega$, and Ψ is such that*

$$\alpha_i \equiv \|(-\operatorname{div} \vec{F}_i(\cdot, \nabla\psi^i))_+\|_{L^2}^2 \tag{7}$$

and

$$\beta_i \equiv \|\nabla(-\operatorname{div} \vec{F}_i(\cdot, \nabla\psi^i))_+\|_{L^p}^p \tag{8}$$

$i = 1,2$, are finite. Then problem (6) has a solution $U \in (W_0^{1,p}(\Omega))^2$. If, in addition, $(-\operatorname{div} F_i(\cdot, \nabla\psi^i))_+ \in L^\infty(\Omega)$, then U also belongs to $(L^\infty(\Omega))^2$.

We give the proof of this theorem at the end of this section. Let us introduce the corresponding penalized system of equations. Let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(t) \equiv 0$ for all $t \geq 0$ and $\eta'(t) \geq 0$ for all $t \in \mathbb{R}$. Consider a solution $U_\varepsilon = (u_\varepsilon^1, u_\varepsilon^2)^t \in (W_0^{1,p}(\Omega) \cap L^2(\Omega))^2$, $\varepsilon > 0$ of:

$$A \begin{bmatrix} \operatorname{div} \vec{F}_1(x, \nabla u_\varepsilon^1) \\ \\ \operatorname{div} \vec{F}_2(x, \nabla u_\varepsilon^2) \end{bmatrix} - BU_\varepsilon = \begin{bmatrix} -\frac{1}{\varepsilon}\eta(u_\varepsilon^1 - \psi^1) \\ \\ -\frac{1}{\varepsilon}\eta(u_\varepsilon^2 - \psi^2) \end{bmatrix},\ \text{in } \Omega\,. \tag{9}$$

Below we establish uniform *a priori* estimates for U_ε in $(W_0^{1,p} \cap L^2)^2(\Omega)$.

Theorem 1.2 *Let U_ε be a solution of (9). Then there exists a constant $Q > 0$, independent of ε, such that*

$$\|\nabla U_\varepsilon\|_{L^p}^p + \|U_\varepsilon\|_{L^2}^2 \leq Q\left(\|\Psi_+\|_{L^2}^2 + \|\nabla\Psi_+\|_{L^p}^p + + \sum_{i=1}^{2}(\alpha_i + \beta_i)\right). \tag{10}$$

Proof: The entries of the matrices A and B will be denoted by A_{ij} and B_{ij} respectively, and those of A^{-1} and $A^{-1}B$, by A^{ij} and M^{ij}, respectively.

Multiply (9) by $-A^{-1}$. Take the inner product of the result with $(u_\varepsilon^1, u_\varepsilon^2)^t$, then integrate by parts over Ω. Then (ii) and 4c) imply that the left-hand-side exceeds

$$b_0\|\nabla U_\varepsilon\|_{L^p}^p + \lambda_0\|U_\varepsilon\|_{L^2}^2, \tag{11}$$

whereas the right-hand-side is

$$\frac{1}{\varepsilon}\int_\Omega \sum_{i,j=1}^{2} A^{ij}\, u_\varepsilon^i\, \eta(u_\varepsilon^j - \psi^j)\, dx. \tag{12}$$

For the diagonal terms of (12), using 4a)-4b), we have:

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_\Omega A^{ii}\, u_\varepsilon^i\, \eta(u_\varepsilon^i - \psi^i)\, dx \leq \frac{1}{\varepsilon}\int_\Omega A^{ii}\, \psi_+^i\, \eta(u_\varepsilon^i - \psi^i)\, dx \qquad (13)\\
&= \int_\Omega \psi_+^i\, A^{ii} \sum_{k=1}^{2}(-A_{ik}\operatorname{div}\vec{F}_k(x, \nabla u_\varepsilon^k) + B_{ik}\, u_\varepsilon^k)\, dx\\
&\leq Q\sum_{k=1}^{2}(\|\nabla\psi_+^i\|_{L^p}\|\vec{F}_k(\cdot, \nabla u_\varepsilon^k)\|_{L^{p'}} + \|\psi_+^i\|_{L^2}\|u_\varepsilon^k\|_{L^2})\\
&\leq Q(\delta_1\|\nabla U_\varepsilon\|_{L^p}^p + \delta_2\|U_\varepsilon\|_{L^2}^2) + Q'(\|\nabla\psi_+^i\|_{L^p} + \|\psi_+^i\|_{L^2}^2)
\end{aligned}$$

where the last two inequalities follow from (ii) and Young's inequality – with δ_1 and δ_2, small parameters to be chosen later.

Next we estimate the non-diagonal terms of (12). We use the equations (9) together with 4b) to write:

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_\Omega A^{ij}\, u_\varepsilon^i\, \eta(u_\varepsilon^j - \psi^j)\, dx \qquad (14)\\
&\quad\leq \frac{1}{\varepsilon}\int_\Omega \frac{A^{ij}}{M^{ji}}\, \eta(u_\varepsilon^j - \psi^j)\, (\operatorname{div}\vec{F}_j(x, \nabla u_\varepsilon^j) - M^{jj}\, u_\varepsilon^j)\, dx\,.
\end{aligned}$$

The first term on the right side of (14) can be estimated using condition (iii): add and subtract the quantity $\operatorname{div}\vec{F}_j(x, \nabla\psi^j)$ and integrate by parts observing (4a)–4b). The result is bounded from above by:

$$-\frac{1}{\varepsilon}\int_\Omega \frac{A^{ij}}{M^{ji}}\, \eta(u_\varepsilon^j - \psi^j)(-\operatorname{div}\vec{F}_j(x, \nabla\psi^j))_+\, dx$$

$$
\begin{aligned}
&= \int_\Omega \frac{A^{ij}}{M^{ji}}(-\operatorname{div}\vec{F}_j(x,\nabla\psi^j))_+\left[\sum_{k=1}^{2} A_{jk}\operatorname{div}\vec{F}_k(x,\nabla u_\varepsilon^k) - B_{jk}\,u_\varepsilon^k\right]\\
&\le Q(\delta_1\|\nabla U_\varepsilon\|^p_{L^p} + \delta_2\|U_\varepsilon\|^2_{L^2}) + Q\left(\sum_{j=1}^{2}(\alpha_j+\beta_j)\right),
\end{aligned}
$$

again by Young's inequality.

The second term on the right-hand-side of (14), can be estimated from above, using 4b) and (ii), by:

$$
\begin{aligned}
&-\frac{1}{\varepsilon}\int_\Omega \frac{A^{ij}}{M^{ji}}\,M^{jj}\,\eta(u_\varepsilon^j-\psi^j)\psi_+^j\\
&= \int_\Omega \frac{A^{ij}}{M^{ji}}\,M^{jj}\psi_+^j\left[\sum_{k=1}^{2} A_{jk}\operatorname{div}\vec{F}_k(x,\nabla u_\varepsilon^k) - B_{jk}\,u_\varepsilon^k\right]dx\\
&\le Q(\delta_1\|\nabla U_\varepsilon\|^p_{L^p} + \delta_2\|U_\varepsilon\|^2_{L^2}) + Q'(\|\nabla\Psi_+\|^p_{L^p} + \|\Psi_+\|^2_{L^2})\,.
\end{aligned}
$$

Putting together (11) and the estimates (13) and (14) for the diagonal and non-diagonal terms of (12) (and choosing δ_1 and δ_2 sufficiently small) we obtain (10). ∎

Next we show that the penalized system (9) has a solution, at least for $p > 2n/(2n+2)$. We use the Leray-Schauder fixed point theorem (see [10]).

Theorem 1.3 *Suppose the penalty function $\eta \in C^\infty(\mathbb{R})$ is such that $\eta'(t) \le 1$, for all $t \in \mathbb{R}$. Let $p > 2n/(n+2)$. Then there exists a solution $U_\varepsilon = (u_\varepsilon^1, u_\varepsilon^2)^t \in (W_0^{1,p}(\Omega))^2$ of system (9).*

Proof: We first note that if $G = (G^1, G^2) \in (L^p(\Omega))^2$, $p \ge 2$ and $v^i \in L^p(\Omega)$, $i = 1, 2$, then each of the equations:

$$
\begin{aligned}
-\operatorname{div}F_1(x,\nabla u^1) + M^{11}u^1 &= G^1 - M^{12}v^2\\
-\operatorname{div}F_2(x,\nabla u^2) + M^{22}u^2 &= G^2 - M^{21}v^1
\end{aligned}
\tag{15}
$$

has a solution $u^i \in W_0^{1,p}(\Omega)$. To see this first note that the operators

$$A_i(\varphi) \equiv -\operatorname{div}\vec{F}_i(x,\nabla\varphi) + M^{ii}\varphi$$

are pseudo-monotone and semi-coercive from $W_0^{1,p}$ to $W^{-1,p'}$, for $i = 1, 2$. Indeed, pseudo-monotonicity follows from Lemma 4.12 in [18] since it can be easily checked, using the structure conditions (i) and (iii), that these operators are bounded (from $W_0^{1,p}$ to $W^{-1,p'}$), hemicontinuous and monotone. The semi-coercivity of A_i follows from condition 4b). Next we use Theorem 4.18 in [18] to conclude that (15) has a solution $u^i \in W_0^{1,p}(\Omega)$.

Similarly, when $2n/(n+2) < p < 2$, and $G^i, v^i \in L^2(\Omega)$ it also follows that (15) has a solution $u_i \in W_0^{1,p}(\Omega)$.

Use (15) to define the solution operator $T(v) = u$, from $(L^p(\Omega))^2$ into itself for $p \geq 2$, and from $(L^2(\Omega))^2$ into itself for $2n/(n+2) < p < 2$. Recall that, to use the Leray-Schauder fixed-point theorem, we need to show that the solutions of $v = \sigma T(v)$ are uniformly bounded in $W_0^{1,p}(\Omega)$ for any $0 \leq \sigma \leq 1$. This uniform bound can be obtained by deriving estimates in the same way as was done in the proof of Theorem 1.2 and by using condition 4c) on the eigenvalues of $A^{-1}B$. Therefore we can apply the Leray-Schauder fixed point theorem to conclude that

$$\begin{bmatrix} -\operatorname{div} \vec{F}_1(x, \nabla u^1) \\ \\ -\operatorname{div} \vec{F}_2(x, \nabla u^2) \end{bmatrix} + A^{-1}B \begin{bmatrix} u^1 \\ \\ u^2 \end{bmatrix} = A^{-1} \begin{bmatrix} \frac{1}{\varepsilon}\,\eta(w^1 - \psi^1) \\ \\ \frac{1}{\varepsilon}\,\eta(w^2 - \psi^2) \end{bmatrix} \equiv \begin{bmatrix} G^1 \\ \\ G^2 \end{bmatrix} \tag{16}$$

has a solution in $(W_0^{1,p}(\Omega))^2$ for $w^i \in L^p(\Omega)$, if $p \geq 2$ or for $w^i \in L^2(\Omega)$ if $2n/(n+2) < p < 2$.

Now consider the solution operator defined by (16), $S(w) = u$. This operator is compact from L^p to L^p, if $p \geq 2$, and from L^2 to L^2 if $2n/(n+2) < p < 2$, since $W_0^{1,p}$ is compactly imbedded in L^2. Once again, it is possible to derive estimates in the same way as was done in the proof of Theorem 1.2 to show that the solutions of $w = \sigma S(w)$ are uniformly bounded in $W_0^{1,p}(\Omega)$ for any $0 \leq \sigma \leq 1$. Therefore we can apply the Leray-Schauder fixed point theorem and it is easy to see that the fixed point lies in $(W_0^{1,p}(\Omega))^2$. Hence there exists a solution U_ε in $(W_0^{1,p}(\Omega))^2$ of (9), as we wished. ∎

Remark. To conclude the above argument, we used strongly that $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^2(\Omega)$ for $p > 2n/(n+2)$. It is possible to prove a reverse Hölder inequality for system (16), and conclude that $U_\varepsilon \in (L^{2+\varepsilon}_{\text{loc}}(\Omega))^2$. With this, we can conclude compactness again in L^2_{loc} and hence obtain a solution to (9) for $1 < p \leq 2n/(n+2)$. However, since we are not able to pass to the limit as $\varepsilon \to 0$ in this case, we choose not to pursue this here.

Observe that any solution U_ε to the penalized system (9) must satisfy

$$\langle \mathcal{L}U_\varepsilon,\ V - U_\varepsilon \rangle \geq 0 \tag{17}$$

for all $V \in \mathbb{K}$. Our goal is to pass to the limit, as $\varepsilon \to 0$, in (17), at least for some subsequence. From Theorem 1.2, we can extract a subsequence which converges $W^{1,p}$-weakly to some $U \in (W_0^{1,p}(\Omega))^2$ as well as L^2-strongly to U (observe that here we need to have $p > 2n/(n+2)$). Further regularity is needed in order to show that this U satisfies (6). Below we establish *a priori* higher regularity estimates.

Lemma 1.4 *Let $2 \leq q < \infty$. Then, every solution U_ε to (9) satisfies*

$$\|\frac{1}{\varepsilon}\eta(u^i_\varepsilon - \psi^i)\|_{L^q} \leq Q\left[\|(-\operatorname{div} \vec{F}_i(\cdot, \nabla\psi^i))_+\|_{L^q} + \|\psi^i_+\|_{L^q} + \|u^j_\varepsilon\|_{L^q}\right], \tag{18}$$

for each $i = 1, 2$, $j \neq i$, and for some constant Q independent of ε.

Proof: Set $f_r(t) = |t|^{r-1}t$ for $r \geq 1$ Using 4a), it is easy to see that the q-th power of the left side of (18) is at most

$$-\frac{1}{A^{ii}} \int_\Omega f_{q-1}\left(\frac{1}{\varepsilon}\eta(u^i_\varepsilon - \psi^i)\right)\left[\operatorname{div}\vec{F}_i(x,\nabla u^i_\varepsilon) - \sum_{k=1}^{2} M^{ik}u^k_\varepsilon\right] dx. \tag{19}$$

The first term of (19) can be handled exactly as in Theorem 1.2 – by adding and subtracting the quantity

$$-\frac{1}{A^{ii}}\, f_{q-1}\left(\frac{1}{\varepsilon}\eta(u^i_\varepsilon - \psi^i)\right)\operatorname{div}\vec{F}_i(x,\nabla\psi^i)$$

and integrating by parts – to yield terms that are dominated by

$$\frac{1}{A^{ii}} \int_\Omega f_{q-1}\left(\frac{1}{\varepsilon}\eta(u^i_\varepsilon - \psi^i)\right)\left(-\operatorname{div}\vec{F}_i(x,\nabla\psi^i)\right)_+ dx. \tag{20}$$

The remaining terms of (19) can be estimated by

$$\begin{aligned} &\frac{1}{A^{ii}}\int_\Omega f_{q-1}\left(\frac{1}{\varepsilon}\eta(u^i_\varepsilon - \psi^i)\right)[M^{ii}\psi^i_+ + M^{ij}u^j_\varepsilon]\, dx \\ \leq\;& Q\|\frac{1}{\varepsilon}\eta(u^i_\varepsilon - \psi^i)\|^{q-1}_{L^q}[\|\psi^i_+\|_{L^q} + \|u^j_\varepsilon\|_{L^q}]\,. \end{aligned} \tag{21}$$

Hölder's inequality on (20) plus (21) yield (18). ∎

Hereafter, fix a subsequence of solutions of (9), $\{U_{\varepsilon_k}\}$, converging weakly in $(W^{1,p}(\Omega))^2$ and strongly in $(L^2(\Omega))^2$, as well as almost everywhere to $U \in (W^{1,p}(\Omega))^2$.

Using Lemma 1.4 we immediately have that the weak limit $U \in \mathbb{K}$. This follows since

$$\int_\Omega |\eta(u^i - \psi^i)|^2\, dx \leq \liminf_{\varepsilon_k\to 0}\,(\varepsilon_k)^2\int_\Omega |\frac{1}{\varepsilon_k}\eta(u^i_{\varepsilon_k} - \psi^i)|^2\, dx = 0.$$

Hence $u^i \geq \psi^i$ a.e. on Ω. Before we give the proof of Theorem 1.1 we will need to show the sequence $\{U_{\varepsilon_k}\}$ is compact in $W^{1,p}$.

Lemma 1.5 *Let U_{ε_k} be the sequence fixed above. Then the sequence $\{\nabla U_{\varepsilon_k}\}$ converges strongly in L^p.*

Proof: Multiply the difference of (9) for $\varepsilon = \varepsilon_k$ and ε_l by $(U_{\varepsilon_k} - U_{\varepsilon_l})^t A^{-1}$ and integrate over Ω, to get:

$$\begin{aligned} &\int_\Omega \sum_{i=1}^{2}[\vec{F}_i(x,\nabla u^i_{\varepsilon_k}) - \vec{F}_i(x,\nabla u^i_{\varepsilon_l})]\cdot[\nabla u^i_{\varepsilon_k} - \nabla u^i_{\varepsilon_l}], dx + \lambda\|U_{\varepsilon_k} - U_{\varepsilon_l}\|^2_{L^2} \\ \leq\;& Q\|U_{\varepsilon_k} - U_{\varepsilon_l}\|_{L^2}\sum_{i=1}^{2}\left(\|\frac{1}{\varepsilon_k}\,\eta(u^i_{\varepsilon_k} - \psi^i)\|_{L^2} + \|\frac{1}{\varepsilon_l}\,\eta(u^i_{\varepsilon_l} - \psi^i)\|_{L^2}\right) \quad (22) \\ \leq\;& Q'\|U_{\varepsilon_k} - U_{\varepsilon_l}\|_{L^2}\,. \end{aligned}$$

Above Q' depends on ψ^i but not on k and l. With this we conclude that the first term of (22) tends to zero as $k, l \to \infty$.

In Lemma 2.7 of [14] it was shown that, if $\mathcal{A}$ is a mapping satisfying the same structure conditions as $\vec{F}_i$, $i = 1, 2$, then

$$\lim_{k\to\infty} \int_\Omega [\mathcal{A}(x, \nabla v_k) - \mathcal{A}(x, \nabla v)] \cdot [\nabla v_k - \nabla v]\, dx = 0$$

if and only if $\nabla v_k \to \nabla v$ strongly in $L^p(\Omega)$. A simple modification of the proof of Lemma 2.7 [14] together with (22) gives that $\{\nabla U_{\varepsilon_k}\}$ is a Cauchy sequence in L^p. ∎

Proof of Theorem 1.1: Start by observing that

$$\langle \mathcal{L} U_{\varepsilon_k},\ V - U_{\varepsilon_k} \rangle = \int_\Omega (\nabla V - \nabla U_{\varepsilon_k})^t A \begin{bmatrix} \vec{F}_1(x, \nabla u^1_{\varepsilon_k}) \\ \\ \vec{F}_2(x, \nabla u^2_{\varepsilon_k}) \end{bmatrix} - (V - U_{\varepsilon_k})^t B U_{\varepsilon_k}.$$

Hence, to pass to the limit in (17) it is enough to show that the terms $\int_\Omega (\nabla v^i - \nabla u^i_{\varepsilon_k}) A_{ij} \vec{F}_j(x, \nabla u^j_{\varepsilon_k})$ converge to $\int_\Omega (\nabla v^i - \nabla u^i) A_{ij} \vec{F}_j(x, \nabla u^j)$. Use Egorov's theorem to show that, for any $\varphi \in L^p$, then

$$\int_\Omega |\varphi| |\nabla u^j_{\varepsilon_k}|^{p-1}\, dx \to \int_\Omega |\varphi| |\nabla u^j|^{p-1}\, dx$$

as $\varepsilon_k \to 0$, passing to a further subsequence if needed. Then, use the structure condition (ii) and the Generalized Dominated Convergence theorem (see Theorem 16, page 89 in [16]) to conclude that $\vec{F}_j(x, \nabla u^j_{\varepsilon_k})$ converges weakly in $L^{p'}$ to $\vec{F}_j(x, \nabla u^j)$. Since $\nabla v^i - \nabla u^i_{\varepsilon_k}$ converges L^p-strongly to $\nabla v^i - \nabla u^i$, we have what we wished. ∎

We postpone the proof of boundedness of the solutions in Theorem 1.1 to Section 3; see Lemma 3.1 and the subsequent remarks.

2 Regularity restrictions on the obstacle

In this section we will restrict ourselves to the p-Laplacian operator, Δ_p, for which $\vec{F}_i = |\nabla u^i|^{p-2} \nabla u^i$. We examine the condition that α_i and β_i, in (7) and (8) respectively, be finite.

For the p-Laplacian, this means:

$$(-\Delta_p \psi)_+ \in L^2(\Omega) \tag{23}$$

and

$$\nabla(-\Delta_p \psi)_+ \in L^p(\Omega), \tag{24}$$

respectively. These conditions can be quite restrictive. To illustrate this, consider the following example of a $C^\infty_c(\Omega)$-obstacle for which (23) and (24) may fail. Let $0 \in \Omega$, $x = (x_1, x_2, ..., x_n)$, and set $\psi_0(x) = \phi(x)(1 - x_1^2)$ for some $\phi \in C_c^\infty(\Omega)$, with $\phi \equiv 1$ in a neighborhood of the origin. Then in this neighborhood $-\Delta_p\psi_0 = (p-1)2^{p-1}|x_1|^{p-2}$. A simple calculation shows that (23) holds only for $p > \frac{3}{2}$ and (24), only for $p > \frac{3}{2} + \frac{\sqrt{5}}{2} \approx 3.736$ and for $p = 2$. In Theorem 2.1 we prove that this is, in fact, the worst case scenario. Subsequently, we discuss a weaker condition that replaces (24) and contains (23).

Theorem 2.1 *Let $q \geq 1$. Then:*

a) For any $p > \max\{3/2, 2 - 1/q\}$ there exists a constant $Q_1 > 0$ such that

$$\|\Delta_p\psi\|_{L^q} \leq Q_1 \, \|\psi\|_{C^3}^{p-1},$$

for all $\psi \in C_c^2(\Omega) \cap C^3(\Omega)$.

b) For any $p > 3 - 1/q$ there exists a constant $Q_2 > 0$ such that

$$\|\nabla\Delta_p\psi\|_{L^q} \leq Q_2 \, \|\psi\|_{C^3}^{p-1},$$

for all $\psi \in C_c^2(\Omega) \cap C^3(\Omega)$.

It is immediate that a) holds for all $p \geq 2$ and b) for all $p \geq 3$. Also, it follows from the proof below that one can formulate an important case of part a) above as:

$$\Delta_p : \quad W_{\mathrm{loc}}^{3,q}(\Omega) \to L_{\mathrm{loc}}^2(\Omega) \tag{25}$$

is a bounded nonlinear operator for all $p > \frac{3}{2}$, if $q \geq 2n(p-1)/(n + 4p - 6)$.

For the proof of Theorem 2.1, we will need the following estimate.

Lemma 2.2 *If $\psi \in C_c^2(\Omega) \cap C^3(\Omega)$, then*

$$\int_\Omega |\nabla^2\psi|^2 [\, 1 + |\nabla\psi|^2 \,]^{r/2} \, dx \leq \frac{\sqrt{n}}{1 - |r|} \int_\Omega |\nabla^3\psi| [\, 1 + |\nabla\psi|^2 \,]^{(r+1)/2} \, dx$$

whenever $|r| < 1$. Here $|\nabla^k\psi|$ denotes the l^2-norm of all k-th order derivatives of ψ.

Proof: Integrate by parts to get:

$$\begin{aligned} & \int_\Omega |\psi_{x_i x_j}|^2 [\, 1 + |\nabla\psi|^2 \,]^{r/2} \, dx \qquad (26) \\ = \quad & - \int_\Omega \psi_{x_i x_j x_j} \psi_{x_i} [\, 1 + |\nabla\psi|^2 \,]^{r/2} \, dx \\ & - r \int_\Omega \psi_{x_i x_j} \psi_{x_i} [1 + |\nabla\psi|^2]^{(r-2)/2} \sum_k \psi_{x_k} \psi_{x_k x_j} \, dx \end{aligned}$$

Thus the left side of (26) is at most

$$\begin{aligned} & \sqrt{n} \int_\Omega |\nabla^3\psi| |\nabla\psi| [\, 1 + |\nabla\psi|^2 \,]^{r/2} \, dx \\ & + \quad |r| \int_\Omega |\nabla^2\psi|^2 |\nabla\psi|^2 [\, 1 + |\nabla\psi|^2 \,]^{(r-2)/2} \, dx \end{aligned}$$

and hence the lemma follows. ∎

Corollary 2.3 *If* $\psi \in C_c^2(\Omega) \cap C^3(\Omega)$, *then*

$$\int_\Omega |\nabla^2\psi|^2 |\nabla\psi|^r \, dx \le \frac{\sqrt{n}}{1-|r|} \int_\Omega |\nabla^3\psi| \; |\nabla\psi|^{r+1} \, dx \quad \forall |r| < 1 \, .$$

Proof: In Lemma 2.2, replace ψ by ψ/ε, $\varepsilon > 0$, and then let $\varepsilon \to 0$. ∎

We are now ready to give

Proof of Theorem 2.1: We use Lemma 2.2 together with an interpolation theorem due to E.M. Stein ("interpolation for an analytic family of operators"; see [17]). We begin by defining the family of operators T_z: for $v \in L^2(\Omega)$ and $\psi \in C_c^2(\Omega) \cap C^3(\Omega)$, set

$$T_z v \equiv |\nabla^2\psi| \cdot [\, \varepsilon^2 + |\nabla\psi|^2 \,]^{\frac{s-2}{2}(1-z)} \cdot v \, .$$

Above $\varepsilon \in (0,1]$, $\frac{3}{2} < s < 2$, and $z = \sigma + i\eta$ is a complex variable with $0 \le \sigma \le 1$ and $\eta \in \mathbb{R}$. Then, using Lemma 2.2 we have

$$\begin{aligned} \int_\Omega |\, T_{i\eta}\, v| \, dx &\le \|v\|_{L^2} \, \| \, |\nabla^2\psi| \, [\, \varepsilon^2 + |\nabla\psi|^2 \,]^{(s-2)/2} \|_{L^2} \\ &\le C_0 \|v\|_{L^2} \, \|\psi\|_{C^3}^{s-1} \, . \end{aligned}$$

Also

$$\int_\Omega |T_{1+i\eta}\, v| \, dx \le \|v\|_{L^1} \, \|\nabla^2\psi\|_{L^\infty} \le \|v\|_{L^1} \, \|\psi\|_{C^3} .$$

The Stein Interpolation Theorem gives ($0 < \sigma < 1$)

$$\int_\Omega |T_\sigma v| \, dx \le C_\sigma \, \|v\|_{L^r} \, \|\psi\|_{C^3}^{p-1} ,$$

where $p = (1-\sigma)s + 2\sigma$ and $r = 2/(1+\sigma)$. By duality we obtain:

$$\| \, |\nabla^2\psi| \, [\, \varepsilon^2 + |\nabla\psi|^2 \,]^{(p-2)/2} \|_{L^{2/(1-\sigma)}} \le C_\sigma \, \|\psi\|_{C^3}^{p-1} . \tag{27}$$

Note that $p > \frac{3}{2} + \frac{\sigma}{2} = 2 - (1-\sigma)/2 = 2 - 1/q$. The result in a) follows by letting $\varepsilon \to 0$ in (27), since all the terms of $\Delta_p\psi$ behave like $|\nabla^2\psi| \; |\nabla\psi|^{p-2}$.

To prove b), note that all terms of $\nabla\Delta_p\psi$ behave like either $|\nabla^3\psi| \; |\nabla\psi|^{p-2}$ or $|\nabla^2\psi|^2 |\nabla\psi|^{p-3}$. Thus, if we now set

$$T_z v \equiv |\nabla^2\psi|^2 \, [\, \varepsilon^2 + |\nabla\psi|^2 \,]^{(s-3)/2 + z/2} \cdot v \, ,$$

for $2 < s < 3$, then we can use Lemma 2.2 to obtain an estimate which implies:

$$T_{i\eta} : \; L^\infty \to L^1 ,$$

and one easily has:

$$T_{1+i\eta}: \ L^1 \to L^1.$$

Thus, again by the Stein Interpolation Theorem, we have

$$\int_\Omega |T_\sigma v| \, dx \leq C_\sigma \, \|v\|_{L^r} \, \|\psi\|_{C^3}^{p-1}, \tag{28}$$

where now $p = s + \sigma$ and $r = 1/\sigma, \quad 0 < \sigma < 1$. Sending $\varepsilon \to 0$ in the dual statement to (28) gives

$$\| \, |\nabla^2\psi|^2 \, |\nabla\psi|^{p-3}\|_{L^{1/(1-\sigma)}} \leq C_\sigma \, \|\psi\|_{C^3}^{p-1}.$$

This, together with the form of $\nabla\Delta_p\psi$, yields b), since $p > 2 + \sigma =$ with $3 - (1-\sigma) = 3 - 1/q$. $\blacksquare$

To see (25), we can assume without loss of generality that $u \in C_c^3(\Omega)$. Since all terms of $\Delta_p u$ are dominated by some constant multiple of $|\nabla^2 u| \cdot |\nabla u|^{p-2}$, we apply Corollary 2.1 with $r = 2(p-2)$. Now Hölder's inequality, with the resulting q-norm on $|\nabla^3 u|$ and $(2p-3)q'$-norm on $|\nabla u|, \quad q' = q/(q-1)$, yields the result, since Sobolev's inequality implies that $(2p-3)q'$ can not, in general, exceed $nq/(n-2q)$, at least when $q \leq n/2$.

We conclude this section with a discussion of a weaker condition on the obstacles which still implies existence in the special case of the p-Laplacian. Let $e \subset \Omega$ be a Borel set and recall the definition of the p-conductor capacity $C_p(e)$:

$$C_p(e) \equiv \inf\{\int |\nabla\phi|^p \, dx \,|\, \phi \in C_c^\infty(\Omega), \, \phi \geq 1 \text{ on } e, \quad \bar{e} \subset \Omega\}.$$

We observe that it suffices to replace condition (24) by the weaker requirement:

$$\int_\Omega (-\Delta_p\psi)_+^p \, dC_p < \infty. \tag{29}$$

The integral in (29) is the usual Choquet integral, taken in the sense

$$\int_0^\infty C_p(\Omega \cap [(-\Delta_p\psi)_+ > t]) \, dt^p. \tag{30}$$

The Choquet integral arises as a capacity functional: if f is a smooth non-negative function on Ω with compact support in Ω, then the integral $\int_\Omega f^p \, dC_p$ is comparable to

$$\inf \int |\nabla\phi|^p \, dx, \tag{31}$$

where the infimum is taken over all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq f$, a.e. on Ω. For this and related results see [1] and [3].

Using the functional (31) it follows that condition (29) can be used as a replacement for (24). Indeed, whenever an integral of the form

$$Q \int \eta(u-\psi)\Delta_p\psi \, dx \tag{32}$$

appears in the *a priori* estimates of Section 1, with Q a positive constant, then for any $\phi \in W_0^{1,p}(\Omega)$ for which $\phi \geq (-\Delta_p\psi)_+$ a.e. on Ω, we estimate (32) by

$$-Q \int \eta(u-\psi)\phi\, dx\,.$$

Then, substituting for η, as in our proofs in Section 1, we integrate by parts, obtaining an estimate in terms of $\|\nabla\phi\|_p$. Finally, (31) relates this estimate to (29) and (30).

It should also be noted that

$$\left[\int_\Omega (-\Delta_p\psi)_+^2\, dx\right]^{1/2} \leq Q\left[\int_\Omega (-\Delta_p\psi)_+^p\, dC_p\right]^{1/p},$$

whenever $p > 2n/(n+2)$. This follows from the Sobolev inequality. Thus both conditions on ψ in Theorem 1.1, (23) and (24), can be replaced by the single condition (29) when $p > 2n/(n+2)$.

Condition (29) is a bit more satisfying than (23) and (24) since (29) deals only with two derivatives of ψ. Also (29) clearly holds for smooth ψ when $p \geq 2$. The following example shows that it is possible to have an obstacle ψ for which $(-\Delta_p\psi)_+$ is unbounded, (29) is finite, and $p < 2$. Let $\bar{x} = (x_1, x_2, 0, 0, ..., 0)$ and fix $0 \in \Omega$, $\phi \in C_c^\infty(\Omega)$, with $\phi \equiv 1$ in some neighborhood of 0. Set $\psi(x) = \phi(x)(1-|\bar{x}|^\theta)$, $2 < \theta < 3$. Then an easy calculation gives that $(-\Delta_p\psi)_+$ behaves like $Q|\bar{x}|^\alpha$, $\alpha = (\theta-1)(p-2)+(\theta-2) < 0$, for $p < \theta/(\theta-1)$, while (29) holds for $p > 2/(\theta-1)$. (This last result is a consequence of the estimates for the capacity of an n-rectangle given in [1].)

3 Additional results

3.1 Boundedness of solutions

Here we will give an outline of the proof that solutions of problem (6) under the assumptions of Theorem 1.1, are bounded. We do this by applying the Moser iteration method to solutions of (9). Again, this result is only valid for $p > 2n/(n+2)$. Of course, only $p \leq n$ are of eventual interest here.

Lemma 3.1 *Let U_ε be a solution of (9) and assume that the obstacle Ψ satisfies the hypothesis in Theorem 1.1. In addition, suppose $(-\mathrm{div}\, F_i(\cdot, \nabla\psi^i))_+ \in L^\infty(\Omega)$. Then there exists a constant Q, independent of ε, such that*

$$\|U_\epsilon\|_{L^\infty}^s \leq Q(\|U_\epsilon\|_{L^2}+1) \tag{33}$$

for $s = 1 + \frac{n}{2} - \frac{n}{p}$; $p > 2n/(n+2)$.

Proof: We begin as in the proof of Theorem 1.2, but this time we take the inner product with the function $(-f_r(u_\varepsilon^1), -f_r(u_\varepsilon^2))^t$, where f_r is as in the proof of Lemma 1.4. Observe that we have:

$$|u_\varepsilon^i|^{r-1}|\nabla u_\varepsilon^i|^p = |\nabla(f_{(r-1)/p+1}(u_\varepsilon^i))|^p.$$

Hence we can use the Sobolev inequality to write:

$$K_r \left(\int_\Omega |U_\varepsilon|^{(r+p-1)\sigma}\, dx \right)^{1/\sigma} \leq Q \left(\int_\Omega |U_\varepsilon|^{r+1}\, dx + \int_\Omega |g_\varepsilon|^{r+1}\, dx \right) \qquad (34)$$

where Q is independent of ε and r, and $g_\varepsilon = \left(\frac{1}{\varepsilon}\eta(u^1_\varepsilon - \psi^1),\ \frac{1}{\varepsilon}\eta(u^2_\varepsilon - \psi^2)\right)$; $K_r = b_0 r p^p/(r+p-1)^p$ and $\sigma = n/(n-p)$.

Using Lemma 1.4, we have:

$$\|g_\varepsilon\|^q_{L^q} \leq Q(\|U_\varepsilon\|^q_{L^q} + 1), \qquad (35)$$

where Q depends on ψ_i, but is independent of ε. Thus, we can write (34) as

$$\left(\int_\Omega |U_\varepsilon|^{(r+p-1)\sigma}\, dx \right)^{1/\sigma} \leq \frac{Q}{K_r}(\|U_\varepsilon\|^{r+1}_{L^{r+1}} + 1). \qquad (36)$$

Now iterate (36), first taking $r = r_1 = 1$ and then $r = r_j = q_{j-1} - 1$, where

$$q_j = 2\sigma^j + (p-2)\sum_{k=1}^{j} \sigma^k = -\frac{(p-2)\sigma}{\sigma - 1} + \sigma^j \left[2 + \frac{(p-2)\sigma}{\sigma-1} \right].$$

Thus, for example, we have for $r = r_2$

$$\begin{aligned} \left(\int_\Omega |U_\varepsilon|^{(p\sigma+p-2)\sigma}\, dx \right)^{1/\sigma} &\leq \frac{Q}{K_{r_2}} \left\{ \frac{Q^\sigma}{K^\sigma_{r_1}} (\|U_\varepsilon\|^2_{L^2} + 1)^\sigma + 1 \right\} \\ &\leq \frac{Q}{K_{r_2}} \cdot \frac{Q}{K_{r_1}} \cdot 4^\sigma (\|U_\varepsilon\|^{2\sigma}_{L^2} + 1) \end{aligned}$$

since we can assume $Q/K_{r_1} \geq 1$ and apply the estimate $(a+1)^\sigma + 1 \leq 2^\sigma(a^\sigma + 1) + 1 \leq 4^\sigma(a^\sigma + 1)$, for any $a \geq 0$. And then in general,

$$\|U_\varepsilon\|^{q_N/\sigma}_{L^{q_N}} \leq \frac{\prod_{j=1}^{N} (4Q)^{\sigma^{j-1}}}{\prod_{j=1}^{N} K^{\sigma^{j-1}}_{r_j}} (\|U\|^{2\sigma^{N-1}}_{L^2} + 1). \qquad (37)$$

Now take the $2\sigma^{N-1}$ root of both sides of (37) and let $N \to \infty$. This yields the desired result since

$$\frac{q_N}{2\sigma^N} \to 1 + \frac{(p-2)\sigma}{2(\sigma-1)} = 1 + \frac{n}{2} - \frac{n}{p}.$$

Note $1 + \frac{n}{2} - \frac{n}{p} > 0$ if and only if $p > 2n/(n+2)$. ∎

Lemma 3.1 together with the estimates (10) and (33) give the final claim of Theorem 1.1, namely that $U \in (L^\infty(\Omega))^2$. Note that in order to guarantee that the exponents q_j are increasing without bound, it is necessary to require $p\sigma > 2$ i.e. $p > 2n/(n+2)$.

3.2 Maximum principles

In this subsection we will discuss a small class of non-monotone systems for which the components of the solutions to the corresponding obstacle problems are comparable and non-negative. As we observed in the Introduction, this is referred to as a maximum principle. These results complement those obtained in [6] where the case $p = 2$ was studied.

We will first consider a particular example. Let:

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -2 \\ -(2+\theta) & 1+2/\theta \end{bmatrix},$$

where $\theta = 2^{1/(p-1)}$, $2 < p < \infty$. Now the penalized system (9) with $F_i = |\zeta|^{p-2}\zeta$, $i = 1, 2$, implies that:

$$\left.\begin{array}{c} \Delta_p u^1_\varepsilon - \Delta_p u^2_\varepsilon - 2u^1_\varepsilon + 2u^2_\varepsilon \geq 0 \\ \\ -2\Delta_p u^1_\varepsilon + \Delta_p u^2_\varepsilon + (2+\theta)u^1_\varepsilon - (1+2/\theta)u^2_\varepsilon \geq 0 \end{array}\right\} \text{ in } \Omega. \tag{38}$$

Thus we have:

$$\left.\begin{array}{c} -\Delta_p u^2_\varepsilon + 2u^2_\varepsilon \geq -\Delta_p u^1_\varepsilon + 2u^1_\varepsilon \\ \\ -\Delta_p \tilde{u}^1_\varepsilon + (1+2/\theta)\tilde{u}^1_\varepsilon \geq -\Delta_p u^2_\varepsilon + (1+2/\theta)u^2_\varepsilon \end{array}\right\} \text{ in } \Omega, \tag{39}$$

where $\tilde{u}^1 = \theta u^1_\varepsilon$. The following lemma then implies

$$u^2_\varepsilon \geq u^1_\varepsilon \geq \frac{1}{\theta} u^2_\varepsilon, \text{ in } \Omega. \tag{40}$$

Lemma 3.2 *If $W_j \in W^{1,p}_0 \cap L^2(\Omega)$ and satisfies*

$$-\Delta_p W_1 + \lambda W_1 \geq -\Delta_p W_2 + \lambda W_2, \quad \textit{in} \ \Omega$$

with $\lambda \geq 0$, then $W_1 \geq W_2$, a.e. Ω.

Proof: Set $S = \Omega \cap [W_2 \geq W_1]$. Using the function $(W_2 - W_1)_+$, we can write

$$\int_S (|\nabla W_2|^{p-2}\nabla W_2 - |\nabla W_1|^{p-2}\nabla W_1)(\nabla W_1 - \nabla W_2)\,dx + \lambda \int_S (W_1 - W_2)^2\,dx \leq 0\,.$$

This easily implies $W_1 \geq W_2$ in Ω. ∎

We deduce from above that the solutions $(u^1_\varepsilon, u^2_\varepsilon)$ of (38) satisfy

$$u^1_\varepsilon \geq 0, \quad u^2_\varepsilon \geq 0, \quad \text{in } \Omega. \tag{41}$$

Finally, appealing to the proof of Theorem 1.1, we conclude that (40) and (41) remain valid in the limit as $\varepsilon \to 0$ and $p > 2$.

The same conclusions deduced above for this special example, namely (40) and (41), remain valid for any system satisfying the following six conditions. We use the notation introduced in the proof of Theorem 1.2.

1. $\det(A) < 0$
2. $A^{ij} < 0$, $i, j = 1, 2$
3. $M^{ij} = (A^{-1}B)_{ij} > 0$, $i = j$; $M^{ij} < 0$, $i \neq j$
4. The minimum eigenvalue of $(A^{-1}B)_S$ is greater than zero
5. $B_{11} = -\sigma B_{12}$, $B_{22} = -\xi B_{21}$, where now
$$\sigma = \left| \frac{A_{1,1}}{A_{1,2}} \right|^{1/(p-1)}, \quad \xi = \left| \frac{A_{2,2}}{A_{2,1}} \right|^{1/(p-1)}$$
6. $M^{11} + M^{12}\sigma \geq 0$ and $M^{21}\xi + M^{22} \geq 0$.

Conditions 1–4 imply (via Theorem 1.1) that problem (6) has a solution in $(W_0^{1,p}(\Omega) \cap L^\infty(\Omega))^2$, $p > 2n/(n+2)$. Conditions 5 and 6 imply (40) and (41) for $p > 2$. These conditions are mutually contradictory if $1 < p \leq 2$.

Any symmetric matrix A satisfying both conditions 1 and 2 has, of course, eigenvalues of opposite signs. Therefore these matrices give rise to operators of the form $A\Delta_p - B$ which are not monotone and for which there is a maximum principle property.

3.3 The scalar problem

We conclude this section with some remarks about the scalar case. Consider solving $\langle -\Delta_p u + \lambda u,\ v - u \rangle \geq 0$ with $u \in \mathbb{K} = \{v \in W_0^{1,p}(\Omega) \cap L^2(\Omega) \mid v \geq \psi \text{ a.e. } \Omega\}$ by finding:

$$\min \int_\Omega \left(\frac{1}{p} |\nabla u|^p + \frac{\lambda}{2}\, u^2 \right) dx, \tag{42}$$

where the minimum is taken over all $u \in \mathbb{K}$. For $\lambda \geq 0$ the existence of a minimizer is immediate for all $p > 1$. However, if $\lambda < 0$, it is easy to see that problem (42) has no solution for $1 < p < 2$. To see this, just choose $u_k = \psi_+ + \varphi_k$ where
$$\varphi_k(x) = \min(|x|^{-\alpha}, k) - 1$$
on the ball $\Omega = B(0,1)$, centered at the origin and of radius 1. Then $\|\varphi_k\|_2^2$ behaves like $k^{2-n/\alpha}$ as $k \to \infty$ for $\alpha > n/2$. Also, $\|\nabla\varphi_k\|_p^p$ is $\mathrm{O}(1)$ for $\alpha < n/p - 1$,and it is $\mathrm{O}(k^{(1+1/\alpha)p - n/\alpha})$ when $\alpha > n/p - 1$, as $k \to \infty$. Thus for $p < 2n/(n+2)$, (42) is $-\infty$ for $\alpha \in \left(\frac{n}{2}, \frac{n}{p} - 1\right)$, whereas it is $-\infty$ for $2n/(n+2) < p < 2$ when α satisfies
$$\frac{n}{p} - 1 \leq \frac{n}{2} \leq \frac{p}{2-p} < \alpha.$$

For $p \geq 2$, (42) has a solution. To see this, start by observing that any minimizing sequence is bounded *a priori* in L^2. This follows from the Poincaré and Hölder inequalities, as long as $-\lambda$ is sufficiently small ($-\lambda < Q_p$, where Q_p is the constant appearing in the Poincaré inequality). Hence it is bounded in $W^{1,p}$. The compactness of the imbedding $W^{1,p} \subset L^2$ then allows passage to the limit using L^2-strong convergence and $W^{1,p}$-weak lower semi-continuity of the $W_0^{1,p}$-norm.

Acknowledgments. The authors would like to thank Professors Djairo G. de Figueiredo and Milton C. Lopes-Filho for their helpful comments and discussions.

References

[1] Adams, D.R., *Sets and functions of finite L^p-capacity*, Ind. Univ. Math. J., **27** (1978), 611–627.

[2] Adams, D.R., *Weakly elliptic systems with obstacle constraints: part I - a 2×2 model problem*, Partial Differential Equations with Minimal Smoothness and Applications, IMA Volumes in Math. Vol. **42**, Springer-Verlag 1992, 1–14.

[3] Adams, D.R. and Hedberg, L., *Function Spaces and Potential Theory*, Springer-Verlag 1996.

[4] Adams, D.R. and Nussenzveig Lopes, H.J., *Weakly elliptic systems of variational inequalities: a 2×2 model problem with obstacles in both components*, Annali di Mat. Pura ed Appl., **169** (1995), 183–201.

[5] Berman, A. and Plemmons, R.J., *Nonnegative matrices in the mathematical sciences*, Computer Science and Applied Math. Series, Academic Press, 1979.

[6] de Figueiredo, D. and Mitidieri, E., *Maximum principles for linear elliptic systems*, Rendiconti Mat. Trieste, **22** (1990), 36–66.

[7] Fuchs, M., *Existence via partial regularity for degenerate systems of variational inequalities with natural growth*, Comment Math. Univ. Carolinae, **33** (1992), 427–435.

[8] Fuchs, M., *Smoothness for systems of degenerate variational inequalities with natural growth*, Comment. Math. Univ. Carolinae, **33** (1992), 33–41.

[9] Fuchs, M., *On the existence of weak solutions for degenerate systems of variational inequalities with critical growth*, Comment. Math. Univ. Carolinae, **35** (1994), 445–449.

[10] Gilbarg, D. and Trudinger, N., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1977.

[11] Heinonen, J., Kilpelainen, T. and Martio, O., *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Math. Monographs, Clarendon Press, 1993.

[12] Hildebrandt, S. and Widman, O., *Variational inequalities for vector-valued functions*, J. Reine Angew. Math., **309** (1979), 181–220.

[13] Kinderlehrer, D. and Stampacchia, G., *An introduction to variational inequalities*, Academic Press, 1990.

[14] Li, Gongbao and Martio, O., *Stability in obstacle problems*, Math. Scand. **75** (1994), 87–100.

[15] Park, D., *Classical Dynamics and its Quantum Analogues*, Lecture Notes in Physics, Springer-Verlag, 1979.

[16] Royden, H. L., *Real Analysis*, 2^{nd} edition, The Macmillan Company, 1968.

[17] Stein, E. and Weiss, G., *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[18] Troianello, G., *Elliptic Differential Equations and Obstacle Problems*, Univ. Series in Math. Plenum Press, 1987.

David R. Adams
Department of Mathematics, University of Kentucky.
Patterson Office Tower, Lexington, KY 40506, USA
E-mail address: dave@ms.uky.edu

Helena J. Nussenzveig Lopes
Departamento de Matematica, IMECC-UNICAMP.
Caixa Postal 6065, Campinas, SP 13081-970, Brazil
E-mail address: hlopes@ime.unicamp.br