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On Tykhonov's theorem for convergence of solutions of slow and fast systems *

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Abstract

Slow and fast systems gain their special structure from the presence of two time scales. Their analysis is achieved with the help of Singular Perturbation Theory. The fundamental tool is Tykhonov's theorem which describes the limiting behaviour, for compact interval of time, of solutions of the perturbed system which is a one-parameter deformations of the so-called unperturbed system. Our aim here is to extend this description to the solutions of all systems that belong to a small neighbourhood of the unperturbed system. We investigate also the behaviour of solutions on the infinite time interval. Our results are formulated in classical mathematics. They are proved within Internal Set Theory which is an axiomatic approach to Nonstandard Analysis.

1 Introduction

Let us consider an initial value problem (IVP) of the form

$$\begin{aligned} \varepsilon\dot{x} &= F(x,y,\varepsilon) \quad x(0)=\alpha_\varepsilon\,, \\ \dot{y} &= G(x,y,\varepsilon) \quad y(0)=\beta_\varepsilon\,, \end{aligned} \tag{1}$$

where the dot $(\dot{})$ means d/dt, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and parameter ε is a positive real number. We consider the behaviour of solutions when ε is small. The small parameter ε multiplies the derivative so the usual theory of continuous dependence of the solutions with respect to the parameters can not be applied. The analysis of such systems is achieved with the help of the so called *Singular Perturbation Theory*. The purpose of Singular Perturbation Theory is to investigate the behaviour of solutions of (1) as $\varepsilon \to 0$ for $0 \le t \le T$ and also for $0 \le t < +\infty$. The vectors x and y are the fast and slow components of the

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system. This system is called a *fast and slow system*. If we use the *fast time*, $\tau = t/\varepsilon$, Problem (1) becomes

$$\begin{aligned} x' &= F(x,y,\varepsilon) \quad x(0) = \alpha_\varepsilon\,, \\ y' &= \varepsilon G(x,y,\varepsilon) \quad y(0) = \beta_\varepsilon\,, \end{aligned} \tag{2}$$

where $' = d/d\tau$. This problem is called the *perturbed problem*. It is a *regular perturbation* of the *unperturbed problem*

$$\begin{aligned} x' &= F(x,y,0) \quad x(0) = \alpha_0\,, \\ y' &= 0 \quad y(0) = \beta_0\,. \end{aligned} \tag{3}$$

Hence, first x varies very quickly and is approximated by the solution of the *boundary layer equation*

$$x' = F(x,\beta_0,0) \quad x(0) = \alpha_0\,, \tag{4}$$

and y remains close to its initial value β_0. The system of differential equations

$$x' = F(x,y,0)\,, \tag{5}$$

in which y is a parameter, is called the *fast equation*. A solution of (5) may behave in one of several ways: it may be unbounded as $\tau \to \infty$, it may tend toward an equilibrium point, or it may approach a more complex attractor. Obviously, if the fast equation has multiple stable equilibria, the asymptotic behaviour of a solution is determined by its initial value. Assume the second case occurs, that is, the solutions of (5) tend toward an equilibrium $\xi(y)$, where $x = \xi(y)$ is a root of equation

$$F(x,y,0) = 0\,. \tag{6}$$

The manifold of equation (6) is called the slow manifold: it is the set of equilibrium points of the fast equation (5); the surface $\mathcal{L}$ of equation $x = \xi(y)$ is a component of the slow manifold. The solution of (3) is defined for all $\tau \geq 0$ and tends to $(\xi(\beta_0), \beta_0)$, namely to a point of $\mathcal{L}$. Hence a fast transition brings the solution of problem (1) near the slow manifold. Then, a slow motion takes place near the slow manifold, and is approximated by the solution of the *reduced problem*

$$\dot{y} = G(\xi(y),y,0) \quad y(0) = \beta_0\,. \tag{7}$$

The preceding description is definitely heuristic and imprecise. In a more rigorous description we usually consider ε as a parameter that tends to 0 and we assume that Problem (1) has a unique solution $x(t,\varepsilon)$, $y(t,\varepsilon)$. Let $y_0(t)$ be the solution of the reduced Problem (7), which is assumed to be defined for $0 \leq t \leq T$, then we have $\lim_{\varepsilon\to 0} y(t,\varepsilon) = y_0(t)$ for $0 \leq t \leq T$. We also have $\lim_{\varepsilon\to 0} x(t,\varepsilon) = \xi(y_0(t))$, but the limit holds only for $0 < t \leq T$, since there is a boundary layer at $t = 0$, for the x-component. Indeed, let $x_0(\tau)$ be the solution of the boundary layer equation (4) then $\lim_{\varepsilon\to 0} x(\varepsilon\tau, \varepsilon) = x_0(\tau)$ for $0 \leq \tau < +\infty$. This description of the solution of Problem (1) was given by Tykhonov [26], under the hypothesis that the equilibrium point $\xi(y)$ of equation (5) is asymptotically stable for all y and that the asymptotic stability is uniform with respect to y (see Section 2.3). A highly recommended classical reference for these matters is Chapter X (especially Section 39) of Wasow's book [29].

Hence, Singular Perturbation Theory describes the solutions of (2), which is a one-parameter deformation of (3). Actually, as noticed by Arnold (see [1], footnote page 157), the behaviour of the perturbed problem solutions "takes place in all systems that are close to the original unperturbed system. Consequently, one should simply study neighbourhoods of the unperturbed problem in a suitable function space. However, here and in other problems of perturbation theory, for the sake of mathematical convenience, in the statements of the results of an investigation such as an asymptotic result, we introduce (more or less artificially) a small parameter ε and, instead of neighborhoods, we consider one-parameter deformations of the perturbed systems. The situation here is as with variational concepts: the directional derivative (Gateaux differential) historically preceded the derivative of a mapping (the Fréchet differential)".

The aim of this paper is to define a suitable function space of IVPs, and to study small neighbourhoods of the unperturbed problem. This paper is organized as follows. In Section 2 we describe a topology on the set of IVPs and we give the asymptotic behaviour of the solutions of an IVP that lies in a small neighbourhoods of an IVP which satisfies various hypotheses (Theorem 1). We also investigate the solutions behaviour on the infinite time interval (Theorem 2). For the proofs of Theorem 1 and 2, we use *Internal Set Theory* (IST), which is an axiomatic approach of A. Robinson's *Nonstandard Analysis* (NSA) [22], proposed by E. Nelson [21]. Section 3 starts with a short tutorial on IST. Then we present the nonstandard translates (Theorems 3 and 4) in the language of IST of Theorems 1 and 2. This section ends with an external discussion of the notion of uniform asymptotic stability, which is the crucial assumption for the validity of the results. In Section 4 we give the proofs of Theorems 3 and 4. We recall that IST is a *conservative extension* of ordinary mathematics. This means that any statement of ordinary mathematics which is a theorem of IST was already a theorem of ordinary mathematics, so there is no need to translate the proofs.

We want to emphasize that Theorems 3 and 4 were obtained directly from [18, 25, 27]. Afterwards, we noticed that the classical translations of these results are nothing more than considering neighbourhoods, as suggested by Arnold. NSA allowed also the discovery and good understanding of new phenomena which are not covered by Tykhonov's theory, namely the so called *canard* solutions. These solutions are related to the important phenomenon of delayed loss of stability in dynamical bifurcations [17]. For more informations on the applications of NSA to the asymptotic theory of differential equations, the reader is referred to [2, 3, 6, 8, 9, 19, 25].

2 Singular Perturbations

2.1 Slow and Fast Vectors Fields

Our main problem is to study IVPs for fast and slow systems of the form

$$\begin{aligned} \varepsilon\dot{x} &= f(x,y) \quad x(0)=\alpha \\ \dot{y} &= g(x,y) \quad y(0)=\beta\,, \end{aligned} \tag{8}$$

where $f : D \to \mathbb{R}^n$ and $g : D \to \mathbb{R}^m$ are continuous, D is an open subset of $\mathbb{R}^{n+m}$, and $(\alpha,\beta) \in D$. We denote by

$$\begin{aligned} \mathcal{T} \quad = \quad & \big\{(D,f,g,\alpha,\beta) : D \text{ open subset of } \mathbb{R}^{n+m}, \\ & (f,g) : D \to \mathbb{R}^{n+m} \text{ continuous },\ (\alpha,\beta)\in D\big\}\,. \end{aligned}$$

Our aim is to study Problem (8) when ε is small and (D,f,g,α,β) is sufficiently close to an element $(D_0,f_0,g_0,\alpha_0,\beta_0)$ satisfying various hypothesis. The hypothesis, which are denoted by the letter H, are listed below. The system of differential equations

$$x' = f_0(x,y)\,, \tag{9}$$

in which y is a parameter, will be called the *fast equation*.

(H1). *For all y, the fast equation (9) has the uniqueness of the solutions with prescribed initial conditions.*

We assume that the we are given an n-dimensional compact manifold $\mathcal{L}$, which is contained in the set

$$f_0(x,y) = 0 \tag{10}$$

of equilibrium points of the fast equation (9). The manifold $\mathcal{L}$ is given as the graph of a function, that is, there is a continuous mapping $\xi : Y \to \mathbb{R}^n$, Y being a compact domain in $\mathbb{R}^m$, such that $(\xi(y),y) \in D_0$ for all $y \in Y$ and $\mathcal{L} = \{(x,y) : x = \xi(y),\ y \in Y\}$.

(H2). *The set Y is a compact domain. The function ξ is continuous. For all $y \in Y$, $x = \xi(y)$ is an* isolated root *of equation (10), that is, $f_0(\xi(y),y)=0$, and there exists a number $\delta > 0$ such that the relations $y \in Y$, $\|x-\xi(y)\| < \delta$ and $x \neq \xi(y)$ imply $f_0(x,y) \neq 0$.*

It is not excluded that equation (10) may have other roots beside $\xi(y)$. The manifold defined by equation (10) is called the *slow manifold*. We recall the concept of *uniform asymptotic stability* of equilibrium points of equations depending on parameters.

Definition 1. *The equilibrium point $x = \xi(y)$ of the equation (9) is said to be*
1. Stable (in the sense of Liapunov) if for every $\mu > 0$ there exists a η with the property that any solution $x(\tau)$ of (9) for which $\|x(0) - \xi(y)\| < \eta$ can be continued for all $\tau > 0$ and satisfies the inequality $\|x(\tau) - \xi(y)\| < \mu$.
Asymptotically stable if it is stable and, in addition, $\lim_{\tau\to\infty} x(\tau) = \xi(y)$ for all solutions such that $\|x(0) - \xi(y)\| < \eta$.
2. Attractive if it admits a basin of attraction, that is, a neighbourhood $\mathcal{V}$ with the property that any solution $x(\tau)$ of (9) for which $x(0) \in \mathcal{V}$ can be continued for all $\tau > 0$ and satisfies $\lim_{\tau\to\infty} x(\tau) = \xi(y)$.
Moreover we say that the basin of attraction of the equilibrium point $x = \xi(y)$ is uniform over Y if there exists $a > 0$ such that for all $y \in Y$, the ball $\mathcal{B} = \{x \in \mathbb{R}^n : \|x - \xi(y)\| \leq a\}$, of center $\xi(y)$ and radius a, is a basin of attraction of $\xi(y)$.

It is easy to see that an equilibrium point is asymptotically stable if and only if it is stable and attractive. We must require that

(H3). *For each $y \in Y$, the point $x = \xi(y)$ is an asymptotically stable equilibrium point of the fast equation (9) and the basin of attraction of $x = \xi(y)$ is uniform over Y.*

The system of differential equations

$$\dot{y} = g_0(\xi(y), y)\,, \tag{11}$$

defined on the interior Y_0 of Y, will be called *the slow equation.* Since the set Y is compact, we restricted the slow equation on Y_0 to avoid non essential technicalities with the maximal interval of definition of a solution.

(H4). *The slow equation (11), has the uniqueness of the solutions with prescribed initial conditions.*

(H5). *The point β_0 is in Y_0. The point α_0 is in the basin of attraction of the equilibrium point $x = \xi(\beta_0)$.*

We refer to the problem

$$x' = f_0(x, \beta_0) \quad x(0) = \alpha_0\,, \tag{12}$$

consisting of the fast equation (9), where $y = \beta_0$, together with the initial condition $x(0) = \alpha_0$ as the boundary layer equation. Let $x_0(\tau)$ be the solution of the boundary layer equation. According to the hypothesis (H5), $x_0(\tau)$ is defined for all $\tau \geq 0$ and $\lim_{\tau\to\infty} x_0(\tau) = \xi(\beta_0)$. We refer to the problem

$$\dot{y} = g_0(\xi(y), y) \quad y(0) = \beta_0\,, \tag{13}$$

consisting of the slow equation (11) together with the initial condition $y(0) = \beta_0$, as the *reduced problem.* Let $y_0(t)$ be the solution of the reduced problem. Let $I = [0, \omega)$, $0 < \omega \leq +\infty$ be its maximal positive interval of definition.

Our result asserts that the curve C consisting of two continuous arcs C_1 and C_2, where C_1 is the arc $x = x_0(\tau)$, $y = \beta_0$, $0 \le \tau < +\infty$, and C_2 is the arc $x = \xi(y_0(t))$, $y = y_0(t)$, $t \in I$, gives an approximation of the solutions $((x(t), y(t))$ of Problem (8), when ε is small enough and (D, f, g, α, β) is close to $(D_0, f_0, g_0, \alpha_0, \beta_0)$. The closeness is measured by a topology on the set $\mathcal{T}$. To have a convenient definition of this topology, we introduce the notation $\|h\|_\Delta = \sup_{x\in\Delta} \|h(x)\|$, where h is a function defined on a set Δ, with values in a normed space.

Definition 2. *The* topology of uniform convergence on compacta *on the set $\mathcal{T}$ is the topology for which the neighbourhood system of an element $(D_0, f_0, g_0, \alpha_0, \beta_0)$ is generated by the sets*

$$\begin{aligned} V(\Delta, a) \quad = \quad & \{(D, f, g, \alpha, \beta) \in \mathcal{T} : \Delta \subset D,\ \|f - f_0\|_\Delta < a, \\ & \|g - g_0\|_\Delta < a,\ \|\alpha - \alpha_0\| < a,\ \|\beta - \beta_0\| < a\} \end{aligned}$$

where Δ is a compact subset of D_0 and a is a real positive number.

We are now in a position to state our main result.

Theorem 1. *Let $f_0 : D_0 \to \mathbb{R}^n$, $g_0 : D_0 \to \mathbb{R}^m$ and $\xi : Y \to \mathbb{R}^n$ be continuous functions and let (α_0, β_0) be in D_0. Let hypotheses (H1) to (H5) be satisfied. Let $x_0(\tau)$ be the solution of the boundary layer equation (12). Let $y_0(t)$ be the solution of the reduced problem (13). Let T be in I, I being the positive interval of definition of y_0. For every $\eta > 0$, there exists $\delta > 0$ and a neighbourhood $\mathcal{V}$ of $(D_0, f_0, g_0, \alpha_0, \beta_0)$ in $\mathcal{T}$ with the properties that for all $\varepsilon < \delta$, and all $(D, f, g, \alpha, \beta) \in \mathcal{V}$, any solution $(x(t), y(t))$ of the problem (8) is defined at least on $[0, T]$ and there exists $L > 0$ such that $\varepsilon L < \eta$, $\|x(\varepsilon\tau) - x_0(\tau)\| < \eta$ for $0 \le \tau \le L$, $\|x(t) - \xi(y_0(t))\| < \eta$ for $\varepsilon L \le t \le T$ and $\|y(t) - y_0(t)\| < \eta$ for $0 \le t \le T$.*

Let us discuss now the approximations for $t \in [0, +\infty)$. Let $y_\infty \in Y_0$ be an equilibrium point of the slow equation (11), that is, $g_0(\xi(y_\infty), y_\infty) = 0$.

(H6). *The point $y = y_\infty$ is an asymptotically stable equilibrium point of equation (11) and β_0 lies in the basin of attraction of y_∞.*

When (H6) is satisfied, the solution $y_0(t)$ of the reduced problem is defined for all $t \ge 0$ and satisfies the property $\lim_{t\to\infty} y_0(t) = y_\infty$. In this case the approximation given by Theorem 1 holds for all $t \ge 0$.

Theorem 2. *Let $f_0 : D_0 \to \mathbb{R}^n$, $g_0 : D_0 \to \mathbb{R}^m$ and $\xi : Y \to \mathbb{R}^n$ be continuous functions. Let y_∞ be in Y_0 and (α_0, β_0) be in D_0. Assume that hypothesis (H1) to (H6) hold. Let $x_0(\tau)$ be the solution of the boundary layer equation (12). Let $y_0(t)$ be the solution of the reduced problem (13). For every $\eta > 0$, there exists $\delta > 0$ and a neighbourhood $\mathcal{V}$ of $(D_0, f_0, g_0, \alpha_0, \beta_0)$ in $\mathcal{T}$ with the properties that for all $\varepsilon < \delta$, and all $(D, f, g, \alpha, \beta) \in \mathcal{V}$, any solution $(x(t), y(t))$ of the problem (8) is defined for all $t \ge 0$ and there exists $L > 0$ such that*

$\varepsilon L < \eta$, $\|x(\varepsilon\tau) - x_0(\tau)\| < \eta$ for $0 \le \tau \le L$, $\|x(t) - \xi(y_0(t))\| < \eta$ for $t \ge \varepsilon L$ and $\|y(t) - y_0(t)\| < \eta$ for $t \ge 0$.

The proofs Theorems 1 and 2 are postponed to Section 3.2.

2.2 Non-autonomous systems

The problem (8) contains an apparently more general situation

$$\begin{aligned} \varepsilon\dot{x} &= f(x, y, t) \quad x(t_0) = \alpha\,, \\ \dot{y} &= g(x, y, t) \quad y(t_0) = \beta\,, \end{aligned} \tag{8$'$}$$

where f and g are defined on an open set $D \subset \mathbb{R}^{n+m+1}$ and $(\alpha, \beta, t_0) \in D$. To see this, we consider t as a dependent slow variable and append the trivial equation $\dot{t} = 1$. The fast equation is

$$x' = f_0(x, y, t) \tag{9$'$}$$

where y and t are parameters. The component $\mathcal{L}$ of the slow manifold

$$f_0(x, y, t) = 0 \tag{10$'$}$$

is given as the graph of a function $x = \xi(y, t)$, where (y, t) belongs to a compact domain $Y \subset \mathbb{R}^m \times \mathbb{R}$. The slow equation, considered on the interior Y_0 of Y, is

$$\dot{y} = g_0(\xi(y, t), y, t). \tag{11$'$}$$

There is no change in the formulation of hypotheses (H1) to (H4), except that equations (9), (10) and (11) are replaced by equations (9$'$), (10$'$) and (11$'$). The formulation of hypothesis (H5) is: $(\beta_0, t_0) \in Y_0$ and α_0 lies in the basin of attraction of $\xi(y_0, t_0)$. Thus Theorem 1 applies to problem (8$'$). Let $x_0(\tau)$ be the solution of the boundary layer equation $x' = f_0(x, \beta_0, t_0)$, $x(0) = \alpha_0$. Let $y_0(t)$ be the solution, defined on $[t_0, T]$, of the reduced problem $\dot{y} = g_0(\xi(y, t), y, t)$, $y(t_0) = \beta_0$. For any $\eta > 0$ and any solution $(x(t), y(t))$ of problem (8$'$), there exists $L > 0$ such that $\varepsilon L < \eta$, $\|x(t_0 + \varepsilon\tau) - x_0(\tau)\| < \eta$ for $0 \le \tau \le L$, $\|x(t) - \xi(y_0(t))\| < \eta$ for $t_0 + \varepsilon L \le t \le T$ and $\|y(t) - y_0(t)\| < \eta$ for $t_0 \le t \le T$, as long as ε is small enough and (D, f, g, α, β) is in a small neighbourhood of $(D_0, f_0, g_0, \alpha_0, \beta_0)$.

When the function $g_0(\xi(y, t), y, t)$ depends nontrivially on the variable t, Theorem 2 does not apply to Problem (8$'$), because Hypothesis (H6) does never hold for the non-autonomous equation (11$'$). In that case one could require that y_∞ is a stationary solution, that is, $g_0(\xi(y_\infty, t), y_\infty, t) \equiv 0$. However, since Y is assumed to be a compact set, the limiting behaviour, as $t \to +\infty$ is not relevant. It would be necessary to generalize Theorems 1 and 2 to noncompact pieces of the slow manifold.

2.3 Deformations: Tykhonov's theorem

As explained in the introduction, the classical Tykhonov's theorem concerns the one-parameter deformation (2) of the unperturbed problem (3), under the assumption of uniqueness of solutions of system (2), so that we can consider the solution $(x(t,\varepsilon), y(t,\varepsilon))$ as depending on the parameter ε and discuss its limit as $\varepsilon \to 0$. Actually, Tykhonov formulated his result only for systems for which the right-hand side does not depend on ε. In [26] Tykhonov requires that the equilibrium point $x = \xi(y)$ of system (5) is asymptotically stable for all $y \in Y$, but he does not require the uniformity of the basin of attraction over Y. The number η whose existence is assumed in Definition 1 will, in general, depend on μ and also on y. This brings the following definition.

Definition 3. *The equilibrium point $x = \xi(y)$ of the equation (9) is said to be uniformly asymptotically stable over Y if for every $\mu > 0$ there exists a η with the property that for any $y \in Y$, any solution $x(\tau)$ of (9) for which $\|x(0) - \xi(y)\| < \eta$ can be continued for all $\tau > 0$ and satisfies the inequality $\|x(\tau) - \xi(y)\| < \mu$ and $\lim_{\tau\to\infty} x(\tau) = \xi(y)$.*

Tykhonov [26] proves that η may be chosen independently of y as long as Y is compact, that is, the asymptotic stability is uniform over Y. However, this is false, and simple examples show that η is not always bounded away from zero on the compact set Y. We are therefore forced to introduce the hypothesis that the asymptotic stability of $x = \xi(y)$ is uniform over Y (see [29], p. 255). These matters have been fully discussed by Hoppensteadt [12] (see also [28]).

Assume that $x = \xi(y)$ is uniformly asymptotically stable over Y. It is easy to show that the basin of attraction is uniform over Y. Conversely, Hoppensteadt [12] proved that if Y is compact, then the asymptotic stability of $x = \xi(y)$ for all $y \in Y$, together with the existence of a uniform basin of attraction over Y, imply that the asymptotic stability is uniform over Y (see also the remark following Lemma 6 of the present paper). Thus, to formulate Tykhonov's theorem under the hypothesis that $x = \xi(y)$ is uniformly asymptotically stable over Y as given by Wasow [29], is the same as formulating it under the hypothesis that $x = \xi(y)$ is asymptotically stable for all $y \in Y$ and has a uniform basin of attraction over Y as given by Hoppensteadt [11, 12] or as given in the present paper (see also [10], p. 235). Note that the proof of Hoppensteadt [11] is based on construction of Lyapunov functions and is quite different from Tykhonov original proof [26]. The reader who is not acquainted with Russian language should consult Wasow [29] who follows the presentation given in [26].

We notice that Tykhonov paper deals also with systems of the form

$$\begin{aligned} \varepsilon_j \dot{x}_j &= f_j(x_1, \cdots, x_k, y) \quad j = 1, \cdots, k \\ \dot{y} &= g(x_1, \cdots, x_k, y)\,, \end{aligned}$$

where $x_j \in \mathbb{R}^{n_j}$, $y \in \mathbb{R}^m$ and $\varepsilon_1, \cdots, \varepsilon_k$ are small positive parameters. Tykhonov gives the behaviour of solutions when $\varepsilon_1 \to 0$ and $\varepsilon_{j+1}/\varepsilon_j \to 0$, $j = 1, \cdots, k-1$.

Tykhonov defines a hierarchy of boundary layer equations and reduced problems that approximate the solutions at various time scales. Such systems have been also studied by Hoppensteadt [14, 15]. They will be considered in a forthcoming paper with emphasis on the underlying functional spaces and topologies.

Of course, Theorem 2 may also be formulated in terms of one-parameter deformations. We obtain approximations, on the infinite time interval, of the perturbed problem solutions, under the assumption that the reduced problem has an asymptotically stable equilibrium point. This result is neither presented in Tykhonov's paper nor in Wasow's book. However, Hoppensteadt in a series of papers [11, 13, 15, 16] studied extensively the approximations on the infinite time interval. His studies concern also non autonomous systems.

3 Nonstandard results

3.1 A short tutorial on Internal Set Theory

Internal Set Theory (IST) is an axiomatic approach to Nonstandard Analysis, proposed by Nelson [21]. We adjoin to ordinary mathematics (say ZFC) a new undefined unary predicate *standard* (st). The axioms of IST are the usual axioms of ZFC plus three others which govern the use of the new predicate. Hence *all theorems of* ZFC *remain valid*. What is new in IST is an addition, not a change. We call a formula of IST *external* if it involves the new predicate "st"; otherwise, we call it *internal*. Thus internal formulas are the formulas of ZFC. IST is a *conservative extension* of ZFC, that is, every internal theorem of IST is a theorem of ZFC. Some of the theorems which are proved in IST are external and can be reformulated so that they become internal. Indeed, there is a *reduction algorithm* which reduces any external formula A of IST to an internal formula A', with the same free variables, which satisfies $A \equiv A'$, that is, $A \Leftrightarrow A'$ for all standard values of the free variables. We give the reduction of the frequently occurring formula $\forall x\ (\forall^{\rm st} y\ A \Rightarrow \forall^{\rm st} z\ B)$, where A and B are internal formulas:

$$\forall x\ (\forall^{\rm st} y\ A \Rightarrow \forall^{\rm st} z\ B)\ \equiv\ \forall z\ \exists^{\rm fin} y'\ \forall x\ (\forall y \in y'\ A \Rightarrow B). \tag{14}$$

A real number x is called *infinitesimal*, denoted by $x \simeq O$, if $|x| < \varepsilon$ for all standard $\varepsilon > 0$, *limited* if $|x| < r$ for some standard r, *appreciable* if it is limited and not infinitesimal, and *unlimited*, denoted by $x \simeq \mp\infty$, if it is not limited. Let (E, d) be a standard metric space. Two points x and y in E are called *infinitely close*, denoted by $x \simeq y$, if $d(x, y) \simeq 0$.

We may not use external formulas in the axiom schemes of ZFC, in particular we may not use external formulas to define subsets. The notations $\{x \in \mathbb{R} : x \text{ is limited}\}$ or $\{x \in \mathbb{R} : x \simeq 0\}$ are not allowed. Moreover, we can prove that there does not exist subsets L and I of $\mathbb{R}$ such that, for all x in $\mathbb{R}$, x is in L if and only if x is limited, or x is in I if and only if x is infinitesimal. This result is frequently used in proofs. Suppose that we have shown that a certain internal property A holds for every limited r; then we know that A holds for some unlimited r, for otherwise we could let $L = \{x \in \mathbb{R} : A\}$. This is called the *Cauchy principle*. It has the following consequence

Lemma 1. (Robinson's Lemma). *Let $r(t)$ be a real function such that $r(t) \simeq 0$ for all limited $t \geq 0$, then there exists an unlimited positive number ν such that $f(t) \simeq 0$ for all $t \in [0, \nu]$.*

Proof. The set of all s, such that $|r(t)| < 1/s$ for all $t \in [0, s]$, contains all limited $s \geq 1$. By the Cauchy principle, it must contain some unlimited ν. $\square$

Let X be a standard topological space. A point x in X is said to be *infinitely close* to a standard point x_0, denoted by $x \simeq x_0$, if x is in every standard neighbourhood of x_0. Let A be a standard open subset of X. A point $x \in X$ is said to be *nearstandard in* A if there exists a standard $x_0 \in A$ such that $x \simeq x_0$. We recall that A is compact if and only if any point $x \in A$ is nearstandard in A. We recall that A is open if and only if any point $x \in X$ which is nearstandard in A, belongs to A. For more informations on the nonstandard approach to topological spaces, the reader is referred to [24].

3.2 Perturbations

Classically, the intuitive notion of perturbation can only be described via deformations or neighbourhoods. The first benefit we gain from NSA is a natural and useful notion of perturbation. A perturbation of a standard object is a nonstandard object which is (infinitely) close to it in some sense to be precised. Since a perturbation is a simple nonstandard object, its properties can be investigated directly, and do not require to use extra-properties with respect to the parameters of the deformation as in the classical approach.

Definition 4. *An element $(D, f, g, \alpha, \beta) \in \mathcal{T}$ is said to be a perturbation of the standard element $(D_0, f_0, g_0, \alpha_0, \beta_0) \in \mathcal{T}$ if D contains all the nearstandard point in D_0, $f(x, y) \simeq f_0(x, y)$ and $g(x, y) \simeq g_0(x, y)$ for all (x, y) which is nearstandard in D_0 and $\alpha \simeq \alpha_0$, $\beta \simeq \beta_0$.*

We note that $f_0(x, y)$ and $g_0(x, y)$ are well defined for all nearstandard points (x, y) in D_0. Indeed, D_0 is a standard open set so it contains all the nearstandard points (x, y) in D_0. With this notion we can reformulate Theorems 1 and 2 as follows.

Theorem 3. *Let $f_0 : D_0 \to \mathbb{R}^n$, $g_0 : D_0 \to \mathbb{R}^m$ and $\xi : Y \to \mathbb{R}^n$ be standard continuous functions. Let (α_0, β_0) be standard in D_0. Assume that hypothesis (H1) to (H5) hold. Let $x_0(\tau)$ be the solution of the boundary layer equation (12). Let $y_0(t)$ be the solution of the reduced problem (13). Let T be standard in I, I being the positive interval of definition of y_0. Let $\varepsilon > 0$ be infinitesimal. Let (D, f, g, α, β) be a perturbation of $(D_0, f_0, g_0, \alpha_0, \beta_0)$. Any solution $(x(t), y(t))$*

of Problem (8) is defined at least on $[0,T]$ *and there exists* $L > 0$ *such that* $\varepsilon L \simeq 0$, $x(\varepsilon\tau) \simeq x_0(\tau)$ *for* $0 \leq \tau \leq L$, $x(t) \simeq \xi(y_0(t))$ *for* $\varepsilon L \leq t \leq T$ *and* $y(t) \simeq y_0(t)$ *for* $0 \leq t \leq T$.

Theorem 4. *Let* $f_0 : D_0 \to \mathbb{R}^n$, $g_0 : D_0 \to \mathbb{R}^m$ *and* $\xi : Y \to \mathbb{R}^n$ *be standard continuous functions. Let* $y_\infty \in Y_0$ *and* $(\alpha_0, \beta_0) \in D_0$ *be standard. Assume that hypothesis (H1) to (H6) hold. Let* $x_0(\tau)$ *be the solution of the boundary layer equation (12). Let* $y_0(t)$ *be the solution of the reduced problem (13). Let* $\varepsilon > 0$ *be infinitesimal. Let* (D, f, g, α, β) *be a perturbation of* $(D_0, f_0, g_0, \alpha_0, \beta_0)$*. Any solution* $(x(t), y(t))$ *of Problem (8) is defined for all* $t \geq 0$ *and and there exists* $L > 0$ *such that* $\varepsilon L \simeq 0$, $x(\varepsilon\tau) \simeq x_0(\tau)$ *for* $0 \leq \tau \leq L$, $x(t) \simeq \xi(y_0(t))$ *for* $t \geq \varepsilon L$ *and* $y(t) \simeq y_0(t)$ *for* $t \geq 0$.

The proofs of Theorems 3 and 4 are postponed to Section 4. Theorems 3 and 4 are external statements. As we have recalled, Nelson [21] proposed a reduction algorithm that reduces external theorems to equivalent internal forms. Let us show that the reduction of Theorem 3 (resp. Theorem 4) is Theorem 1 (resp. Theorem 2). We need the following result.

Lemma 2. *The element* $(D, f, g, \alpha, \beta) \in \mathcal{T}$ *is a perturbation of the standard element* $(D_0, f_0, g_0, \alpha_0, \beta_0) \in \mathcal{T}$ *if and only if* (D, f, g, α, β) *is infinitely close to* $(D_0, f_0, g_0, \alpha_0, \beta_0)$ *for the topology of uniform convergence on compacta.*

Proof. Let (D, f, g, α, β) be a perturbation of $(D_0, f_0, g_0, \alpha_0, \beta_0)$. Let Δ be a standard compact subset of D_0. Any $(x, y) \in \Delta$ is nearstandard in Δ, and so in D_0. Then $\Delta \subset D$, $\alpha \simeq \alpha_0$, $\beta \simeq \beta_0$ and $f(x, y) \simeq f_0(x, y)$, $g(x, y) \simeq g_0(x, y)$ for all $(x, y) \in \Delta$. Let $a > 0$ be a standard real number. Then $\|f - f_0\|_\Delta < a$, $\|g - g_0\|_\Delta < a$, $\|\alpha - \alpha_0\| < a$ and $\|\beta - \beta_0\| < a$. Hence $(D, f, g, \alpha, \beta) \in V(\Delta, a)$ for all standard compact $\Delta \subset D_0$ and all standard $a > 0$, that is, $(D, f, g, \alpha, \beta) \simeq (D_0, f_0, g_0, \alpha_0, \beta_0)$ for the topology of uniform convergence on compacta. Conversely, let (D, f, g, α, β) be infinitely close to $(D_0, f_0, g_0, \alpha_0, \beta_0)$ for the topology of uniform convergence on compacta. Let (x, y) be nearstandard in D_0. There exists a standard element $(x_0, y_0) \in D_0$ such that $(x, y) \simeq (x_0, y_0)$. Let Δ be a standard compact neighbourhood of (x_0, y_0) such that $\Delta \subset D_0$. Then $\Delta \subset D$, $(x, y) \in \Delta$ and $\|f(x, y) - f_0(x, y)\| < a$, $\|g(x, y) - g_0(x, y)\| < a$ for any standard $a > 0$. Thus $(x, y) \in D$ and $f(x, y) \simeq f_0(x, y)$, $g(x, y) \simeq g_0(x, y)$. Since $\alpha \simeq \alpha_0$ and $\beta \simeq \beta_0$, we obtain that (D, f, g, α, β) is a perturbation of $(D_0, f_0, g_0, \alpha_0, \beta_0)$. $\square$

Proof of Theorem 1. We adopt the following abbreviations: u is the variable $(D_0, f_0, g_0, \alpha_0, \beta_0)$, v is the variable (D, f, g, α, β), and B is the formula

> If $\eta > 0$ then any solution $(x(t), y(t))$ of Problem (8) is defined at least on $[0, T]$ and there exists $L > 0$ such that $\varepsilon L < \eta$, $\|x(\varepsilon\tau) - x_0(\tau)\| < \eta$ for $0 \leq \tau \leq L$, $\|x(t) - \xi(y_0(t))\| < \eta$ for $\varepsilon L \leq t \leq T$ and $\|y(t) - y_0(t)\| < \eta$ for $0 \leq t \leq T$.

According to the Lemma 2, to say that v is a perturbation of u is the same as saying v is in any standard neighbourhood of u. To say that "any solution $(x(t), y(t))$ of Problem (8) is defined at least on $[0, T]$ and there exists $L > 0$ such that $\varepsilon L \simeq 0$, $x(\varepsilon\tau) \simeq x_0(\tau)$ for $0 \le \tau \le L$, $x(t) \simeq \xi(y_0(t))$ for $\varepsilon L \le t \le T$ and $y(t) \simeq y_0(t)$ for $0 \le t \le T$" is the same as saying $\forall^{\mathrm{st}}\eta\ B$. Then Theorem 3 asserts that

$$\forall\varepsilon\ \forall v(\forall^{\mathrm{st}}\delta\ \forall^{\mathrm{st}}\mathcal{V}\ \varepsilon < \delta\ \&\ v \in \mathcal{V} \Rightarrow \forall^{\mathrm{st}}\eta\ B).$$

In this formula, u, ξ and T are standard parameters, v ranges over $\mathcal{T}$, ε, δ, and η range over the strictly positive real numbers and $\mathcal{V}$ ranges over the neighbourhoods of u. By (14), this is equivalent to

$$\forall\eta\ \exists^{\mathrm{fin}}\delta'\ \exists^{\mathrm{fin}}\mathcal{V}'\ \forall\varepsilon\ \forall v\ (\forall\delta \in \delta'\ \forall\mathcal{V} \in \mathcal{V}'\ \varepsilon < \delta\ \&\ v \in \mathcal{V} \Rightarrow B).$$

For δ' and $\mathcal{V}'$ finite sets, $\forall\delta \in \delta'\ \forall\mathcal{V} \in \mathcal{V}'\ \varepsilon < \delta\ \&\ v \in \mathcal{V}$ is the same as $\varepsilon < \delta\ \&\ v \in \mathcal{V}$ for $\delta = \min\delta'$ and $\mathcal{V} = \cap_{V\in\mathcal{V}'}V$, and so our formula is equivalent to

$$\forall\eta\ \exists\delta\ \exists\mathcal{V}\ \forall\varepsilon\ \forall v\ (\varepsilon < \delta\ \&\ v \in \mathcal{V} \Rightarrow B).$$

This shows that for any standard u, ξ and any standard $T \in I$, the statement of Theorem 1 holds, thus by transfer, it holds for any u, ξ and any $T \in I$. □

The reduction of Theorem 4 to Theorem 2 follows almost verbatim the reduction of Theorem 3 to Theorem 1 and is left to the reader.

3.3 Uniform asymptotic stability

The external characterizations of the notion of stability and attractivity of the equilibrium point $\xi(y)$ of equation (9), given in Definition 1, are as follows.

Lemma 3. *Assume f, ξ and y are standard. The equilibrium point $x = \xi(y)$ of the equation (9) is*
1. Stable if and only if any solution $x(\tau)$ of (9) for which $x(0) \simeq \xi(y)$ can be continued for all $\tau > 0$ and satisfies $x(\tau) \simeq \xi(y)$.
2. Attractive if it admits a standard basin of attraction, that is, a standard neighbourhood $\mathcal{V}$ with the property that any solution $x(\tau)$ of system (9) for which $x(0)$ is standard in $\mathcal{V}$ can be continued for all $\tau > 0$ and satisfies $x(\tau) \simeq \xi(y)$ for all $\tau \simeq +\infty$.

Proof. 1. Let B be the formula "Any solution $x(\tau)$ of equation (9) for which $x(0) = \alpha$ can be continued for all $\tau > 0$ and satisfies the inequality $\|x(\tau) - \xi(y)\| < \mu$". The characterization of stability in the lemma is

$$\forall\alpha\ (\forall^{\mathrm{st}}\eta\ \|\alpha - \xi(y)\| < \eta \Rightarrow \forall^{\mathrm{st}}\mu\ B).$$

In this formula f, ξ and y are standard parameters and η, μ range over the strictly positive real numbers. By (14), this is equivalent to

$$\forall\mu\ \exists^{\mathrm{fin}}\eta'\ \forall\alpha\ (\forall\eta \in \eta'\ \|\alpha - \xi(y)\| < \eta \Rightarrow B).$$

For η' a finite set, $\forall\eta \in \eta'$ $\|\alpha - \xi(y)\| < \eta$ is the same as $\|\alpha - \xi(y)\| < \eta$ for $\eta = \min \eta'$, and so our formula is equivalent to

$$\forall\mu\ \exists\eta\ \forall\alpha\ (\|\alpha - \xi(y)\| < \eta \Rightarrow B).$$

This is the usual definition of stability.
2. By transfer, the attractivity of an equilibrium is equivalent to the existence of a standard basin of attraction. The characterization of a standard basin of attraction $\mathcal{V}$ in the lemma is that any solution $x(\tau)$ of system (9) for which $x(0)$ is standard in $\mathcal{V}$ can be continued for all $\tau > 0$ and satisfies

$$\forall\tau\ (\forall^{\text{st}} r\ \tau > r \Rightarrow \forall^{\text{st}}\mu\ \|x(\tau) - \xi(y)\| < \mu).$$

In this formula ξ and $x(\cdot)$ are standard parameters and r, μ range over the strictly positive real numbers. By (14), this is equivalent to

$$\forall\mu\ \exists^{\text{fin}} r'\ \forall\tau\ (\forall r \in r'\ \tau > r \Rightarrow \|x(\tau) - \xi(y)\| < \mu).$$

For r' a finite set $\forall r \in r'$ $\tau > r$ is the same as $\tau > r$ for $r = \max r'$, and so our formula is equivalent to

$$\forall\mu\ \exists r\ \forall\tau\ (\tau > r \Rightarrow \|x(\tau) - \xi\| < \mu).$$

We have shown that for all standard α in $\mathcal{V}$ (and consequently, by transfer, for all α in $\mathcal{V}$) any solution $x(\tau)$ of problem (9) for which $x(0) = \alpha$, can be continued for all $\tau > 0$ and satisfies $\lim_{\tau\to+\infty} x(\tau) = \xi(y)$. This is the usual definition of a basin of attraction. □

Let hypothesis (H1) be satisfied. Let $\pi(\tau, \alpha, y)$ be the unique noncontinuable solution of equation (9) such that $\pi(0, \alpha, y) = \alpha$. This solution is defined on the interval $I(\alpha, y)$. It follows from the basic theorems of differential equations that the function π is continuous with respect to the initial condition α and the parameters y. The external formulation of this result is as follows.

Lemma 4. *Assume f_0 is standard. Let y_0 and α_0 be standard, then for all standard $\tau \in I(\alpha_0, y_0)$ and all $\alpha \simeq \alpha_0$, $y \simeq y_0$, we have $\tau \in I(\alpha, y)$ and $\pi(\tau, \alpha, y) \simeq \pi(\tau, \alpha_0, y_0)$.*

Proof. The reduction of the Lemma 4 is the usual continuity of solutions with respect to initial conditions and parameters. This lemma is a particular case of the *Short Shadow Lemma* (see Section 4). □

Lemma 5. *Assume that hypothesis (H1) is satisfied. Assume that f, ξ and y are standard. Then the equilibrium point $x = \xi(y)$ is asymptotically stable if and only if there exists a standard $a > 0$ with the property that for any α in the ball $\mathcal{B}$ of center $\xi(y)$ and radius a, the solution $x(\tau)$ of system (9) for which*

$x(0) = \alpha$, can be be continued for all $\tau > 0$ and satisfies $x(\tau) \simeq \xi(y)$ for all $\tau \simeq +\infty$.

Proof. Assume that ξ is asymptotically stable. Then it is attractive, and so it admits a standard basin of attraction $\mathcal{V}$. Let $a > 0$ be standard such that the closure of the ball $\mathcal{B}$ of center $\xi(y)$ and radius a is included in $\mathcal{V}$. Let $\alpha \in \mathcal{B}$ and let α_0 be standard in $\mathcal{V}$ such that $\alpha \simeq \alpha_0$. Let $x(\tau) = \pi(\tau, \alpha, y)$ and $x_0(\tau) = \pi(\tau, \alpha_0, y)$. By the attractivity of $\xi(y)$, the solution $x_0(\tau)$ is defined for all $\tau > 0$ and satisfies $x_0(\tau) \simeq \xi(y)$ for all $\tau \simeq +\infty$. By Lemma 4, $x(\tau) \simeq x_0(\tau)$ for all limited $\tau > 0$. By Robinson's Lemma, there exists $\nu \simeq +\infty$ such that $x(\tau) \simeq x_0(\tau)$ for all $\tau \in [0, \nu]$. Thus $x(\tau) \simeq \xi(y)$ for all unlimited $\tau \leq \nu$. By stability of $\xi(y)$ we have $x(\tau) \simeq \xi(y)$ for all $\tau > \nu$. Hence $x(\tau) \simeq \xi(y)$ for all $\tau \simeq +\infty$. Conversely, assume $\xi(y)$ satisfies the property in the lemma. By Lemma 3, the ball $\mathcal{B}$ is a standard basin of attraction of $\xi(y)$. Hence $\xi(y)$ is attractive. Let $\alpha \simeq \xi(y)$. Then, by hypothesis, $x(\tau) \simeq \xi(y)$ for all $\tau \simeq +\infty$, and, by Lemma 4, $x(\tau) \simeq \pi(\tau, \xi(y), y) = \xi(y)$ for all limited τ. By Lemma 3, $\xi(y)$ is stable. Thus $\xi(y)$ is asymptotically stable. □

Let us return now to the discussion of uniform asymptotic stability over Y of the equilibrium $\xi(y)$ of equation (9). Assume f_0 and ξ are standard. According to Lemma 5, hypothesis (H3) is equivalent to

(H3′). *There exists a standard $a > 0$ with the property that for all standard $y \in Y$, any solution $x(\tau)$ of (9) for which $\|x(0) - \xi(y)\| < a$ can be continued for all $\tau > 0$ and satisfies $x(\tau) \simeq \xi(y)$ for all $\tau \simeq +\infty$.*

The following result is not used in the present paper. We give it as a complement of the previous discussions on uniform asymptotic stability. Moreover this result is connected also to the discussion following Definition 3, on the various hypothesis under which Tykhonov's theorem can be formulated.

Lemma 6. *Let hypothesis (H1) be satisfied. Assume f_0 and ξ are standard and Y compact. If the equilibrium point $x = \xi(y)$ of the equation (9) is asymptotically stable and the basin of attraction is uniform over Y then there exists a standard $a > 0$ with the property that for all $y \in Y$ any solution $x(\tau)$ of (9) for which $\|x(0) - \xi(y)\| < a$ can be continued for all $\tau > 0$ and satisfies $x(\tau) \simeq \xi(y)$ for all $\tau \simeq +\infty$.*

Proof. Assume $\xi(y)$ is asymptotically stable and has a basin of attraction which is uniform over Y. Let $a > 0$ such that for all $y \in Y$, any solution $x(\tau)$ of system (9) for which $\|x(0) - \xi(y)\| < a$ can be continued for all $\tau > 0$ and satisfies $\lim_{\tau \to +\infty} x(\tau) = \xi(y)$. By transfer, there exists a standard $a > 0$ with this property. Let $y \in Y$ and let α be such that $\|\alpha - \xi(y)\| < a$. Let $x(\tau) = \pi(\tau, \alpha, y)$. Since Y is a standard compact set, there exists y_0 standard in Y and α_0 standard in the ball of center $\xi(y_0)$ and radius a such that $y \simeq y_0$ and $x(0) \simeq \alpha_0$. Let $x_0(\tau) = \pi(\tau, \alpha_0, y_0)$. By the Lemma 3, the solution $x_0(\tau)$ can be continued for all $\tau > 0$ and satisfies $x_0(\tau) \simeq \xi(y_0)$ for all $\tau \simeq +\infty$.

According to the Lemma 4, $x(\tau) \simeq x_0(\tau)$ for all limited $\tau > 0$. According to the Robinson's Lemma, there exists $\nu \simeq +\infty$ such that $x(\tau) \simeq x_0(\tau)$ for all $\tau \in [0, \nu]$. By attractivity $x_0(\nu) \simeq \xi(y_0)$. Then $x(\nu) \simeq \xi(y_0) \simeq \xi(y)$. According to the Lemma 4

$$x(\nu + s) \simeq \xi(y_0) \simeq \xi(y), \quad \text{for all limited } s. \tag{15}$$

Assume there exists $\nu_1 > \nu$ such that $x(\nu_1) \not\simeq \xi(y)$, that is $\gamma = \|x(\nu_1) - \xi(y)\|$ is appreciable. Let m be the smallest value of $\tau \in [\nu, \nu_1]$ such that $\|x(m) - \xi(y)\| = \gamma$. Thus $x(m) \not\simeq \xi(y)$. If $s = m - \nu$ was limited then, by property (15), one would have $x(m) = x(\nu + s) \simeq \xi(y)$, which contradicts $x(m) \not\simeq \xi(y)$. Thus s is unlimited and $x(m + \tau)$ lies in the ball $\mathcal{B} = \{x : \|x - \xi(y)\| \leq \gamma\}$ for all $\tau \in [-s, 0]$. By the Lemma 4 we have $x(m + \tau) \simeq x_0(m + \tau)$ for all limited τ. Hence $x_0(m + \tau)$ lies in the (standard) ball $\mathcal{B} = \{x : \|x - \xi(y_0)\| \leq \gamma_0\}$, where $\gamma_0 > 0$ is the standard part of the appreciable number γ, for all standard negative τ and then (by transfert) for all $\tau \leq 0$. Let $\tau_0 \simeq -\infty$. According to Lemma 5, $x_0(m + \tau_0 + \tau) \simeq \xi(y_0)$ for all $\tau \simeq +\infty$. For $\tau = -\tau_0$, we obtain $x_0(m) \simeq \xi(y_0)$, a contradiction with $x_0(m) \not\simeq \xi(y_0)$. □

The external characterizations of the notion of uniform asymptotic stability over Y of the equilibrium point $\xi(y)$ of equation (9), given in Definition 3, is as follows: there exists a standard $a > 0$ with the property that for all $y \in Y$ any solution $x(\tau)$ of (9) for which $\|x(0) - \xi(y)\| < a$ can be continued for all $\tau > 0$ and satisfies $x(\tau) \simeq \xi(y)$ for all $\tau \simeq +\infty$. Thus, the previous lemma asserts that if Y is compact, then the asymptotic stability of $x = \xi(y)$ for all $y \in Y$, together with the existence of a uniform basin of attraction over Y, imply that the asymptotic stability is uniform over Y. The proof by Hoppensteadt [12] of this result is based on a theorem of Massera [20] on construction of Liapunov functions for asymptotically stable equilibrium points.

4 Proofs of Theorems 3 and 4

4.1 Preliminary lemmas

Let us discuss now the *Short Shadow Lemma*, which is the fundamental tool in regular perturbation theory. Let us consider the initial value problems:

$$\dot{x} = F_0(x), \quad x(0) = \alpha_0\,, \tag{16}$$

$$\dot{x} = F(x), \quad x(0) = \alpha\,, \tag{17}$$

where $F_0 : D_0 \to \mathbb{R}^d$ and $F : D \to \mathbb{R}^d$ are continuous functions, D_0 and D open subsets of $\mathbb{R}^d$, F_0 standard, $\alpha \in D$ and α_0 standard in D_0. Problem (17) is said to be a regular perturbation of problem (16) when D contains all the nearstandard points in D_0, $\alpha \simeq \alpha_0$, and $F(x) \simeq F_0(x)$ for all x nearstandard in D_0. We assume that problem (16) has a unique solution. Let $\phi_0 : I \to \mathbb{R}^d$

be its noncontinuable solution. Will the solutions of problem (17) also exist on I and be close to ϕ_0 ? This question is answered by the Short Shadow Lemma, which is one of the first results of nonstandard asymptotic analysis of differential equations. This result appeared in the nonstandard literature under various formulations (see [4, 5, 7, 8, 19]). For our purpose it is more convenient to adopt the formulation given in [23, 25].

Lemma 7. (Short Shadow Lemma) *Let problem (17) be a regular perturbation of problem (16). Then for any nearstandard $t \in I$, any solution ϕ of problem (17) is defined and satisfies $\phi(t) \simeq \phi_0(t)$.*

The perturbed equation may depend also on the time $t \in]a,b[$, where $]a,b[$ is a possibly nonstandard interval. In that case problem (17) is replaced by problem

$$\dot{x} = F(x,t) \quad x(t_0) = \alpha, \tag{18}$$

where $F : D\times]a,b[\to \mathbb{R}^d$ is continuous and $t_0 \in]a,b[$. Problem (18) is said to be a regular perturbation of problem (16) when D contains all the nearstandard points in D_0, $\alpha \simeq \alpha_0$, and $F(x,t) \simeq F_0(x)$ for all $t \in]a,b[$ and x nearstandard in D_0.

Lemma 8. *Let problem (18) be a regular perturbation of problem (16). Then for any nearstandard $s \in I$, any solution ϕ of problem (18) is defined and satisfies $\phi(t_0+s) \simeq \phi_0(s)$ as long as $t_0 + s \in]a,b[$.*

Proof. This result is a corollary of the Short Shadow Lemma. To see this, we consider t as a dependant parameter and append the trivial equation $\dot{t} = 1$. The unperturbed equation would be considered as defined on the standard set $D_0 \times \Lambda_0$, where Λ_0 is the standard open interval whose standard elements are those elements s_0 of $\mathbb{R}$ such that $s + t_0 \in \Lambda$ for all $s \simeq s_0$. $\square$

Let $\mu > 0$. The set $\|x - \xi(y)\| \le \mu,\ y \in Y$ will be called a *μ-tube* around $\mathcal{L}$. The set $\|x - \xi(y)\| = \mu,\ y \in Y$ constitutes the *lateral boundary* of the μ-tube. Let $f_0 : D_0 \to \mathbb{R}^n$, $g_0 : D_0 \to \mathbb{R}^m$ be standard continuous functions on the standard open subset D_0 of $\mathbb{R}^{n+m}$. Let $f : D \to \mathbb{R}^n$, $g : D \to \mathbb{R}^m$ be continuous perturbations of f_0 and g_0, that is D contains all the nearstandard points in D_0 and $f(x,y) \simeq f_0(x,y)$, $g(x,y) \simeq g_0(x,y)$ for all nearstandard (x,y) in D_0. For the proof of Theorem 3, we need the following results. The first result (Lemma 9) states that any solution $(x(t), y(t))$ of system

$$\begin{aligned} \varepsilon\dot{x} &= f(x,y)\,, \\ \dot{y} &= g(x,y)\,, \end{aligned} \tag{19}$$

that comes infinitely close to the slow manifold $\mathcal{L}$ remains infinitely close to it as long as $y(t)$ is nearstandard in Y_0, that is, y is not infinitely close to the boundary of Y. The second result (Lemma 10) states that the y-component of a solution $(x(t), y(t))$ of (19) that is infinitely close to the slow manifold $\mathcal{L}$, is infinitely close to a solution of the slow equation (11).

Lemma 9. *Let hypothesis (H1) to (H3) be satisfied. Let $(x(t), y(t))$ be a solution of (19) with the properties that $y(t)$ is nearstandard in Y_0 for $t_0 \le t \le t_1$, and $x(t_0) \simeq \xi(y(t_0))$ then $x(t) \simeq \xi(y(t))$ for $t_0 \le t \le t_1$.*

Proof. Let y_0 be standard in Y_0 such that $y(t_0) \simeq y_0$. Thus $x(t_0) \simeq \xi(y_0)$. As a function of τ, $(x(t_0+\varepsilon\tau), y(t_0+\varepsilon\tau))$ is a solution of system

$$\begin{aligned} x' &= f(x,y)\,, \\ y' &= \varepsilon g(x,y)\,, \end{aligned} \tag{20}$$

where $' = d/d\tau$, with initial condition $(x(t_0), y(t_0))$. This problem is a regular perturbation of system

$$\begin{aligned} x' &= f(x,y)\,, \\ y' &= 0\,, \end{aligned} \tag{21}$$

with initial condition $(\xi(y_0), y_0)$. By the Short Shadow Lemma

$$x(t_0+\varepsilon\tau) \simeq \xi(y_0),\ y(t_0+\varepsilon\tau) \simeq y_0 \quad \text{for all limited } \tau. \tag{22}$$

Since $y(t)$ is nearstandard in Y_0, for all $t \in [t_0, t_1]$, Y is a standard compact neighbourhood of $y[t_0, t_1]$. Let a be a standard positive number satisfying the properties in hypothesis (H3′). Assume the statement of the Lemma is false. Then there must exist s, $t_0 < s < t_1$ such that $x(s) \not\simeq \xi(y(s))$, that is, $\gamma = \|x(s) - \xi(y(s))\|$ is appreciable. We may choose s such that γ is standard and $0 < \gamma \le a$. Let $\mathcal{T}$ be the γ-tube around $\mathcal{L}$. Since $(x(t_0), y(t_0))$ belongs to $\mathcal{T}$, $(x(s), y(s))$ belongs to the lateral boundary of $\mathcal{T}$, and no point $y(t)$ with $t \in [t_0, t_1]$ belongs to the boundary of Y, there must exist the smallest value m of t, in $[t_0, t_1]$, such that $(x(m), y(m))$ belongs to the lateral boundary of $\mathcal{T}$, that is $\|x(m) - \xi(y(m))\| = \gamma$. By compactness of this boundary, there exists a standard (x_1, y_1) belonging to the lateral boundary of $\mathcal{T}$, that is, $\|x_1 - \xi(y_1)\| = \gamma$, such that $(x(m), y(m)) \simeq (x_1, y_1)$. If $\tau_0 = (m - t_0)/\varepsilon$ was limited then, by property (22), one would has $x(m) = x(t_0+\varepsilon\tau_0) \simeq \xi(y(t_0))$, $y(m) = y(t_0+\varepsilon\tau_0) \simeq y(t_0)$. Since ξ is standard continuous, $\xi(y(m)) \simeq \xi(y(t_0)) \simeq x(m)$, which contradicts $x(m) \not\simeq \xi(y(m))$. Thus τ_0 is unlimited and $(x(m+\varepsilon\tau), y(m+\varepsilon\tau))$ belongs to $\mathcal{T}$ for all $\tau \in [-\tau_0, 0]$. As a function of τ, $(x(m+\varepsilon\tau), y(m+\varepsilon\tau))$ is a solution of system (20) with initial condition $(x(m), y(m))$. This problem is a regular perturbation of system (21) with initial condition (x_1, y_1). The solution of this problem is $(x_1(\tau), y_1)$, where $x_1(\tau) = \pi(\tau, x_1, y_1)$. By the Short Shadow Lemma we have

$$x(m+\varepsilon\tau) \simeq x_1(\tau) \quad y(m+\varepsilon\tau) \simeq y_1 \text{ for all limited } \tau \le 0\,.$$

By Robinson's Lemma, there exists $\tau_1 \simeq -\infty$, which can be chosen such that $-\tau_0 \le \tau_1$, satisfying $x(m+\varepsilon\tau_1) \simeq x_1(\tau_1)$. Thus $x_1(\tau_1)$ belongs to the ball of center $\xi(y_1)$ and radius a. According to hypothesis (H3′), $x_1(\tau_1+\tau) \simeq \xi(y_1)$ for all $\tau \simeq +\infty$. Take $\tau = -\tau_1$, then $x(m) \simeq x_1(0) = x_1(\tau_1 - \tau_1) \simeq \xi(y_1) \simeq \xi(y(m))$. This is a contradiction with $x(m) \not\simeq \xi(y(m))$. ☐

Lemma 10 . *Let hypothesis (H4) be satisfied. Let y^0 be standard in Y_0. Let $(x(t), y(t))$ be a solution of (1) with the property that $x\,(t) \simeq \xi\,(y\,(t))$ for $t_0 \leq t \leq t_1$ and $y(t_0) \simeq y^0$. Let $y_0(t)$ be the solution of the slow equation (11) with initial condition y^0 which is assumed to be defined on the standard interval $0 \leq t \leq T$. Then $y(t_0+s) \simeq y_0(s)$ for all $s \leq T$ such that $t_0+s \leq t_1$.*

Proof. We can write $x(t) = \xi(y(t)) + \alpha(t)$, where $\alpha(t) \simeq 0$ for $t_0 \leq t \leq t_1$. By the second equation in (19) this implies that

$$\frac{dy}{dt}(t) = g(\xi(y(t)) + \alpha(t), y(t)),$$

that is, $y(t)$ is a solution of the non-autonomous equation

$$\frac{dy}{dt} = g(\xi(y) + \alpha(t), y). \tag{23}$$

The solution under consideration can be written $y(t_0+s)$, its initial condition is then $y(t_0)$. The problem consisting of equation (23) together with the initial condition $y(t_0)$ is a regular perturbation (for $t_0 \leq t \leq t_1$) of the problem consisting of the slow equation (11) together with the initial condition y^0, whose solution $y_0(s)$ is assumed to be defined and limited on the interval $[0, T]$. By Lemma 8, $y(t_0+s) \simeq y_0(s)$ as long as $s \leq T$ and $t_0+s \leq t_1$. □

4.2 Proof of Theorem 3

Let $(x(t), y(t))$ be a solution of problem (8), then $(x(\varepsilon\tau), y(\varepsilon\tau))$ is a solution of the problem consisting of system (20) together with initial conditions $x(0) = \alpha$, $y(0) = \beta$. This problem is a regular perturbation of the problem consisting of system (21) together with initial conditions $x(0) = \alpha_0$, $y(0) = \beta_0$, which is nothing more than the boundary layer equation (12). According to the hypothesis (H5), the solution $x_0(\tau)$ of the boundary layer equation is defined for all $\tau \geq 0$, and

$$x_0(\tau) \simeq \xi(\beta_0) \quad \text{for } \tau \simeq +\infty. \tag{24}$$

By the Short Shadow Lemma $x(\varepsilon\tau)$ and $y(\varepsilon\tau)$ are defined and satisfy $x(\varepsilon\tau) \simeq x_0(\tau)$, $y(\varepsilon\tau) \simeq \beta_0$ for all limited τ. By Robinson's Lemma, there exists $L \simeq +\infty$ which can be chosen such that $\varepsilon L \simeq 0$, with the properties that

$$x(\varepsilon\tau) \simeq x_0(\tau), \;\; y(\varepsilon\tau) \simeq \beta_0, \quad \text{for } 0 \leq \tau \leq L. \tag{25}$$

According to (24), when $t_0 = \varepsilon L$, the solution has come infinitely close to the slow manifold $\mathcal{L}$. Let t_1 be the largest value (maybe $t_1 = +\infty$) such that $y(t)$ lies in Y for $t_0 \leq t \leq t_1$. By Lemma 9, the solution remains infinitely close to $\mathcal{L}$ for $t_0 \leq t \leq t_1$ as long as $y(t)$ is nearstandard in Y_0. By Lemma 10, $y(t)$ is infinitely close to the solution $y_0(t)$ of the reduced problem, as long as $t \leq T$

and $t \leq t_1$. If $t_1 < T$, then $y\left(t_1\right) \simeq y_0\left(t_1\right)$ and $y_0\left(t_1\right)$ is nearstandard in Y_0, thus $y(t) \in Y_0$ for some $t > t_1$, which contradicts the definition of t_1. Thus $t_1 \geq T$, and $y(t)$ is defined for $0 \leq t \leq T$ and satisfies

$$y(t) \simeq y_0(t) \quad \text{for } 0 \leq t \leq T. \tag{26}$$

Since $x(t) \simeq \xi(y(t))$ for $t_0 \leq t \leq T$, the approximation (26) of $y(t)$ implies that

$$x(t) \simeq x_0(t) = \xi(y_0(t)) \quad \text{for } t_0 \leq t \leq T. \tag{27}$$

Thus (25), (26) and (27) complete the proof of the theorem.

4.3 Proof of Theorem 4

According to hypothesis (H6), the solution $y_0(t)$ of the reduced problem is defined for all $t \geq 0$ and satisfies $y_0(t) \simeq y_\infty$ for $t \simeq +\infty$. By Theorem 3, the approximations

$$\begin{gathered} y(t) \simeq y_0(t) \quad \text{for } 0 \leq t \leq T\,, \\ x(t) \simeq x_0(t) = \xi(y_0(t)) \quad \text{for } \varepsilon L \leq t \leq T\,, \end{gathered}$$

hold for each limited $T > 0$. By Robinson's Lemma, they hold also for some $T \simeq +\infty$. Thus $x(T) \simeq x_\infty$ and $y(T) \simeq y_\infty$. Starting from $(x(T), y(T))$ and applying again Theorem 3, one obtains

$$x(T+s) \simeq x_\infty, \quad y(T+s) \simeq y_\infty \quad \text{for all limited } s \geq 0\,. \tag{28}$$

It remains to prove that $x(t)$ and $y(t)$ are defined for all $t \geq T$ and satisfy

$$x(t) \simeq x_\infty, \quad y(t) \simeq y_\infty \quad \text{for } t \geq T.$$

Assume that this is false. Then there must exist $s \geq T$ such that $y(s) \not\simeq y_\infty$, that is $\mu = \|y(s) - y_\infty\|$ is appreciable. We can chose s so that the ball $\mathcal{B}$ of center y_∞ and radius μ is included in the basin of attraction of y_∞. Let m be the smallest value $t \geq T$ such that $\|y(m) - y_\infty\| = \mu$. Then $y(t)$ is limited for all $T \leq t \leq m$. By Lemma 9, $x(t) \simeq \xi(y(t))$ for $T \leq t \leq m$. If $s_0 = m - T$ was limited then, by property (28), one would have $x(m) = x(T + s_0) \simeq x_\infty$, $y(m) = y(T + s_0) \simeq y_\infty$, which contradicts $y(m) \not\simeq y_\infty$. Thus s_0 is unlimited and $(x(m+s), y(m+s))$ belongs to the ball $\mathcal{B}$ for all $s \in [-s_0, 0]$. Let $y_1(s)$ be the solution of the slow equation (11) with initial condition $y_1(0) = y(m)$. By Lemma 10 one has

$$y(m+s) \simeq y_1(s) \quad \text{for all limited } s \leq 0\,.$$

By Robinson's Lemma, there exists $s_1 \simeq -\infty$, which one can choose such that $-s_0 \leq s_1$, satisfying $y(m + s_1) \simeq y_1(s_1)$. Thus $y_1(s_1)$ is limited. By the asymptotic stability of y_∞, $y_1(s_1 + s) \simeq y_\infty$ for all $s \simeq +\infty$. Take $s = -s_1$, then $y(m) = y_1(0) = y_1(s_1 - s_1) \simeq y_\infty$. This is a contradiction with $y(m) \not\simeq y_\infty$.

References

[1] V.I. Arnold (Ed.), *Dynamical Systems* V, Encyclopedia of Mathematical Sciences, Vol. 5, Springer-Verlag, 1994.

[2] E. Benoît (Ed.), *Dynamic Bifurcations*, Proceedings, Luminy 1990, Lecture Notes in Math. 1493, Springer Verlag, 1992.

[3] I.P. van den Berg, *Nonstandard Asymptotic Analysis*, Lecture Notes in Math. 1249, Springer Verlag, 1987.

[4] I.P. van den Berg, On solutions of polynomial growth of ordinary differential equations, *J. Diff. Equations*, 81:368-402, 1989.

[5] F. Diener, Sauts des solutions des équations $\varepsilon\ddot{x} = f(t, x, \dot{x})$, *SIAM. J. Math. Anal.*, 17:533-559, 1986.

[6] F. Diener, M. Diener (Eds.), *Nonstandard Analysis in Practice*, Universitext, Springer Verlag, 1995.

[7] F. Diener, G. Reeb, *Analyse non standard*, Hermann, 1989.

[8] M. Diener, C. Lobry, (Editeurs), *Analyse non standard et représentation du réel*, OPU, Alger, CNRS, Paris, 1985.

[9] M. Diener, G. Wallet (Editeurs), *Mathématiques finitaires et analyse non standard*, Publication mathématique de l'Université de Paris 7, Vol. 31-1 et 31-2, 1989.

[10] W. Eckhaus, *Asymptotic Analysis of Singular Perturbations*, Studies in Mathematics and its Applications, vol. 9, North Holland Publishing Company, 1979.

[11] F. Hoppensteadt, Singular perturbations on the infinite interval, *Trans. Amer. Math. Soc.* 123:521-535, 1966.

[12] F. Hoppensteadt, Stability in systems with parameters, *J. Math. Anal. Appls.*, 18:129-134, 1967.

[13] F. Hoppensteadt, Asymptotic stability in singular perturbation problems, *J. Diff. Equations*, 4:350-358, 1968.

[14] F. Hoppensteadt, On systems of ordinary differential equations with several parameters multiplying the derivatives, *J. Diff. Equations*, 5:106-116, 1969.

[15] F. Hoppensteadt, Properties of solutions of ordinary differential equations with small parameters, *Comm. Pure Appl. Math.*, 24:807-840, 1971.

[16] F. Hoppensteadt, Asymptotic stability in singular perturbation problems II: problems having matched asymptotic expansions solutions, *J. Diff. Equations*, 15:510-521, 1974.

[17] C. Lobry, A propos du sens des textes mathématiques, un exemple: la théorie des "bifurcations dynamiques", *Annales Institut Fourier*, 42 (1992), 327-351.

[18] C. Lobry, T. Sari, S. Touhami, Fast and Slow Feedback in Systems Theory, to appear in *J. Biological Systems.*

[19] R. Lutz, M. Goze, *Nonstandard Analysis: a practical guide with applications*, Lectures Notes in Math. 881, Springer-Verlag, 1982.

[20] J.L. Massera, Contribution to stability theory, *Ann. Math.*, 64:182-206, 1956.

[21] E. Nelson, Internal Set Theory: a new Approach to Nonstandard Analysis, *Bull. Amer. Math. Soc.*, 83:1165-1198, 1977.

[22] A. Robinson, *Nonstandard Analysis*, American Elsevier, New York, 1974.

[23] T. Sari, Stroboscopy and averaging, in *Colloque Trajectorien à la mémoire de G. Reeb et J.L. Callot, Strasbourg-Obernai, 12-16 juin 1995.* A. Fruchard et A. Troesch, Editeurs. Prépublication de l'IRMA, Strasbourg, 1995.

[24] T. Sari, General Topology, in *Nonstandard Analysis in Practice*, F. Diener and M. Diener (Eds.), Universitext, Springer-Verlag, 1995.

[25] T. Sari, *Nonstandard perturbation Theory of Differential Equations*, presented as an invited talk at the International Research Symposium on Nonstandard Analysis and its Applications, ICMS, Edinburgh, 11-17 August 1996 (`http://www.geometrie.univ-mulhouse.fr/sari/papers.html`).

[26] A. N. Tykhonov, Systems of differential equations containing small parameters multiplying the derivatives, *Mat. Sborn.* 31:575-586, 1952.

[27] S. Touhami, *Rétroactions rapides, lentes et retardées dans les systèmes dynamiques "entrée-sortie"*, Thèse, Université de Nice, 1996.

[28] A. B. Vasil'eva, V. F. Butuzov, *Asymptotic expansions of the solutions of singularly perturbed equations*, Nauka, Moscou, 1973 (in Russian).

[29] W. Wasow, *Asymptotic expansions for ordinary differential equation*, Robert E. Kriger Publishing Company, New York, 1976.

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