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On the prescribed-period problem for autonomous Hamiltonian systems *

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Abstract

Asymptotically quadratic and subquadratic autonomous Hamiltonian systems are considered. Lower bounds for the number of periodic solutions with a prescribed minimal period are obtained. These bounds are expressed in terms of the numbers of frequencies corresponding to the critical points of the Hamiltonian. Results are based on a global analysis of families of periodic solutions emanating from these points.

1 Introduction

We consider the autonomous Hamiltonian system

$$J\dot{x} = H_x(x), \qquad J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \tag{1.1}$$

where $x \in \mathbb{R}^{2n}$ and I_n is the identity matrix of order n. Here, the Hamiltonian $H(x)$ is assumed to be analytical.

The prescribed-period problem for system (1.1) consists of finding conditions on $H(x)$ that guarantee the existence of periodic solutions of a given period T. During the recent years there appeared a considerable amount of work devoted to this problem, see for example [11]. The most interesting results are those that establish existence of solutions having minimal period T. For convex asymptotically subquadratic and superquadratic Hamiltonians, the existence of at least one such solution was proved by Clarke and Ekeland [3], Ambrosetti and Mancini [1], Ekeland and Hofer [4], Girardi and Matseu [5]. For second-order Hamiltonian systems, with an even potential function, some multiplicity results were obtained by Van Groesen [12]. In these articles, the solution is obtained as a minimum of a functional, through the use of variational techniques.

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In this paper, we investigate the prescribed-period problem with minimal period T. The results obtained are based on an examining the behavior of families of periodic solutions emanating from the equilibrium points. Note this such approach to nonlocal analysis of autonomous Hamiltonian systems was previously utilized in [15, 16].

Now we present the main results.

Suppose there are positive constants α and r such that for $\|x\| > r$,

$$(H) \qquad -\alpha I_{2n} < H_{xx}(x) < \alpha I_{2n}$$

i.e., $-\alpha\|y\|^2 < (H_{xx}(x)y, y) < \alpha\|y\|^2$ for $y \neq 0$, where $\|y\| = (y, y)^{1/2}$ is the Euclidean norm of y.

We assume that $H(x)$ has a finite number of critical points x^p, $p = 1, \ldots, k$ in $\mathbb{R}^{2n}$ ($H_x(x^p) = 0$). Also assume that they are non-degenerate ($\det H_{xx}(x^p) \neq 0$).

Let $\pm i\omega_r^p$ ($\omega_r^p > 0$, $r = 1, \ldots, r_p \leq n$) denote the purely imaginary eigenvalues of the matrix $J^{-1}H_{xx}(x^p)$, and let

$$l_r^p = i(Jx_r^p, x_r^p)\,, \tag{1.2}$$

where x_r^p is the eigenvector associated with the eigenvalue $i\omega_r^p$.

As known in [13], l_r^p is real and $l_r^p \neq 0$ for simple eigenvalues. The quantity ω_r^p is called a frequency of *first kind*, or of *second kind*, if $l_r^p > 0$ or $l_r^p < 0$, respectively.

Denote by d^p the number of negative eigenvalues of the matrix $J^{-1}H_{xx}(x^p)$, and by $n_1^p(T)$ and $n_2^p(T)$ the numbers of the frequencies of first and second kind satisfying the inequality $\omega_r^p > 2\pi/T$. Let

$$m(T) = \left|\sum_{p=1}^{k}(-1)^{d^p}[n_1^p(T) - n_2^p(T)]\right|\,. \tag{1.3}$$

The following theorem gives a lower bound for the number of T-periodic solutions.

Theorem 1 *For $T < 2\pi/\alpha$, equation (1.1) has at least $m(T)$ periodic solutions with minimal period T.*

The proof (given in Section 3) is based on the following arguments. It appears that at least $m(T)$ Lyapunov families of periodic solutions $x_j(t, s)$ with the initial minimal periods $T_j(0) < T$ are continuable in s to an extent that $T_j(s) > T$; therefore, $x_j(t, s) = T$ for some $s = s_j$.

2 Preliminary remarks

First let us recall some known facts relating to periodic solutions of autonomous Hamiltonian systems. For any solution $x(t)$ of (1.1), with

$$H(x(t)) \equiv h = \text{constant}\,, \tag{2.1}$$

and using this integral, equation (1.1) is reduced to a system of order $2n-1$ depending on the parameter h. As a result, a periodic solution $x(t)$ belongs to a one-parameter family. The existence of such families in a neighbourhood of an equilibrium position x^p is established by the Lyapunov center theorem [8]. Namely, if for some k, the values ω_r^p satisfy the condition

$$\frac{\omega_r^p}{\omega_k^p} \neq m, \quad m = 1, 2, \dots; \quad r = 1, \dots, r_p,\, r \neq k, \tag{2.2}$$

then there exists a unique family of periodic solutions $x_k^p(t,s) = x_k^p(t+T_k^p(s), s)$ such that $x_k^p(t,s) \to x^p$ and $T_k^p(s) \to 2\pi/\omega_k^p$ as $s \to 0$. Thus the r_p families bifurcate from an equilibrium point x^p.

Under continuation of such a family in the parameter s branching can occur. It appears (J.Mallet-Paret and J.Yorke [9] that from the corresponding continuum of solutions one can chose a one-parameter family of solutions ("snake") $x_k^p(t,s)$ possessing the following properties.

The snake may terminate at an equilibrium point x^q; in other words, different families emanating from the points x^p and x^q may coalesce (it is possible that $q = p$). Otherwise, the snake is continuable to an arbitrary large value of the modulus

$$M_k^p(s) = T_k^p(s) + |h_k^p(s)| + \max_t \|x_k^p(t,s)\| \tag{2.3}$$

where $h_k^p(s) = H(x_k^p(t,s))$. The minimal period of the snake $T_k^p(s)$ is continuous but at most countable number of points s_r where it drops by a factor $q > 1$ (at such a point, a family with the minimal period $T_k^p(s_r)/q$ may branch off the snake). The left and right limits of $T_k^p(s)$ at s_r are equal.

By scaling the time $t = \tau T$, equation (1.1) is reduced to

$$Jx' = TH_x(x), \qquad x' = dx/d\tau\,, \tag{2.4}$$

so that 1-periodic solutions of (2.4) correspond to T–periodic solutions of (1.1). The variational equation associated with such a solution $x(\tau)$ is

$$Jy' = TA(\tau)y, \qquad A(\tau) = H_{xx}(x(\tau)). \tag{2.5}$$

Let $\rho_k,\ k = 1, \dots, 2n$ be the Floquet multipliers of (2.5) (the eigenvalues of the monodromy matrix $W(T)$ where $W(t)$ is the matrix of a fundamental system of solutions satisfying the condition $W(0) = I_{2n}$).

Since (2.4) is autonomous, (2.5) has always a multiplier $\rho = 1$ corresponding to the periodic solution $y_1(\tau) = x'(\tau)$. In view of integral (2.1), the multiplicity of this multiplier equals two [10]. Hereafter, we assume $\rho_1 = \rho_2 = 1$. If the corresponding elementary divisors of the matrix $W(T)$ are simple, there exists one more 1-periodic solution $y_2(\tau)$. In the case of non-simple divisors the second solution is of the form $y_2(\tau) = f_2(\tau) + \tau y_1(\tau)$ where $f_2(\tau+1) = f_2(\tau)$. Substituting this solution in (2.5), we find that $f_2(\tau)$ satisfies the equation

$$Jf_2' = TA(\tau)f_2 - Jy_1(\tau)\,. \tag{2.6}$$

If $\rho_k \neq 1$ for $k > 2$, the corresponding invariant subspaces of the matrix $W(T)$ are J-orthogonal [13], i.e.,

$$(Jy_r(0), y_q(0)) = 0 \qquad \text{for } r \leq 2,\ q > 2\,. \tag{2.7}$$

If $\rho_k = 1$ for some $k > 2$, relation (2.7) is reached by an appropriate choice of the vectors associated with the multiplier $\rho = 1$.

Setting $r = 1$ in (2.7), and taking into account that $Jy_1 = H_x$ and that $(Jy_1, y_1) = 0$, we find

$$(H_x(x(0)), y_q(0)) = 0 \qquad \text{for } q \neq 2. \tag{2.8}$$

As $H_x(x(0)) \neq 0$, from (2.8) it follows

$$(H_x(x(0)), y_2(0)) \neq 0. \tag{2.9}$$

Fixing the energy h and one of the coordinates (so that the trajectory of $x(\tau)$ is transversal to the corresponding $(2n-2)$–dimensional disc B), we obtain the Poincare map $G(v, h)$ with $G(v_0, h) = v_0$ where v_0 corresponds to the solution $x(\tau)$. The eigenvalues of the matrix of partial derivatives $G_v(v_0, h)$ are the multipliers $\rho_3, \dots, \rho_{2n}$ of equation (2.5).

Let $x(t, s)$ be a snake, then

$$v_0(s) = G(v_0(s), h(s)). \tag{2.10}$$

Differentiating (2.10), we obtain

$$D(s)v_{0s}(s) = h_s(s)G_h(v_0(s), h(s)) \tag{2.11}$$

where $D(s) = [I_{2n-2} - G_v(v_0(s), h(s))]$, $h_s(s) = dh(s)/ds$,
$G_h(v_0, h) = \partial G(v_0, h)/\partial h$, and $v_{0s}(s) = dv_0(s)/ds$.

As is seen from (2.11), if $h_s(s_k) = 0$, then $U(s_k) = \det D(s_k) = 0$ (and, therefore, $\rho_q(s_k) = 1$ for some $q > 2$ [9]). Generically, $h(s)$ is a Morse function ($h_{ss}(s_k) \neq 0$ for $h_s(s_k) = 0$ [2]) and $v_{0s}(s_k) \neq 0$, so the functions $h_s(s)$ and $U(s)$ change their signs at the same points s_k.

As is mentioned above, the frequencies ω_k^p are classified into that of first or second kind depending on the sign of l_k^p (this complies with the Krein's classification [7] of the corresponding multipliers $\rho_k^p = \exp(i\omega_k^p T)$ of the linearized system). If $H_{xx}(x^p) > 0$ or $H_{xx}(x^p) < 0$ (i.e., x^p is a minimum or a maximum of $H(x)$) all eigenvalues of the matrix $J^{-1}H_{xx}(x^p)$ are purely imaginary [13]. Taking into account that $J^{-1}H_{xx}(x^p)x_k^p = i\omega_k^p x_k^p$, we find

$$l_k^p = i(Jx_k^p, x_k^p) = (H_{xx}(x^p)x_k^p, x_k^p)/\omega_k^p,$$

so $l_k^p > 0$ or $l_k^p < 0$ and, therefore, all frequencies ω_k^p, $k = 1, \dots, n$ are of first or second kind, correspondingly. If the matrix $H_{xx}(x^p)$ is indefinite, there exist frequencies of each kind. Note that bilateral bounds for their numbers expressed in the numbers of positive and negative eigenvalues of the matrix $H_{xx}(x^p)$ are obtained in [17].

3 Proof of main Theorem

First let us establish some preliminary results. Let $x_k^p(t,s)$ be a Lyapunov family ($x_k^p(t,s) \to x^p$, $T_k^p(s) \to 2\pi/\omega_k^p$ as $s \to 0$). The following lemma describes the behavior of the corresponding energy $h_k^p(s)$ in the vicinity of x^p depending on the kind of the frequency ω_k^p.

Lemma 1 *For small s, the function $h_k^p(s)$ increases or decreases with s if the frequency ω_k^p is of first or second kind, respectively.*

Proof . Setting $x = x_k^p(t,s)$ in (2.1) and differentiating it with respect to s, we obtain

$$h_{ks}^p(s) = dh_k^p(s)/ds = (H_x(x_k^p(t,s)), x_{ks}^p(t,s)) = (J\dot{x}_k^p(t,s), x_{ks}^p(t,s)). \quad (3.1)$$

For small s, one can assume $s = |h - H(x^p)|$, then [8]

$$x_k^p(t,s) = s^{1/2} x_k^p \exp(i\omega_k^p t) + O(s,t) \quad (3.2)$$

where $O(s,t)/s^{1/2} \to 0$ as $s \to 0$. From (3.1) and (3.2) we obtain

$$h_{ks}^p(0) = 1/2i\omega_k^p(Jx_k^p, x_k^p) = 1/2\omega_k^p l_k^p. \quad (3.3)$$

Thus, the sign of $h_{ks}^p(0)$ coincides with that of l_k^p. The lemma is proved. $\square$

Let $h(s)$ and $T(s)$ be the energy and period of a one-parameter family $x(t,s)$.

Lemma 2 *If $h_s(s_k) = 0$, then $T_s(s_k) = 0$.*

Proof Setting $x = x(\tau,s)$ in (2.4) and differentiating it with respect to s, we obtain

$$Jx_s' = TA(\tau,s)x_s + T_s H_x(x(\tau,s)). \quad (3.4)$$

Let $T_s(s) \neq 0$. Taking into account that

$$H_x(x(\tau,s)) = Jx'(\tau,s)/T = Jy_1(\tau,s)/T\,,$$

we find that the function $-x_s(\tau,s)T(s)T_s^{-1}(s)$ satisfies (2.6). Therefore, $x_s(\tau,s)$ may be represented in the form

$$x_s(\tau,s) = -T_s(s)T^{-1}(s)[y_2(\tau,s) - \tau y_1(\tau,s)] + \sum_k a_k y_k(\tau,s)\,, \quad (3.5)$$

where the sum includes the 1–periodic solutions $y_k(\tau,s)$ of (2.5), $x_s(\tau,s) = x_s(\tau+1,s)$.

From (2.8), (2.9) and (3.5) we obtain

$$h_s(s) = (H_x(x(\tau,s)), x_s(\tau,s)) = -T_s(s)T^{-1}(s)(H_x(x(\tau,s)), y_2(\tau,s)) \neq 0\,. \quad (3.6)$$

Thus, $h_s(s) \neq 0$ for $T_s(s) \neq 0$. The lemma is proved. $\square$

Let $x_k^p(t,s)$ and $x_r^q(t,s)$ be Lyapunov families; if they belong to the same snake, they merge under continuation in s. The following lemma (having a dominant role in the proof of the above Theorem) indicates cases when different families certainly belong to different snakes.

Let $\mu_l^m = 1$ and $\mu_l^m = 2$ for a frequency ω_l^m of first and second kind, respectively. Recall that d^m denotes the number of negative eigenvalues of the matrix $J^{-1}H_{xx}(x^m)$.

Lemma 3 *If the values $\mu_k^p + d^p$ and $\mu_r^q + d^q$ are both odd or even, the families $x_k^p(t,s)$ and $x_r^q(t,s)$ belong to different snakes.*

Proof Suppose $x_k^p(t,s)$ and $x_r^q(t,s)$ belong to the same snake $x(t,s)$. We also assume that $x(t,s) = x_k^p(t,s)$ for small s, then $x(t,s_*-s) = x_r^q(t,s)$, $h(0) = H(x^p)$ and $h(s_*) = H(x^q)$ for some s_*. Let s_k, $k = 1,2,\dots$ be successive critical points of $h(s)$ on $(0,s_*)$, $(h_s(s_k) = 0)$. As is shown above, $U(s_k) = 0$; generically, $h_{ss}(s_k) \neq 0$ and $U_s(s_k) \neq 0$.

Suppose first that $\mu_k^p = \mu_r^q$ (i.e., the frequencies ω_k^p and ω_r^q are of the same kind). By Lemma 1, the signs of $h_{ks}^p(s) = h_s(s)$ and $h_{rs}^q(s) = -h_s(s_*-s)$ for small s coincide; therefore, the total number of the points s_k is odd. It follows that the signs of $U(0)$ and $U(s_*)$ are different. Clearly, $U(s) = (\rho_3(s) - 1)\dots(\rho_{2n}(s)-1)$. The complex multipliers $\rho_k(s)$ are conjugate, so $U(s) > 0$ or $U(s) < 0$ when, respectively, the number of multipliers $\rho_k(s) \in (0,1)$ is even or odd. Observing that $A(\tau,0) = H_{xx}(x^p)$ and $A(\tau,s_*) = H_{xx}(x^q)$, we find that the respective numbers of the multipliers equal d^p and d^q. Hence, one of these values and, therefore, one of the sums $\mu_k^p + d^p$ and $\mu_r^q + d^q$ is odd and another is even.

Suppose now that $\mu_k^p \neq \mu_r^q$. Here the signs of $h_{ks}^p(s)$ and $h_{rs}^q(s)$ are different, so the number of $s_k \in (0,s_*)$ is even. Therefore, the signs of $U(0)$ and $U(s_*)$ coincide, so, both of the values d^p and d^q are odd or even and, thus, as in the previous case, one of the sums $\mu_k^p + d^p$ and $\mu_r^q + d^q$ is odd and another is even. Thus, only under this condition different families may belong to the same snake. The lemma is proved. $\square$

Note that from Lemma 3 it follows that families $x_k^p(t,s)$ and $x_r^p(t,s)$ emanating from the same equilibrium position x^p and corresponding to frequencies ω_k^p and ω_r^p of the same kind cannot merge together.

The above results enable us to prove readily the Theorem.

Proof of Theorem 1 By definition, $n_1^p(T)$ and $n_2^p(T)$ are, respectively, the numbers of the frequencies of first and second kind satisfying the inequality $\omega_k^p > 2\pi/T$. Taking into account Lemma 3, we find that at least $m(T)$ of the corresponding families $x_k^p(t,s)$ with the initial periods $T_k^p(0) < T$ belong to different snakes, i.e., they do not merge as s increases. So, such a family $x_k^p(t,s)$

either meets a family $x_h^q(t,s)$ with the initial period $T_h^q(0) > T$ or is continuable to an arbitrary large value of the modulus $M_k^p(s)$. Clearly, in the first case $T_k^p(s) = T$ for some s. Let us prove that the same is true for the second case.

Let $X(t) = \|x(t)\|$, $X_- = X(t_-) = \min_t X(t)$ and $X_+ = X(t_+) = \max_t X(t)$ where $x(t) = x(t+T)$ is a solution of (1.1), $0 \le t_- < t_+ < T$. By (H), there exist constants γ and c such that $\|H_x(x)\| < \gamma\|x\| + c$ for all $x \in \mathbb{R}^{2n}$. Taking into account that $\|H_x(x)\| = \|\dot{x}\|$, we find (see [6], ch.4, lemma 4.1)

$$X_+ \le [X_- + c(t_+ - t_-)]\exp[\gamma(t_+ - t_-)].$$

Thus, if $T_k^p(s)$ remains bounded, then $X_{k-}^p \to \infty$ when $X_{k+}^p \to \infty$ and, therefore, for some s, $\|x_k^p(t,s)\| > r\,, \forall t$.

By (H), the value α may serve as a Lipschitz constant for the function $J^{-1}H_x(x)$ with $\|x\| > r$; so from a theorem by Yorke [14] it follows that

$$T_k^p(s) > 2\pi/\alpha\,. \tag{3.7}$$

As mentioned above, at some points s_r the minimal period of the solution $x_k^p(t,s_r)$ may be equal $T_k^p(s_r)/q$ where q is an integer [9]. Let us show that $T_k^p(s_r)$ is the minimal period of $x_k^p(t,s_r)$ for some $s \ne s_r$. Really, for $T = T_k^p(s_r)/q$, variational equation (2.4) has a multiplier $\rho'_m = \exp(2\pi i/q)$ $\quad (m > 2)$ [9]; so, for $T = T_k^p(s_r)$, the corresponding multiplier $\rho_m = (\rho'_m)^q = 1$. Therefore, $h_s(s_r) = 0$ and, by Lemma 2, $T_{ks}^p(s_r) = 0$. Generically, s_r is a local extremum of the function $T_k^p(s)$, so, for $T \in (2\pi/\omega_k^p, 2\pi/\alpha)$, there exists $s \ne s_r$ such that $T_k^p(s) = T_k^p(s_r)$ is the minimal period of $x_k^p(t,s)$. The theorem is proved.

4 Conclusion

Theorem 1 gives a lower bound for the number of periodic solutions with a prescribed minimal period T. Note that if a system is asymptotically subquadratic (i.e., $H(x)\|x\|^{-2} \to 0$ as $\|x\| \to \infty$), then the value α in (H) may be taken as small as one likes, so Theorem 1 enables one to establish the existence of periodic solutions with an arbitrary large minimal period.

Suppose that a system has a unique equilibrium position $x = 0$ and $H_{xx}(0) > 0$ or $H_{xx}(0) < 0$. As shown above, the frequencies ω_k, $k = 1, \dots, n$ are of first or second kind. The corresponding families cannot coalesce as s increases. Therefore, for any $T < 2\pi/\alpha$, there exist at least $n_1(T)$ periodic solutions with the minimal period T where $n_1(T)$ is the number of frequencies $\omega_k > 2\pi/T$. In particular, for an asymptotically subquadratic system, there exist at least n periodic solutions with any minimal period $T > 2\pi/\omega_1$ where ω_1 is the smallest frequency of the linearized system.

Note that these results cannot be improved without additional information about $H(x)$. One can easily construct a Hamiltonian such that $\|x_k(t,s)\| \to \infty$ and the periods $T_k(s)$ increase monotonically to $2\pi/\alpha$ as $s \to \infty$. Clearly, here the number of solutions with a minimal period T equals $n_1(T)$, i.e., coincides with the lower bound obtained.

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