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GLOBAL ATTRACTOR AND FINITE DIMENSIONALITY FOR A CLASS OF DISSIPATIVE EQUATIONS OF BBM'S TYPE

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Abstract

In this work we study the Cauchy problem for a class of nonlinear dissipative equations of Benjamin-Bona-Mahony's type. We discuss the existence of a global attractor and estimate its Hausdorff and fractal dimensions.

§1. Introduction

We consider a family of dispersive equations of Benjamin-Bona-Mahony's type under the effect of dissipation, and we study the existence of a global attractor and its dimension. Our model can be written in the abstract form

$$Mu_t + u_x + uu_x + \alpha Lu = f \tag{1.1}$$

where $-\infty < x < \infty$, $t \geq 0$ and $\alpha \geq 0$. The operators M and L can be differential operators or pseudo-differential operators, and the function f is an external excitation.

In the simplest case, when M and L are the differential operators $M = I - \frac{\partial^2}{\partial x^2}$, $L = -\frac{\partial^2}{\partial x^2}$, the equation (1.1) is the well-known Benjamin-Bona-Mahony model, which describes the unidirectional propagation of weakly nonlinear dispersive long waves where Burger's type dissipation is considered.

If $f \equiv 0$, the existence of global solutions and asymptotic behaviour in time have been studied by several authors. The asymptotic behavior of solutions to the generalized Korteweg-de-Vries-Burgers and Benjamin-Bona-Mahony-Burgers equations in one space dimension was studied by Amick, Bona and Schonbek in [3], by B. Wang and W. Yang in [13], and by Bona and Luo in [7]. These results were generalized by Zhang [14] to multiple spatial dimensions. In [4], [5], [6] the authors considered a family of equations of KdV and BBM's type described by pseudo-differential operators, and studied the asymptotic behaviour in one space dimension.

In [8], Ghidaglia showed that the behaviour of the periodic solution of the KdV equation is described by a global attractor that has finite Hausdorff and fractal dimensions. The author obtained similar results for the Schrödinger equation in [9].

The aim of this work is to investigate the existence of a global attractor and estimate its dimension, using the techniques of [8], [9] and [10]. More specifically,

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we consider the solutions of (1.1) that are periodic in the spatial variable, that is, solutions $u(x,t)$ such that

$$u(x+\beta,t)=u(x,t) \tag{1.2}$$

where β is a real number. In the case that the external excitation f is independent of time, and the orders of pseudo-differential operators M and L are μ and s with $s\geq\mu\geq 2$, we show that the behaviour, for large t, of the infinite dimensional dynamical system (1.1) is, in fact, described by an attractor of finite dimension.

In [11] B. Wang proved the existence of a weak attractor which is also strong, working directly with the BBM equation in H^2. In [12], the same author proved the existence of a global attractor for the generalized Benjamin-Bona-Mahony equation in H^k for every integer $k\geq 2$. He also proved that the attractor has finite Hausdorff and fractal dimensions, and constructed approximate inertial manifolds. Here, we consider a family of dispersive equations of BBM's type, and we present a proof that applies in an abstract context which includes the BBM equation. We also prove that the Hausdorff dimension is finite.

We shall use standard notation. By $L^q(\Omega)$ we shall denote the space of functions in Ω whose q^{th} power is integrable, with the norm $\|g\|^q_{L^q}=\int_\Omega |f(x)|^q dx$, $1\leq q<+\infty$. The norm in $L^2(\Omega)$ we will denote by $\|\cdot\|_{L^2}=\|\cdot\|$. By $L^\infty(\Omega)$ we denote the space of measurable essentially bounded functions in Ω with the norm

$$\|g\|_{L^\infty}=\operatorname*{ess\,sup}_{x\in\Omega}\,|g(x)|\,.$$

For each $\sigma\in\mathbb{R}$ we shall denote by $H^\sigma(\Omega)$ the usual Sobolev space of order σ. By $H^\sigma_p(\Omega)$, $\sigma\geq 0$, $\Omega=(0,1)$ we shall indicate the space of functions periodic in the sense of (1.2), with $\beta=1$. If $g\in H^\sigma_p(\Omega)$ then g has an expansion in Fourier series

$$g(x)=\sum_{k\in\mathbb{Z}} g_k\,\exp(2ki\pi x)\,. \tag{1.3}$$

The norm of g in $H^\sigma_p(\Omega)$ will be denoted by

$$\|g\|^2_\sigma=\sum_{k\in\mathbb{Z}}(1+|k|^2)^\sigma|g_k|^2, \tag{1.4}$$

which is equivalent to $H^\sigma(\Omega)$ norm, $\sigma\geq 0$, according to Temam ([10]). We shall denote by $\dot{L}^2(\Omega)$ and $\dot{H}^\sigma(\Omega)$ the space of functions $g\in L^2(\Omega)$ or $H^\sigma(\Omega)$ such that

$$\int_\Omega g(x)\,dx=0. \tag{1.5}$$

The space $\dot{H}^\sigma_p(\Omega)$, $\sigma\in\mathbb{R}_+$, is the space of functions $g\in L^2(\Omega)$ such that g satisfies (1.5) and

$$\sum_{k\in\mathbb{Z}}(1+|k|^2)^\sigma|g_k|^2<+\infty\,. \tag{1.6}$$

In $\dot{H}^1_p(\Omega)$ the Poincaré inequality holds, that is, if $g\in\dot{H}^1_p(\Omega)$ then

$$\|g\|\leq C(\Omega)\,\|g'\|\,. \tag{1.7}$$

The inequality (1.7) shows that $\dot{H}^1_p(\Omega)$ is a Hilbert space with scalar product of $H^1_0(\Omega)$, and $\|u\|_1 = \{(u,u)_1\}^{1/2}$ is a norm on this space equivalent to that induced by $H^1(\Omega)$.

The operators M and L of (1.1) are pseudo-differential operators of orders μ and s, respectively, with

$$\begin{aligned} M:\ \dot{H}^\mu_p(\Omega) &\to \dot{L}^2_p(\Omega), \qquad \mu \geq 1, \quad \mu \in \mathbb{R} \\ L:\ \dot{H}^s_p(\Omega) &\to \dot{L}^2_p(\Omega), \qquad s \geq 0, \quad s \in \mathbb{R} \end{aligned}$$

and

$$Mg(x) = \sum_{k\in\mathbb{Z}} m(k)\, g_k\, \exp(2k\pi i x) \tag{1.8}$$

$$Lg(x) = \sum_{k\in\mathbb{Z}} \ell(k)\, g_k\, \exp(2k\pi i x) \tag{1.9}$$

where m and ℓ are the principal symbols of the operators M and L respectively. We assume from now on that the symbols m and ℓ are even functions of k that satisfy the growth conditions:

(i) There exist constants c_1, $c_2 > 0$ such that

$$c_1(1+|k|)^\mu \leq m(k) \leq c_2(1+|k|)^\mu. \tag{1.10}$$

(ii) There exist constants c_3, $c_4 > 0$ such that

$$c_3|k|^s \leq \ell(k) \leq c_4|k|^s. \tag{1.11}$$

The domains of operators M and L are given by

$$D(M) = \left\{ g \in \dot{H}^\mu_p(\Omega),\ \sum_{k\in\mathbb{Z}} |m(k)|^2|g_k|^2 < \infty \right\}$$

$$D(L) = \left\{ g \in \dot{H}^s_p(\Omega),\ \sum_{k\in\mathbb{Z}} |\ell(k)|^2|g_k|^2 < \infty \right\}.$$

If X is a Banach space then we denote by $C(0,T;\ X)$ the space of continuous functions $u:\ [0,T] \to X$. Various positive constants will be denoted by C; they may vary from line to line.

This paper is organized as follows. In Section 2 we study the existence and uniqueness of global solutions of the Cauchy problem associated to equation (1.1). Then in Section 3 we provide a priori bounds for the nonlinear semigroups given by the evolution equation, and we use them to establish the existence of a global attractor. Finally, we show in Section 4 that this set has finite Hausdorff dimension.

§2. The Cauchy Problem

In this section we consider Problem (1.1) with initial data $u(x,0) = u_0(x)$. We prove that the Cauchy problem is globally well-posed in the Sobolev space $\dot{H}^r_p(\Omega)$ where $\Omega = (0,1)$ and $r = \max\{\mu, s\}$.

The lemma below is useful in proving the existence of a solution.

Lemma 1.2. *Let $M: \dot{H}^{\mu}_{p}(\Omega) \to \dot{L}^{2}_{p}(\Omega)$, $\mu \geq 1$ satisfy assumptions (1.10) above. Then*

a)

$$M^{-1} \text{ exists.} \tag{2.1}$$

b) $M^{-1}\left(\frac{dg}{dx}\right) \in \dot{H}^{\mu}_{p}(\Omega)$ whenever $g \in \dot{H}^{\mu}_{p}(\Omega)$ and there exists a constant $C > 0$ such that

$$\|M^{-1}\frac{dg}{dx}\|_{\mu} \leq C\,\|g\|_{\mu}. \tag{2.2}$$

The proof of this lemma follows directly from the definition and (1.10).

Theorem 2.1 (Local Existence). *Let $u_0 \in \dot{H}^{r}_{p}(\Omega)$ with $r = \max\{\mu, s\}$, $\mu \geq 2$, $s \geq 0$. Assume $f \in \dot{H}^{r}_{p}(\Omega)$, and suppose M and L satisfy the assumptions (1.10), (1.11). Then, for each $T > 0$, there exists a unique function $u \in C(0,T;\ \dot{H}^{\mu/2}_{p}(\Omega))$, with u and u_t in the class $C(0,T;\ \dot{H}^{r}_{p}(\Omega))$, that solves (1.1) in $\Omega \times [0,T]$ with $u(x,0) = u_0(x)$. The mapping that associates to $u_0 \in \dot{H}^{r}_{p}(\Omega)$ the solution of (1.1) is continuous from $\dot{H}^{r}_{p}(\Omega)$ to $C(0,T;\ \dot{H}^{r}_{p}(\Omega))$.*

Proof First, we consider the linear problem

$$\begin{aligned} w_t + \alpha M^{-1}Lw &= 0 \\ w(x,0) = u_0(x) &\in \dot{H}^{r}_{p}(\Omega) \end{aligned} \tag{2.3}$$

that has a unique solution w given by $w(x,t) = E(t)u_0(x)$, where $\{E(t)\}_{t\geq 0}$ is the strongly continuous semigroup of linear operators generated by $A = -\alpha M^{-1}L$, $\alpha > 0$. The solution w lies in the class $C(0,\infty;\ \dot{H}^{r}_{p}(\Omega)$ and w_t lies in $C(0,\infty;\ \dot{H}^{r}_{p}(\Omega)$.

Next, we consider the nonlinear problem (1.1). Using Lemma 1.2 and the observations concerning the solution of (2.3), we can write the integral equation associated with (1.1):

$$u(x,t) = E(t)u_0(x) - \int_0^t E(t-\sigma)M^{-1}\frac{\partial}{\partial x}\left(u + \frac{u^2}{2}\right)d\sigma + \int_0^t E(t-\sigma)M^{-1}f\,d\sigma. \tag{2.4}$$

Let $R > 0$, $T > 0$, and define the space of functions

$$y_R(T) = \left\{ \sup_{[0,T]} w \in C(0,T;\ \dot{H}^{r}_{p}(\Omega) : \sup_{[0,T]} \|w(\cdot,t) - E(t)u_0(\cdot)\|_r \leq R,\ w(x,0) = u_0(x) \right\}.$$

We define the map $P:\ y_R(T) \to C(0,T;\ \dot{H}^{r}_{p}(\Omega)$ by

$$Pw(x,t) = E(t)u_0(x) - \int_0^t E(t-\sigma)M^{-1}\frac{\partial}{\partial x}\left(u + \frac{u^2}{2}\right)d\sigma + \int_0^t E(t-\sigma)M^{-1}f\,d\sigma \tag{2.5}$$

for all $0 \leq t \leq T$. Using well-known techniques we can easily prove that P is a contraction as long as $T = T_0$ is chosen sufficiently small. Thus, P has a fixed point, which gives us a local solution of the integral equation (2.4). Next, since u satisfies equation (2.4), we can calculate u_t explicitly and obtain that u satisfies (1.1) with $u(x,0) = u_0(x)$, and that $u_t \in C(0, T_0;\ \dot{H}^r_p(\Omega))$.

Multiplying equation (1.1) by u, integrating in space, and using Poincaré and Hölder's inequalities and (1.10)–(1.11), we obtain the estimate

$$\|u(\cdot,t)\|^2_{\mu/2} + \frac{\alpha}{c_1}\int_0^t \|L^{1/2}u(\cdot,\sigma)\|^2\,d\sigma \le C\,\|f\|^2\,T + \|u_0\|^2_{\mu/2} \tag{2.6}$$

for all $0 \le t \le T$ and $\mu \ge 2$. Therefore, $u \in C(0,T;\ \dot{H}^{\mu/2}_p(\Omega)$, $\mu \ge 2$.

Multiplying equation (1.1) by Mu, integrating in space, using Poincaré and Hölder's inequalities and properties (1.10)–(1.11), we have

$$\|u(\cdot,t)\|^2_{\mu} + C_0\int_0^t \|L^{1/2}M^{1/2}u\|^2\,d\sigma \le C_1\,\|u_0\|^2_{\mu} + \|f\|^2 T + C_2\int_0^t \|u(\cdot,\sigma)\|^2_{\mu} d\sigma.$$

From Gronwall's inequality

$$\|u(\cdot,t)\|^2_{\mu} \le C\,\left(\|u_0\|_{\mu}, \|f\|, T\right)\, e^{C_2 T}. \tag{2.7}$$

Therefore, if $s \le \mu$ then $r = \mu$ and consequently $u \in C(0,T;\ \dot{H}^r_p(\Omega)$.

Now, suppose that $s > \mu \ge 2$. Multiplying equation (1.1) by u_t, integrating in space and using the results above, we obtain

$$\frac{1}{2}\,\|M^{1/2}u_t\|^2 + \frac{\alpha}{2}\,\frac{d}{dt}\,\|L^{1/2}u\|^2 \le C\,(\|f\|, T) + \|L^{1/2}u\|^2.$$

Gronwall's inequality and (2.6) imply that $u \in C(0,T;\ \dot{H}^{\frac{s}{2}}_p(\Omega)$ for $s > \mu \ge 2$ and all $0 \le t \le T$.

Finally, multiplying equation (1.1) by Lu_t and using the same sequence of ideas, we obtain

$$\frac{1}{2}\,\|L^{1/2}M^{1/2}u_t\|^2 + \frac{\alpha}{2}\,\frac{d}{dt}\,\|Lu\|^2 \le C\,(\|f\|, + \|u_0\|_r, T) + C_1\|Lu\|^2.$$

From Gronwall's inequality and (2.6) we get

$$u \in C(0,T;\ \dot{H}^s_p(\Omega) \;\text{ for }\; s > \mu \ge 2 \;\text{ and all }\; 0 \le t \le T.$$

Since we know that $u \in C(0,T;\ \dot{H}^r_p(\Omega)$, we can use the integral equation (2.4) to find u_t, and it follows from these that $u_t \in C(0,T;\ \dot{H}^r_p(\Omega)$. Uniqueness is a direct consequence of Gronwall's inequality.

§3. Existence of a global attractor

In this section we study the existence of a global attractor. The first step is to prove the existence of an absorbing set in $\dot{H}^s_p(\Omega)$, $s \ge \mu \ge 2$.

We consider the Cauchy problem

$$\begin{gathered} Mu_t + u_x + uu_x + \alpha Lu = f \\ u(x,0) = u_0(x) \\ u(x+1,\, t) = u(x,t). \end{gathered} \tag{3.1}$$

If the function f is time independent, the system (3.1) is autonomous, and for each $t \in \mathbb{R}^+$ we define the mapping

$$\begin{aligned} E(t) : \dot{H}^r_p(\Omega) \;&\rightarrow\; \dot{H}^r_p(\Omega) \\ u_0 \;&\mapsto\; E(t)u_0 = u(x,t). \end{aligned} \tag{3.2}$$

The family $\{E(t)\}_{t\in\mathbb{R}^+}$ forms a semigroup.

Proposition 3.1. *If $E(t)$ is the mapping defined in (3.2), then there exists a constant $C = C(\|u_0\|_r, \|f\|_r, T)$ such that*

$$\sup_{0\le t\le T} \|E(t)\, u_0\|_r \le C\,(\|u_0\|_r, \|f\|_r, T)$$

with $u_0 \in \dot{H}^r_p(\Omega)$, $f \in \dot{H}^r_p(\Omega)$, $r = \max\{\mu, s\}$, $\mu \ge 2$, $s \ge 2$.

The proof of this proposition follows directly from Theorem 2.1.

The next result is related to the existence of bounded absorbing set for semigroup $\{E(t)\}_{t\ge0}$ in $\dot{H}^s_p(\Omega)$, $s \ge \mu \ge 2$.

Proposition 3.2. *Let $f \in \dot{H}^s_p(\Omega)$, $s \ge \mu \ge 2$. There exists a constant $\rho_0 = \rho_0(\|f\|_0)$ such that for every $R > 0$ there exists $T > 0$, $T = T(R)$ such that*

$$\|E(t)u_0\|_s \le \rho_0 \;\text{ for all }\; u_0 \in \dot{H}^s_{per}(\Omega) \;\text{ with }\; \|u_0\|_s \le R$$

and $t \ge T(R)$, where $E(t)u_0(x) = u(x,t)$ is the solution of Cauchy problem (3.1).

Proof Multiplying the equation in (3.1) by u and integrating in (Ω), we have

$$\frac{d}{dt}\,\|M^{1/2}u\|^2 + \alpha\|L^{1/2}u\|^2 \le C\,\|f\|^2. \tag{3.3}$$

Poincaré's inequality and the fact that $s \ge \mu$ imply that $\|M^{1/2}u\| \le C\|L^{1/2}u\|$. Therefore, there exists a constant $\beta > 0$ such that

$$\frac{d}{dt}\,\|M^{1/2}u\|^2 + \beta\,\|M^{1/2}u\|^2 \le C\,\|f\|^2. \tag{3.4}$$

From (3.4) and properties (1.10)–(1.11) we obtain

$$\|u(t)\|^2_{\mu/2} \le C_0\,\|u_0\|^2_{\mu/2}e^{-\beta t} + C\,\|f\|^2\left(1 - e^{-\beta t}\right) \le C_0R^2e^{-\beta t} + C\|f\|^2(1 - e^{-\beta t}). \tag{3.5}$$

This shows that $E(t)u_0$ is uniformly bounded in $\dot{H}^{\mu/2}_p(\Omega)$ and

$$\|u(t)\|^2_{\mu/2} \le C\|f\|^2 = \rho_1^2 \tag{3.6}$$

for all $t \geq T_0(R) = \frac{1}{\beta} \ln \frac{C_0 R^2}{C\|f\|^2}$.

Multiplying the equation (3.1) by Mu and integrating in Ω, we have, for all $t \geq T_0(R)$,

$$\frac{d}{dt}\|Mu\|^2 + 2\alpha\|L^{1/2}Mu^{1/2}u\|^2 = 2\int_\Omega f\,Mu\,dx - 2\int_\Omega uu_x Mu\,dx. \tag{3.7}$$

Using Hölder's inequality and the embedding $\dot{H}_p^{\mu/2}(\Omega) \hookrightarrow L^\infty(\Omega)$, $\mu \geq 2$, we deduce from (3.7) that

$$\frac{d}{dt}\|Mu\|^2 + 2\alpha\|L^{1/2}Mu^{1/2}u\|^2 \leq 2\|f\|\,\|Mu\| + 2\|u\|_{L^\infty}\|u_x\|\,\|Mu\|. \tag{3.8}$$

Therefore, for $\mu \geq 2$ we have

$$\begin{aligned}\frac{d}{dt}\|Mu\|^2 + 2\alpha\|L^{1/2}Mu^{1/2}u\|^2 &\leq 2\|f\|\,\|Mu\| + C\|u\|^2_{\mu/2}\|Mu\| \\ &\leq \left(2\|f\| + C\|u\|^2_{\mu/2}\right)\|Mu\|. \end{aligned}\tag{3.9}$$

Poincaré's inequality implies that

$$\|Mu\| \leq C\|L^{1/2}M^{1/2}u\| \quad \text{for} \quad u \in \dot{H}_p^s(\Omega), \qquad s \geq \mu \geq 2. \tag{3.10}$$

From (3.10) we have for (3.9)

$$\frac{d}{dt}\|Mu\|^2 + 2\alpha\|L^{1/2}M^{1/2}u\|^2 \leq C\left(\|f\| + C\|u\|^2_{\mu/2}\right)\|L^{1/2}M^{1/2}u\|.$$

Using the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we have

$$\frac{d}{dt}\|Mu\|^2 + \alpha\|L^{1/2}M^{1/2}u\|^2 \leq C\left(\|f\| + C\|u\|^2_{\mu/2}\right)^2.$$

From (3.10) and (3.6) it follows that

$$\frac{d}{dt}\|Mu\|^2 + \beta_1\|Mu\|^2 \leq C\left(\|f\| + \rho_1^2\right)^2, \qquad \beta_1 > 0, \tag{3.11}$$

and for all $t \geq T_0$ we obtain

$$\|Mu\|^2 \leq \|Mu(T_0)\|^2 e^{-\beta_1(t-T_0)} + C\left(\|f\| + \rho_1^2\right)^2\left(1 - e^{-\beta_1(t-T_0)}\right). \tag{3.12}$$

We choose $T_1 = T_1(R) \geq T_0(R)$ such that

$$\|Mu(T_0)\|^2 e^{-\beta_1(t-T_0)} \leq C(\|f\| + \rho_1^2)^2 \tag{3.13}$$

holds for every u_0 satisfying $\|u_0\|_s \leq R$, $s \geq \mu \geq 2$. This is possible since we know from (2.7) that $\|u(T_0)\|_\mu$ is bounded by a quantity that only depends on R and the data of the problem. Then, according to (3.12) and (3.13),

$$\|Mu(t)\|^2 \leq C\left(\|f\| + \rho_1^2\right)^2 = \rho_2^2 \quad \text{for all} \quad t \geq T_1(R). \tag{3.14}$$

If $s = \mu \geq 2$ the proof is concluded.

We now consider the case $s > \mu \geq 2$. Taking into account Lemma 1.2, we can write the equation (3.1) as

$$u_t + M^{-1}u_x + M^{-1}(uu_x) + \alpha M^{-1}Lu = M^{-1}f. \tag{3.15}$$

Multiplying (3.15) by Lu and integrating in space, we have

$$\frac{d}{dt}\|L^{1/2}u\|^2 + 2\alpha\|M^{1/2}Lu\|^2 = 2\left[\left(M^{-1}f, Lu\right) - \left(M^{-1}\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right), Lu\right)\right]. \tag{3.16}$$

Integrating by parts and using the Cauchy-Schwarz inequality yields

$$\frac{d}{dt}\|L^{1/2}u\|^2 + 2\alpha\|M^{-\frac{1}{2}}Lu\|^2 \leq 2\left[\|M^{-\frac{1}{2}}f\|\,\|M^{-\frac{1}{2}}Lu\| + \|M^{-\frac{1}{2}}\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\|\,\|M^{-\frac{1}{2}}Lu\|\right].$$

Using the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we have

$$\frac{d}{dt}\|L^{1/2}u\|^2 + \alpha\|M^{-\frac{1}{2}}Lu\|^2 \leq C\left(\|M^{-\frac{1}{2}}f\|^2 + \|M^{-\frac{1}{2}}\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\|^2\right). \tag{3.17}$$

From (1.10) we obtain

$$\|M^{-\frac{1}{2}}f\| \leq C\|f\|. \tag{3.18}$$

For $\mu \geq 2$ and the embedding $\dot{H}^{\mu}_{p}(\Omega) \hookrightarrow L^{\infty}(\Omega)$, it follows that

$$\|M^{-\frac{1}{2}}\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\|^2 \leq \|u^2\|^2 \leq C\|u\|^2_{L^\infty}\|u\|^2 \leq C\|u\|^4_{\mu}. \tag{3.19}$$

From (3.19), (3.18), and (3.14), we have for (3.17)

$$\frac{d}{dt}\|L^{1/2}u\|^2 + \alpha\|M^{-\frac{1}{2}}Lu\|^2 \leq C\left(\|f\|^2 + \rho_2^4\right) \text{ for all } t \geq T_1. \tag{3.20}$$

On the other hand, Poincaré's inequality implies

$$\|M^{-\frac{1}{2}}Lu\|^2 \geq C\|L^{\frac{1}{2}}u\| \text{ for } s > \mu \geq 2. \tag{3.21}$$

Therefore, from (3.21) we have for (3.20)

$$\frac{d}{dt}\|L^{1/2}u\|^2 + \beta_2\|L^{1/2}u\|^2 \leq C\left(\|f\|^2 + \rho_2^4\right) \tag{3.22}$$

for all $t \geq T_1$. Integrating (3.22) in time for $t \geq T_1$ we obtain

$$|L^{1/2}u(t)\|^2 \leq \|L^{\frac{1}{2}}u(T_1)\|^2 e^{-\beta_2(t-T_1)} + C\left(\|f\|^2 + \rho_2^4\right)\left(1 - e^{-\beta_2(t-T_1)}\right). \tag{3.23}$$

As in (3.13), we choose $T_2 = T_2(R) \geq T_1(R)$ such that

$$\|L^{1/2}u(T_1)\|^2 e^{-\beta_2(t-T_1)} \leq C\left(\|f\|^2 + \rho_2^4\right). \tag{3.24}$$

Therefore, from (3.24) we have

$$\|L^{1/2}u(t)\|^2 \leq C\left(\|f\|^2+\rho_2^4\right)=\rho_3^2 \text{ for all } t\geq T_2. \tag{3.25}$$

Next, we consider the equation

$$Lu_t+LM^{-1}u_x+LM^{-1}(uu_x)+\alpha LM^{-1}Lu=LM^{-1}f. \tag{3.26}$$

Multiplying (3.26) by Lu and integrating in space we obtain, after integration by parts and the use of Hölder's inequality,

$$\frac{d}{dt}\|Lu\|^2+\alpha\|M^{-\frac12}L^{\frac32}u\|^2\leq C\left(\|M^{-\frac12}L^{1/2}f\|^2+\|L^{1/2}M^{-\frac12}\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\|^2\right). \tag{3.27}$$

From (1.10) and (1.11) we have

$$\|M^{-1/2}L^{1/2}f\|\leq C\|f\|_{\frac{s}{2}} \tag{3.28}$$

and

$$\|M^{-1/2}L^{1/2}\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\|\leq C\|u^2\|_{s/2}\leq C\|u\|_{s/2}^2 \tag{3.29}$$

because $\dot{H}_p^s(\Omega)$ is an algebra for $s\geq 1$.

Using (3.28), (3.29), and Poincaré's inequality, we obtain for (3.27)

$$\frac{d}{dt}\|Lu\|^2+\beta_3\|Lu\|^2\leq C\left(\|f\|_{s/2}^2+\|u\|_{s/2}^2\right),\qquad \forall t\geq T_2. \tag{3.30}$$

From (3.25) we have

$$\frac{d}{dt}\|Lu\|^2+\beta_3\|Lu\|^2\leq C\left(\|f\|_{\frac{s}{2}}^2+\rho_3^2\right),\qquad \forall t\geq T_2$$

and

$$\|Lu\|^2\leq\|Lu(T_2)\|^2e^{-\beta_3(t-T_2)}+C\left(\|f\|_{s/2}^2+\rho_3^2\right)\left(1-e^{-\beta_3(t-T_2)}\right) \tag{3.31}$$

for all $t\geq T_2$. Choosing $T=T(R)\geq T_2(R)$, we have

$$\|Lu\|^2\leq C\left(\|f\|_{s/2}^2+\rho_3^2\right)=\rho_4^2 \text{ for all } t\geq T.$$

On the other hand, we know that $\|u\|\leq\rho_1$; therefore,

$$\|u\|^2+\|Lu\|^2\leq\rho_1^2+\rho_4^2.$$

Using (1.11) we deduce that

$$\|u\|_s^2\leq C\left(\rho_1^2+\rho_4^2\right)=\rho_0^2 \text{ for all } t\geq T(R).$$

This completes the proof of Proposition 3.2.

Proposition 3.2 shows that $E(t)\,u_0$ is uniformly bounded in $\dot H^s_p(\Omega)$, for $s \ge \mu \ge 2$ and every $t \ge T(R)$, when $\|u_0\|_s \le R$. In other words, every solution with initial data u_0 in the ball $\{\|u_0\|_s \le R\}$ is absorbed at time $t \ge T(R)$, by the ball

$$B_0 = \left\{ v \in \dot H^s_p(\Omega), \qquad \|v\|_s \le \rho_0 \right\}.$$

It is natural to consider then the w-limit set of B_0, which is defined as

$$w(B_0) = \bigcap_{\ell \ge 0} \overline{\cup_{t\ge \ell} E(t) B_0},$$

where the closure is taken in $\dot H^s_p(\Omega)$.

In order to obtain the existence of a global attractor for the equation (1.1), we next prove that the flow $E(t)$ is uniformly compact, for t large.

Proposition 3.3. *Let $f \in \dot H^s_p(\Omega)$, $s \ge \mu \ge 2$. For every bounded set B of $\dot H^s_p(\Omega)$ there exists $T > 0$, $T = T(B)$, such that $\cup_{t\ge T} E(t) B$ is relatively compact in $\dot H^s_p(\Omega)$.*

Proof The idea is to prove that $E(t) = E_1(t) + E_2(t)$, where the operator $E_1(\cdot)$ is uniformly compact for t large, and the norm of $E_2(\cdot)$ as a bounded operator goes to zero as $t \to \infty$.

Decompose the solution u of (1.1) as $u = v + w$, where $v(x,t) = E_2(t)v(x,0)$ is the solution of the linear problem

$$\begin{gathered} Mv_t + v_x + \alpha L v = 0 \\ v(x,0) = u_0(x), \end{gathered} \tag{3.32}$$

and $w = w(x,t)$ is the solution to

$$\begin{gathered} Mw_t + w_x + \alpha L w = f - uu_x \\ w(x,0) = 0. \end{gathered} \tag{3.33}$$

In order to prove that $E_2(t)$ has decaying norm we consider the equation

$$Lv_t + LM^{-1}v_x + \alpha LM^{-1}Lv = 0. \tag{3.34}$$

Multiplying by Lv, integrating in Ω, and using the properties of operators M and L, we obtain

$$\|E_2(t)\|_{\mathcal{L}(\dot H^s_p(\Omega), \dot H^s_p(\Omega))} \le C_0\, e^{-Ct}, \qquad \forall t \ge 0. \tag{3.35}$$

Now consider the equation

$$LMw_t + Lw_x + \alpha LLw = Lf - L(uu_x).$$

Multiplying by Lw and using the properties of operators M and L and the fact that $\|u(t)\|_s \le \rho_0$, we have

$$\frac{d}{dt}\,\|LM^{1/2}w\|^2 + \beta_5 \|LM^{1/2}w\|^2 \le C\left(\|f\|_s, \rho_0\right), \qquad \beta_5 > 0. \tag{3.36}$$

From (3.36) we can conclude that w is uniformly bounded in $\dot H^{\frac{\mu}{2}+s}_p(\Omega)$. Using the compact embedding from $\dot H^{\frac{\mu}{2}+s}_p(\Omega)$ into $\dot H^s_p(\Omega)$ it follows that $\cup_{t\ge T} E_1(t)\, B$ is relatively compact in $\dot H^s_p(\Omega)$. The proposition follows as in the proof of Theorem 1.1, Chapter 1 of Temam [10].

Theorem 3.2. *Let f, $u_0 \in \dot{H}^s_p(\Omega)$, $s \geq \mu \geq 2$. Then the semigroup $\{E(t)\}_{t\geq 0}$ has a global attractor $\mathcal{A} = w(B)$ in $\dot{H}^s_p(\Omega)$. The set $\mathcal{A}$ is compact in $\dot{H}^s_p(\Omega)$ and has the properties:*

i) The set $\mathcal{A}$ is invariant under $E(t)$, that is, $E(t)\,\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$.

ii) For every bounded set B in $\dot{H}^s_p(\Omega)$, $d(E(t)B, \mathcal{A}) \to 0$ as $t \to +\infty$.

The proof this theorem is a consequence of Temam [10], Theorem 1.1, Chapter I.

§4. Dimension of the global attractor

Our aim in this section is to study the finite dimensionality of the global attractor. In the first part we shall prove the differentiability property of $E(t)$ and in the second part we will establish the finite dimension of the attractor.

We consider the following non-autonomous evolution equation, which corresponds to a linearized version of the equation (1.1):

$$\begin{gathered} Mv_t + v_x + (uv)_x + \alpha L v = 0 \\ v(x,0) = v_0(x) \\ v(x+1,t) = v(x,t) \end{gathered} \tag{4.1}$$

where $u(t) = E(t)u_0$, $u_0 \in \dot{H}^s_p(\Omega)$ is a trajectory solution of (1.1), and $v_0 \in \dot{H}^s_p(\Omega)$. It is not difficult to prove that, since $u \in C^1([0,\infty);\ \dot{H}^s_p(\Omega))$, the problem (4.1) has a unique solution $v \in C^1([0,\infty);\ \dot{H}^s_p(\Omega))$.

Next we show, with the aid of the linearized problem (4.1), that the linear mapping $(DE(t)u_0)\,v_0 \equiv v(t)$ is the uniform differential of $E(t)$.

Theorem 4.1. *For every $0 < R$, $T < \infty$, there exists a positive constant $C = C(R,T)$ such that for all u_0, $v_0 \in \dot{H}^s_p(\Omega)$, $s \geq \mu \geq 2$, that satisfy $\|u_0\|_s \leq R$, $\|u_0 + v_0\|_s \leq R$ and $0 \leq t \leq T$, we have*

$$\|E(t)(u_0 + v_0) - E(t)u_0 - (DE(t)u_0)\,v_0\|_s \leq C\|v_0\|_s^2.$$

Proof Let u_0, $v_0 \in \dot{H}^s_p(\Omega)$, $s \geq \mu \geq 2$, with $\|u_0\|_s \leq R$, $\|u_0+v_0\|_s \leq R$. We consider the solutions $u_1(t) = E(t)u_0$, $u_2(t) = E(t)(u_0 + v_0)$ and $v(t) = (DE(t)u_0)\,v_0$. Then $w = u_2 - u_1 - v$ satisfies the problem

$$\begin{gathered} Mw_t + w_x + u_2 u_{1x} - (u_1 v)_x + \alpha L w = 0 \\ w(0) = 0. \end{gathered} \tag{4.2}$$

Since u_1, u_2, $v \in C^1([0,\infty);\ \dot{H}^s_p(\Omega))$, we may use the sequence of ideas in the proof of Theorem 2.1 to obtain

$$\|w\|_s^2 \leq C(R,T)\|v_0\|^2$$

for the solution w of the problem (4.2). Therefore, $E(t)$ is uniformly differentiable in the bounded sets of $\dot{H}^s_p(\Omega)$. ♢

Now we study how the operators $D(E(t))\,u_0$ transform the m-dimensional volumes in $\dot H^s_p(\Omega)$, $s \ge \mu \ge 2$ where $u_0 \in \mathcal{A}$. Let $v_0^1, v_0^2, \cdots, v_0^m$ in $\dot H^s_p(\Omega)$. We study the evolution of the quantities

$$\|v^1(t) \wedge \cdots \wedge v^m(t)\|_s^2 = \det_{1\le i,j\le m} (v^i(t), v^j(t))_s \tag{4.3}$$

where $v^i(t) = (DE(t)u_0)\, v_0^i$.

The expression (4.3) is the Gram determinant, and it represents the square of $m!$-times the volume of the m-dimensional polyhedron defined by the vectors $v^1(t), \cdots, v^m(t)$. The aim is to show that for sufficiently large m this determinant decays exponentially as $t \to +\infty$. More precisely, we consider an invariant set X which is bounded in $\dot H^s_p(\Omega)$, $s \ge \mu \ge 2$. We have:

Theorem 4.2. *Let $X \subset \dot H^s_p(\Omega)$ be an invariant bounded set. Assume $s \ge \mu \ge 2$. Then there exist positive constants b_0, b_1, γ such that for every $u_0 \in X$, $t \ge 0$ and integer $m \ge 1$, the functions $v^i(t) = (DE(t)u_0)\, v_0^i$ satisfy*

$$\|v^1(t) \wedge \cdots \wedge v^m(t)\|_s \le \|v_0^1 \wedge \cdots v_0^m\|_s\, b_1^{-m} \exp\left(b_0 m^{1-2\mu} - \gamma m\right) t$$

for all $v_0^i \in \dot H^s_p(\Omega)$.

Proof We consider $w^i(t) = v^i(t)\, e^{\gamma t}$, where $\gamma > 0$ is to be chosen. For simplicity, we omit the index i in this part of the proof.

Clearly, $w(t)$ is the unique solution of

$$\begin{gathered} Mw_t + w_x + (uw)_x + \alpha Lw - \gamma Mw = 0 \\ w(0) = v_0. \end{gathered} \tag{4.4}$$

Since M is invertible, we have that

$$Lw_t + LM^{-1}w_x + LM^{-1}(uw)_x + \alpha LM^{-1}Lw - \gamma Lw = 0. \tag{4.5}$$

Multiplying (4.5) by Lw and integrating in space gives

$$\frac{d}{dt}\,\|Lw\|_0^2 + 2\left(LM^{-1}(uw)_x, Lw\right) + 2\alpha\|M^{-1/2}L^{3/2}w\|^2 - 2\gamma\|Lw\|^2 = 0\,. \tag{4.6}$$

Since $\|Lw\| \le C\|M^{-1/2}L^{3/2}w\|$, we can choose $\gamma > 0$ such that

$$\gamma\,\|Lw\|^2 \le 2\alpha\,\|M^{-1/2}L^{3/2}w\|^2.$$

Hence,

$$\frac{d}{dt}\,\|Lw\|^2 \le -2\,\left(LM^{-1}(uw)_x, Lw\right).$$

We now consider the following quadratic forms on $\dot H^s_p(\Omega)$:

$$g(\xi) = \|L\xi\|^2,$$

and

$$z(t,\xi) = -2\,\langle LM^{-1}\,(u(t)\,\xi)_x,\ L\xi\rangle \tag{4.7}$$

for any t. Clearly, by Poincaré's inequality there exist nonnegative numbers b_3 and b_4 such that

$$b_3\,\|\xi\|_s^2 \le g(\xi) \le b_4\,\|\xi\|_s^2,$$

for all $\xi \in \dot{H}_p^s(\Omega)$. Moreover, the function $t \to g\,(e^{\gamma t}(DE(t)\,u_0)\,v_0) = g(w(t))$ is differentiable and its derivative satisfies

$$\frac{d}{dt}\,g(w(t)) \le z(t,w(t)).$$

On the other hand, since the order of M is $\mu \ge 2$, we have that $M^{-\frac{1}{2}}\,\frac{d}{dx}$ is a bounded operator and therefore, by the Schwarz inequality, (4.7) implies that

$$|z(t,\xi)| \le C\|\xi\|_s\|M^{-1/2}\xi\|_s = C\|\xi\|_s\,\left(M^{-1}\xi,\xi\right)_s^{1/2}$$

where we have used that $u(t)$ is bounded in $\dot{H}_p^s(\Omega)$ and also in $L^\infty(\Omega)$. We note that M^{-1} is a continuous linear operator from $\dot{H}_p^s(\Omega)$ to $\dot{H}_p^{s+\mu}(\Omega)$ and, therefore, it is a compact operator on $\dot{H}_p^s(\Omega)$.

The hypotheses of Theorem A in the appendix of the paper [8] by Ghidaglia are then fulfilled, where we have taken $\alpha = b_3$, $\beta = b_4$, $\sigma = \frac{1}{2}$, $q = g$, $r = z$ and $K = M^{-1}$. Here we are using an extension of this theorem to the case where $\frac{dg}{dt} \le r$ (instead of $\frac{dg}{dt} = r$), which follows immediately from the arguments given in [8] (pg. 387).

Thus,

$$\det\,(w^i(t),w^j(t))_s \le \left(\frac{b_4}{b_3}\right)^m \exp\left\{\frac{Ct}{b_3}\sum_{\ell=1}^m K_\ell\right\}\det\,(v_0^i,v_0^j)_s,$$

where $\{K_\ell\}_{\ell=1}^\infty$ are the eigenvalues of the operator M^{-1}, namely $K_\ell = (1+C\ell^{2\mu})^{-1}$. Since $\sum_{\ell=1}^m K_\ell \le Cm^{1-2\mu}$, we conclude that

$$\det_{1\le ij\le m}\left(w^i(t),w^j(t)\right)_s \le Ce^{Ctm^{1-2\mu}}\det_{1\le i,j\le m}(v_0^i,v_0^j)_s.$$

The theorem follows from the fact that $w^i(t) = e^{\gamma t}v^i(t)$.

Theorem 4.3. *The global attractor $\mathcal{A}$ has finite fractal and Hausdorff dimensions in $\dot{H}_p^s(\Omega)$, $s \ge \mu \ge 2$.*

Proof This result is a consequence of an abstract result according to [8]. The main idea is to apply Theorem 4.2 with $X = \mathcal{A}$ and to choose m such that $b_0m^{1-2\mu} - \gamma m < 0$, that is, $m > \left(\frac{b_0}{\gamma}\right)^{\frac{1}{2\mu}}$. For such m, according to Theorem 4.2, the mapping $DE(t)u_0$ contracts m-dimensional volumes in $\dot{H}_p^s(\Omega)$ for sufficiently large t, uniformly for $u_0 \in \mathcal{A}$. With this result, according to Temam [10], Chapter V, and Ghidaglia [8], Theorem 3.2, it follows that $\mathcal{A}$ has finite fractal dimension.

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