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On reaction-diffusion systems *

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Abstract

We consider reaction-diffusion systems which are strongly coupled. we prove that they generate analytic semigroups, find a characterization for the spectrum of the generator, and present some examples.

1 Introduction

The investigation of qualitative properties of solutions to systems of partial differential equations is a fascinating and challenging subject. Even though a systematic study of this subject is yet in its infancy, some results have been appeared in the literature since the early 1950's with classical and pioneer work of Fichera [7, 8] on elliptic systems. More recently, a great contribution to the study of quasilinear parabolic systems was given by Amann [2, 3, 4, 5] and references therein.

In this note we consider systems of reaction-diffusion equations of the form

$$u_t = D\Delta u + f(u), \tag{1}$$

where D is an $N \times N$ real matrix and $f : \mathbb{R}^N \to \mathbb{R}^N$ is a C^2 function.

Except for some publications on the subject, such as the searching for traveling waves solutions and some problems in ecology and epidemic theory, most of the authors assume that the diffusion matrix D is *diagonal* with positive entries, so that the coupling between the equations in (1) is present only on the nonlinearity of the reaction term f. However, cross-diffusion phenomena are not uncommon (see, e.g. [6] and references therein) and even certain mathematical models of vibrations of plates (see the examples in Section 4) can be treated as equations like (1) in which D is not even diagonalizable. It is the main subject of this note to consider the case in which the matrix D is not necessarily diagonal but has eigenvalues in the half plane $\{z \in \mathbf{C} : \operatorname{Re} z > 0\}$. We prove in this case that the semigroup generated by the linear part of (1) is an *analytic* semigroup. We shall consider only the case of Dirichlet boundary conditions, but the method should extend to some other cases without major modifications. For the case of the entire space, see [15].

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2 The linear semigroup

Let Ω be a bounded region in $\mathbb{R}^n$ with smooth boundary and let D be an $N \times N$ real matrix. In this section we are concerned with the system

$$u_t = D\Delta u \tag{1}$$

subjected to Dirichlet boundary condition $u = 0$ on $\partial\Omega$. Our main objective in this section is to give a condition on D in order that (1) generate an analytic semigroup in a Hilbert space, and derive some of its properties. To put this problem into the framework of [10], we need some notation.

Let $X = L^2(\Omega)^N$ be the Hilbert space of square integrable functions $u : \Omega \to \mathbb{R}^N$ with the inner product

$$\langle u, v\rangle = \int_\Omega (u_1(x)\bar{v}_1(x) + \dots + u_N(x)\bar{v}_N(x))\, dx\,.$$

Let $A : D(A) \subset X \to X$ be the linear operator given by $D(A) = (H^2(\Omega) \cap H_0^1(\Omega))^N$ and $Au = -D\Delta u$. Our main result is the following

Theorem 1 *Assume that all eigenvalues of D have positive real part. Then A is sectorial and therefore, $-A$ is the generator of an analytic semigroup $\{e^{-At} : t \geq 0\}$ in X.*

Proof. Let $\theta \in (0, \frac{\pi}{2})$ such that $|\arg\lambda| < \theta$ for any eigenvalue λ of D. We prove that the sector

$$S = \{z \in \mathbf{C} : \theta \leq |\arg z| \leq \pi, z \neq 0\}$$

is in the resolvent set of A and there exists a constant C such that for any $z \in S$,

$$\|(z - A)^{-1}\| \leq \frac{C}{|z|}\,.$$

Let $\{\lambda_j\}_{j=1}^\infty$ and $\{\phi_j\}_{j=1}^\infty$ be the eigenvalues and eigenfunctions of $-\Delta$ in Ω:

$$\begin{cases} \Delta\phi_j + \lambda_j\phi_j = 0 \text{ in } \Omega \\ \phi_j = 0 \text{ on } \partial\Omega. \end{cases}$$

We may assume that $\{\phi_j\}$ is an orthonormal basis of $L^2(\Omega)$.

For $z \in S$ and $f \in X$, let u be given by

$$u = \sum_{j=1}^\infty (z - \lambda_j D)^{-1} f_j \phi_j,$$

where $f_j := \int_\Omega f(x)\phi_j(x)\, dx$. Since $z \in S$ implies $\frac{z}{\lambda_j}$ is not an eigenvalue of D, the matrix $z - \lambda_j D$ is invertible and there exists a constant $C > 0$ such that $\|(z - \lambda_j D)^{-1}\| \leq \frac{C}{|z|}$, for all $j \geq 1$. It follows that the above series is convergent in X, so u is well defined and $\|u\| \leq \frac{C}{|z|}\|f\|$.

Also,

$$\begin{aligned} zu + D\Delta u &= \sum_{j=1}^{\infty} \left[z(z-\lambda_j D)^{-1} f_j - \lambda_j D f_j \right] \phi_j \\ &= \sum_{j=1}^{\infty} (z-\lambda_j D)^{-1}(z-\lambda_j D) f_j \phi_j \\ &= \sum_{j=1}^{\infty} f_j \phi_j = f\,, \end{aligned}$$

so, $u = (z-A)^{-1}f$. Therefore, z is in the resolvent set of A,

$$\|(z-A)^{-1}\| \le \frac{C}{|z|},$$

and the proof is complete. $\diamondsuit$

For $z = 0$ the equation $-Au = f$ is equivalent to the system

$$\begin{cases} \Delta u = D^{-1} f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

Since the solution of this latter is a compact map of f, it follows that A^{-1} is a compact operator. Therefore, the spectrum of A consists only of eigenvalues of finite multiplicity and μ is in the spectrum of A if and only if the equation

$$\begin{cases} D\Delta u + \mu u = 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has nontrivial solution. Writing $u = \sum_{j=1}^{\infty} u_j \phi_j$, this is equivalent to the requirement that the equation

$$\sum_{j=1}^{\infty} (\mu - \lambda_j D) u_j \phi_j = 0$$

has nontrivial solution which in turn is equivalent to $\det(\mu I - \lambda_j D) = 0$ for some $j \ge 1$. Therefore μ is in the spectrum of A if and only if there exist $j \ge 1$ and $1 \le k \le N$ such that $\mu = \lambda_j d_k$, where $d_1, ..., d_N$ are the eigenvalues of D. In short: $\sigma(A) = \cup_{j=1}^{\infty} \lambda_j \sigma(D)$, so the spectrum of A is a countable set located on the rays drawn from the origin to the set of the eigenvalues of D.

Corollary 2 $\{e^{-At} : t \ge 0\}$ *is a compact semigroup in* $\mathcal{L}(X)$ *and there exist constants* $C \ge 1$ *and* $\alpha > 0$ *such that*

$$\|e^{-At}\|_{\mathcal{L}(X)} \le Ce^{-\alpha t}\,,$$

for all $t \ge 0$.

Using Fourier series and the orthonormal basis $\{\phi_j\}_{j=1}^{\infty}$ of $L^2(\Omega)$, it is easy to compute the semigroup generated by $-A$:

$$e^{-At}f = \sum_{j=1}^{\infty} e^{-\lambda_j Dt} f_j \phi_j \,,$$

for all $f \in X$, where $f_j := \int_\Omega f(x)\phi_j(x)\,dx$.

Next we study the spaces of fractional powers $X^\alpha = D(A^\alpha)$ of A. As expected, we prove that each X^α is a product of similar spaces.

Lemma 3 *Let $\alpha \geq 0$. Then $X^\alpha = D((-\Delta)^\alpha)^N$. In particular, $X^{1/2} = H_0^1(\Omega)^N$.*

Proof. Let $\alpha > 0$ and $f \in X$. Then

$$\begin{aligned} A^{-\alpha} f &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} f\, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \sum_{j=1}^{\infty} e^{-\lambda_j Dt} f_j \phi_j \, dt \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{\infty} \left(\int_0^\infty t^{\alpha-1} e^{-\lambda_j Dt}\, dt \right) f_j \phi_j \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{\infty} \lambda_j^{-\alpha} \left(\int_0^\infty s^{\alpha-1} e^{-Ds}\, ds \right) f_j \phi_j \\ &= \sum_{j=1}^{\infty} \lambda_j^{-\alpha} D^{-\alpha} f_j \phi_j \,. \end{aligned}$$

Therefore, $g \in R(A^{-\alpha}) = D(A^\alpha)$ if and only if $\sum_{j=1}^{\infty} \lambda_j^\alpha \|D^\alpha g_j\|^2$ is convergent, so

$$\begin{aligned} D(A^\alpha) &= \{g \in L^2(\Omega)^N : \sum_{j=1}^{\infty} \lambda_j^{2\alpha} \|g_j\|^2 < \infty\} \\ &= D((-\Delta)^\alpha) \times D((-\Delta)^\alpha) \times ...\, times D((-A)^\alpha)\,, \end{aligned}$$

and $A^\alpha g = \sum_{j=1}^{\infty} \lambda_j^\alpha D^\alpha g_j \phi_j$.

We have $X^{1/2} = H_0^1(\Omega)^N$ since $D((-\Delta)^{1/2} = H_0^1(\Omega)$. The proof is complete.

3 Nonlinear problems

Given a vector field $f : \mathbb{R}^N \to \mathbb{R}^N$ satisfying certain growth and regularity assumptions, we can define a map $f^e : X^\alpha \to X$ for some $\alpha > 0$ in such a way that the resulting map f^e is locally Lipschitz continuous. In these cases, the study of well-posedness of the initial value problem for (1) can be put into the framework of abstract evolution equations. We first write (1) as the evolution equation

$$\dot{u} + Au = f^e(u) \tag{2}$$

and then we look for mild solutions of (2), defined as continuous solutions of the integral equation

$$u(t) = e^{-At}u(0) + \int_0^t e^{-A(t-s)} f^e(u(s))\, ds\,.$$

We refer the reader to Henry's book [10] for general results on existence, uniqueness and continuation.

Instead of giving general conditions on f such that (1) defines a local dynamical system on X^α, we consider in the next section some examples where this property can be easily verified.

4 Examples

Example 1. Let D and f satisfy the hypotheses stated in the introduction, and assume also that $\mathrm{Re}\, \sigma(D) > 0$. If n (the dimension of the space variable) is 1, 2 or 3, then, by Theorem 1.6.1 in [10], we have $X^\alpha \subset C^\nu(\Omega)$ for $\frac{3}{4} < \alpha < 1$ and therefore the map $f^e : X^\alpha \to X$ defined by $f^e(u)(x) = f(u(x))$, $x \in \Omega$ is well defined and it is C^1 with f and f' bounded on bounded sets. It follows from the previous results that the system

$$\begin{cases} u_t = D\Delta u + f(u), x \in \Omega, t > 0 \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{3}$$

defines a local dynamical system on X^α. Stability of periodic solutions of system (3) was considered by Leiva [12] for a diagonal matrix D and Neumann boundary conditions.

Example 2. Let $\beta > 0$ and consider the equation

$$u_{tt} - 2\beta\Delta u_t - f\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u + \Delta^2 u = 0, x \in \Omega, t > 0 \tag{4}$$

with boundary conditions

$$u = \Delta u = 0 \text{ for } x \in \partial\Omega.$$

Equation (4) arise in the mathematical study of structural damped nonlinear vibrations of a string or a beam and was considered in [16] and references therein.

As it is well known (see, e.g. [16]), the linear part of (4) generates an exponentially stable analytic semigroup in the space $Y = H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$. It is our purpose here to obtain the same result as a consequence of

Theorem 2.1. To this task, we first change variables $w = \Delta u$, $v = u_t$. In this new variables, the linear part of equation (4) becomes

$$\begin{cases} w_t = \Delta v \\ v_t = -\Delta w + 2\beta\Delta v, \end{cases} \tag{5}$$

or, $z_t = D\Delta z$, with Dirichlet boundary conditions, where $z = \begin{pmatrix} w \\ v \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 1 \\ -1 & 2\beta \end{pmatrix}$.

Since the eigenvalues of D are $d_{1,2} = \beta \pm \sqrt{\beta^2 - 1}$, the spectral condition on D in Theorem 2.1 is satisfied. Since $(u,v) \mapsto (w,v) : H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$ is an isomorphism, it follows that the operator $L(u,v) = (v, -\Delta^2 u + 2\beta\Delta v)$, $D(L) = \{u \in H^4(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\} \times H^2(\Omega) \cap H_0^1(\Omega)$ is also the generator of an exponentially stable analytic semigroup on $H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$.

Now we consider the initial value problem for (4). As usual, we introduce the variable $u_t = v$ and write (4) as the following system

$$\begin{cases} u_t = v \\ v_t = -\Delta^2 u + 2\beta\Delta v + f\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u \end{cases} \tag{6}$$

in the space Y. Letting $\hat{f} : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ be defined by

$$\hat{f}(u) = f\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u$$

and $f^e(u,v) = (0, \hat{f}(u))$, system (6) is in the form (2), where $A = -L$.

Lemma 4 *Assume that $f : [0,\infty) \to \mathbb{R}$ is C^1. Then f^e is locally Lipschitz continuous on Y.*

The proof is straightforward. If we assume that f also satisfies some kind of dissipation condition, then it follows from arguments contained, for example, in Hale [9] and Henry [10]) that (6) defines a dynamical system in Y which has a global attractor (see [16] for more details).

Example 3. Let $M : \mathbb{R} \to \mathbb{R}$ be a C^1 function, $\alpha > 0$, $m \neq 0$ be real constants and consider the system

$$\begin{cases} u_{tt} + \alpha\Delta^2 u - m\Delta\theta = M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u, x \in \Omega, t > 0 \\ \theta_t - \Delta\theta + m\Delta u_t = 0\,, \end{cases} \tag{7}$$

defined on a bounded region $\Omega \subset \mathbb{R}^n$ with boundary conditions

$$u = \Delta u = \theta = 0, x \in \partial\Omega\,.$$

System (7) comes from the thermoelasticity theory and is a model of deflection u of a plate submitted to local variation of its temperature θ. The linear version of (7) was considered in [11, 14, 13].

To put (7) as an evolution equation, we first write (7) as the first order system

$$\begin{cases} u_t = v \\ v_t = -\alpha\Delta^2 u + m\Delta\theta + M\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u, x \in \Omega, t > 0 \\ \theta_t = \Delta\theta - m\Delta v. \end{cases} \tag{8}$$

Let $Y = H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ and define $L : D(L) \subset Y \to Y$ and $f : Y \to Y$ setting

$$D(L) = \{u \in H^4(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\} \times H^2(\Omega) \cap H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega)\,,$$

$$L(u, v, \theta) = (-v, \alpha\Delta^2 u - m\Delta\theta, -\Delta\theta + m\Delta v)\,,$$

and

$$f(u, v, \theta) = (0, M\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u, 0)\,.$$

Letting $z = (u, v, \theta)$, we rewrite (8) as the system

$$\dot{z} + Lz = f(z).$$

The proof of the following lemma was pointed out in [13] and can be found in [14]. Here we give another proof which is an application of Theorem 1.

Lemma 5 *$-L$ is the generator of an analytic semigroup in Y with compact resolvent and there exist constants $C \geq 1$ and $\sigma > 0$ such that*

$$\|e^{-Lt}\|_{\mathcal{L}(Y)} \leq Ce^{-\sigma t},$$

for all $t \geq 0$.

Proof. With the change of variable $w = \Delta u$, the equation $\dot{z} + Lz = 0$ becomes

$$\begin{cases} w_t = \Delta v \\ v_t = -\alpha\Delta w + m\Delta\theta, x \in \Omega, t > 0 \\ \theta_t = \Delta\theta - m\Delta v, \end{cases} \tag{9}$$

or, equivalently, $z_t = D\Delta z$ with Dirichlet boundary conditions. Here, $z = (w, v, \theta)$ and D is the matrix

$$D = \begin{pmatrix} 0 & 1 & 0 \\ -\alpha & 0 & m \\ 0 & -m & 1 \end{pmatrix}.$$

Since the eigenvalues of D are the roots of the characteristic equation

$$z^3 - z^2 + (\alpha + m^2)z - \alpha = 0\,,$$

a simple application of the Routh-Hurwitz criterion shows that Re $\lambda > 0$ for any eigenvalue λ of D. Therefore Theorem 1 applies and we have the result. $\diamondsuit$

If M satisfy the same assumptions as f in the previous example, then the initial value problem for (8) is well posed and, once again using results in [9] and [10], we can prove that (8) has a global attractor in Y.

Example 4. The following example was considered by M. Alves in [1] where she proves the existence of a global attractor of the dynamical system generated by the system

$$\begin{cases} u_{tt} - cv_{tt} + \alpha\Delta^2 u - \Delta u_t - M(\int_\Omega |\nabla u|^2\,dx)\Delta u = 0 \\ -cu_{tt} + \gamma v_{tt} + \delta\Delta^2 v - \beta_0\Delta v - \Delta v_t = 0 \end{cases} \tag{10}$$

defined on a bounded smooth region $\Omega \subset \mathbb{R}^n$ together with the boundary conditions

$$u = \Delta u = v = \Delta v = 0\,.$$

Here, M satisfies the same hypotheses as in Example 3 and α, δ, γ, c and β_0 are positive constants such that $\gamma > c^2$.

In order to put this system as an evolution equation, we proceed in the usual way of writing it as a first order system. Let $u_t = w$ and $v_t = z$, so that (10) becomes

$$\begin{cases} u_t = w \\ w_t = \frac{1}{d}\left[-\alpha\Delta^2 u - \delta c\Delta^2 v + \gamma\Delta w + c\Delta z + c\beta_0\Delta v + \gamma M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u\right] \\ v_t = z \\ z_t = \frac{1}{d}\left[-\alpha c\Delta^2 u - \delta\Delta^2 v + c\Delta w + \Delta z + \beta_0\Delta v + cM\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u\right], \end{cases} \tag{11}$$

where $d = \gamma - c^2$.

Let $Y = H^2(\Omega) \cap H^1_0(\Omega) \times L^2(\Omega) \times H^2(\Omega) \cap H^1_0(\Omega) \times L^2(\Omega)$ and let $p = (u, w, v, z)$. Then (11) can be written as the evolution equation

$$\dot{p} = Lp + f(p)\,,$$

where

$$\begin{aligned} &L(u, w, v, z) \\ &\quad= (w, \frac{1}{d}(-\alpha\gamma\Delta^2 u - \delta c\Delta^2 v + \gamma\Delta w + c\Delta z), z, \frac{1}{d}(-\alpha c\Delta^2 u - \delta\Delta^2 v + c\Delta w + \Delta z) \end{aligned}$$

with domain $D(L) = \{(u, w, v, z) : u, v \in H^4(\Omega), w, z \in H^2(\Omega) \cap H^1_0(\Omega)$ and $u = v = \Delta u = \Delta v = 0$ on $\partial\Omega\}$ and $f : Y \to Y$ is given by

$$\begin{aligned} &f(u, w, v, z) \\ &\quad= \frac{1}{d}\left(0, c\beta_0\Delta v + \gamma M(\int_\Omega |\nabla u|^2\,dx)\Delta u, 0, \beta_0\Delta v + cM(\int_\Omega |\nabla u|^2\,dx)\Delta u\right)\,. \end{aligned}$$

Now we consider the change of variables $(u_1, u_2, v_1, v_2) = (\Delta u, w, \Delta v, z)$, which transform the equation $\dot{p} = Lp$ into the equation

$$q_t = D\Delta q$$

with Dirichlet boundary conditions, where $q = (u_1, u_2, v_1, v_2)$ and D is the matrix

$$D = \frac{1}{d}\begin{pmatrix} 0 & d & 0 & 0 \\ -\alpha\gamma & \gamma & -\delta c & c \\ 0 & 0 & 0 & d \\ -\alpha c & c & -\delta & 1 \end{pmatrix}$$

The characteristic equation of D is

$$(\gamma - c^2)z^4 - (\gamma + 1)z^3 + (\alpha\gamma + \delta + 1)z^2 - (\alpha + \delta)z + \alpha\delta = 0\,.$$

Now, Routh-Hurwitz criterion can be applied to show that Re $z > 0$ for any eigenvalue z of D and therefore, Theorem 1 can be applied to get the following result

Theorem 6 *Let $Y = \left(H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)\right)^2$ and L as above. Then, $-L$ is the generator of an analytic semigroup $\{e^{-Lt} : t \geq 0\}$ in Y and there exist constants $C \geq 1$ and $\sigma > 0$ such that $\|e^{-Lt}\|_{\mathcal{L}(Y)} \leq Ce^{-\sigma t}$ for all $t \geq 0$. Moreover, for each $t > 0$, e^{-Lt} is a compact operator.*

It is now a simple matter to verify that f satisfies the sufficient hypotheses for (11) to be a well posed problem in Y and we quote the reference [1] for the proof that (11) has a global attractor.

Remark All the previous results obtained in this note remain true if we replace $-\Delta$ in (1) by a positive selfadjoint linear operator with compact resolvent. This consideration should be useful to study abstract versions of the above examples.

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