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# Exponentially slow traveling waves on a finite interval for Burgers' type equation *

P. P. N. de Groen & G. E. Karadzhov

**Abstract**

In this paper we study for small positive $\epsilon$ the slow motion of the solution for evolution equations of Burgers' type with small diffusion,

$$u_t = \epsilon u_{xx} + f(u)\, u_x\,, \quad u(x,0) = u_0(x), \quad u(\pm 1, t) = \pm 1, \qquad (\star)$$

on the bounded spatial domain $[-1,1]$; $f$ is a smooth function satisfying $f(1) > 0, f(-1) < 0$ and $\int_{-1}^{1} f(t)dt = 0$. The initial and boundary value problem $(\star)$ has a unique asymptotically stable equilibrium solution that attracts all solutions starting with continuous initial data $u_0$. On the infinite spatial domain $\mathbb{R}$ the differential equation has slow speed traveling wave solutions generated by profiles that satisfy the boundary conditions of $(\star)$. As long as its zero stays inside the interval $[-1,1]$, such a traveling wave suitably describes the slow long term behaviour of the solution of $(\star)$ and its speed characterizes the local velocity of the slow motion with exponential precision. A solution that starts near a traveling wave moves in a small neighborhood of the traveling wave with exponentially slow velocity (measured as the speed of the unique zero) during an exponentially long time interval $(0,T)$. In this paper we give a unified treatment of the problem, using both Hilbert space and maximum principle methods, and we give rigorous proofs of convergence of the solution and of the asymptotic estimate of the velocity.

## 1 Introduction

Slow motion is a phenomenon that may occur in singularly perturbed non-linear parabolic equations, e.g. in reaction-diffusion ([20, 21]), convection-diffusion ([13, 14, 18]) and Cahn-Hilliard ([2]) equations. Typically, the solution of such a problem develops in finite time metastable shock profiles that persist during an exponentially (with respect to the small parameter) long period of time and

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that move with exponentially slow speed. The fact that a shock layer moves slowly and may not exist forever makes it difficult to define notions as metastability and velocity and to track the evolution of the solution rigorously. In this paper we study for small positive $\varepsilon$ the long term behaviour of the shock layer for equations of Burgers' type in one space variable,

$$u_t = \varepsilon u_{xx} + f(u)\, u_x\,, \tag{1.1}$$

on the strip $[-1,1] \times \mathbb{R}^+$, satisfying the boundary conditions

$$u(\pm 1, t) = \pm 1 \tag{1.2}$$

and the continuous and compatible initial condition

$$u(x,0) = u_0(x), \quad u_0(\pm 1) = \pm 1\,. \tag{1.3}$$

The function $f \in C^3(\mathbb{R})$ and its integral $F(x) := \int_{-1}^{x} f(t)\, dt$ satisfy the conditions:

$$f(1) > 0\,, \quad f(-1) < 0\,, \quad F(1) = F(-1) = 0 \quad \text{and} \quad \forall_{|s|<1} : F(s) < 0\,. \tag{1.4}$$

The maximum principle for parabolic equations ([7]) implies the a priori estimate $|u(x,t)| \leq M$ for all initial data satisfying $|u_0(x)| \leq M$ (with $M \geq 1$). Hence the behaviour of $f$ outside the interval $[-M, M]$ is not important. However, for ease of presentation we shall assume that $f''$ and $f'''$ are uniformly bounded on $\mathbb{R}$. In the special case $f(u) = u$, we get Burgers' equation. The special feature $F(-1) = F(1) > F(x)$, $(-1 < x < 1)$ causes slow motion of the shock layer. More generally, the solution $u$ satisfying the boundary conditions $a = u(-1,t) < u(1,t) = b$, shows exponentially slow motion if $F(a) = F(b) > F(x)$, $(a < x < b)$; this case can be reduced to the form (1.2) above by an affine transformation of $u$. According to Theorem 4.4 in Ch. VI of Ladyzhenskaja et al. ([12]) the problem (1.1)–(1.3) has a unique classical solution $u(x,t)$; classical means that $u(x,t)$ is continuous in the closed domain $\overline{Q}_s$, where $Q_s := (-1,1) \times (0,s)$, and the derivatives $u_t$, $u_x$ and $u_{xx}$ are continuous in the open domain $Q_s$, for any $s > 0$.

Numerical methods (see [11]) and formal asymptotic expansions (see [11, 13, 14, 18]) show the existence of an internal shock layer that moves with exponentially slow speed towards a stable (stationary) equilibrium solution. The location and the velocity of this shock layer can be described conveniently (although somewhat arbitrarily) by the position and the speed of the unique zero. It is well known that the exponentially small velocity cannot be determined by "matched asymptotic expansions" and that exponential precision is required. For example, in Burgers' equation ($f \equiv u$), the function $\tanh \frac{x-\xi(t)}{2\varepsilon}$ satisfies equation and boundary conditions approximately up to exponentially small order for any location $\xi(t)$ moving with exponentially slow speed $\xi'(t)$. In order to remove this indeterminacy, Reyna & Ward ([18]) use a projection method, such as was used in [4] in the related problem of resonance in a singularly perturbed turning point problem. This method is based on the fact that the linear operator associated with the first variation around the shock profile has exactly one exponentially small eigenvalue and that small solutions (i.e., solutions not exploding for $\varepsilon\searrow 0$) of the equation of first variation can be found only in the orthogonal complement of the associated eigenfunction. In our approach, the local velocity of the shock layer at a point $x_0$ is derived from the traveling wave solution $\Phi(x-Vt,\varepsilon)$ of (1.1) whose profile $x\mapsto\Phi(x,\varepsilon)$ fits exactly the boundary conditions (1.2) and satisfies $\Phi(x_0,\varepsilon)=0$. Obviously, this does not give another expression for the local velocity, but it has the advantage that such traveling wave solutions can be used in upper and lower estimates by the maximum principle, making rigorous the asymptotic formulae for the slow velocity. Moreover, those profiles are well-suited for the study of metastability.

We are going to explain the long term behaviour of this solution by approximating it with traveling waves. Strictly speaking, there are no traveling waves on a finite interval and we shall use the notion in the sense of a restriction of a true traveling wave solution $\psi(x-vt)$ of equation (1.1), defined on all of $\mathbb{R}^2$, to a finite rectangular subdomain $[-1,1]\times[S\,,\,T]$, where $[S\,,\,T]$ is chosen so that the unique zero of $x\mapsto\psi(x-vt)$ is inside the open interval $(-1\,,\,1)$. Outside this rectangle the traveling wave is not very interesting for our analysis. In Section 2 we show that for each $x_0\in(-1\,,\,1)$ a unique traveling wave $\Phi(x-V(\varepsilon,x_0)t\,,\varepsilon)$ exists, which satisfies $\Phi(x_0\,,\varepsilon)=0$ and $\Phi(\pm1,\varepsilon)=\pm1$, and we derive a precise estimate of its exponentially slow velocity $V$ (i.e., $V$ is of the order $O(e^{-c/\varepsilon})$ for some $c>0$). Moreover, $V$ is a monotone function of $x_0$, which is positive for $x_0\approx-1$ and negative for $x_0\approx1$, implying that the wave moves to the right in the former case and to the left in the latter. The unique "traveling wave" with zero velocity satisfies the boundary conditions (1.2) for all times and is the stationary or equilibrium solution of (1.1)–(1.3); we denote it by $\Phi_e$ and its zero $x_e$ we call the "equilibrium point". The method enables us to show that the error in the asymptotic estimate of $x_e$, derived in [13], is exponentially small; see (2.24).

For the study of convergence, stability, and metastability, we consider variations around a traveling wave profile in Section 3. The spatial operator $\mathcal{A}$ in the linear approximation $v_t=-\mathcal{A}v$ is positive for all $\varepsilon>0$ and selfadjoint in a weighted $L^2(-1,1)$-norm, where the weight is defined by the derivative of the traveling wave profile. The spectrum consists of isolated eigenvalues of multiplicity one. By the technique of [4] we derive in Lemmata 3.1 and 3.2 an exponentially precise estimate of the exponentially small bottom eigenvalue $\lambda_0(\varepsilon)$ and of the corresponding eigenfunction, and we show that the gap between the bottom eigenvalue and the rest of the spectrum is of order $O(1/\varepsilon)$. Using this, we prove in Theorems 3.9 – 3.11 that the equilibrium is *stable*; it attracts all solutions of the problem (1.1)–(1.3) starting in some small neighborhood of it (small, measured in a weighted Sobolev-norm in $H^1(-1,1)$). We also obtain rates of convergence with respect to this norm; in a manifold of codimension one (which is nearly orthogonal to the derivative of the equilibrium solution) convergence is exponentially fast (of the order $O(e^{-\gamma t/\varepsilon})$ for some $\gamma>0$).

As stated above, the problem (1.1)–(1.3) has a unique classical solution $u(x,t)$. In Section 4 we use the strong maximum principle for parabolic differential operators in conjunction with techniques of Bernstein and Filippov (see [12]) in order to extend this slightly, and we prove in Lemma 4.6 that the function $x \mapsto u(x,t)$ is in the Sobolev space $H^2(-1,1)$ uniformly for all $t > 0$. This allows us to show, using Arzela-Ascoli's theorem and a theorem of Friedman ([6]), that $u(\cdot,t)$ converges as $t \to \infty$ to the equilibrium solution in the Sobolev norm of $H^1(-1,1)$ for any continuous initial value $u(x,0)$.

In the final Section 5 we complete the picture. We show by a contraction argument that every solution starting in a neighborhood of a traveling wave profile $\Phi(x,\varepsilon)$ is attracted exponentially rapidly towards a one-dimensional submanifold, which essentially is equal to the derivative of the same traveling wave profile shifted over an exponentially small distance in $x$-direction, and it stays there during an exponentially long time interval $(0, T_\varepsilon)$. As this traveling wave profile $\Phi(x,\varepsilon)$ can be considered as a "snapshot" of a slow moving wave, this indeed shows that solutions of (1.1)–(1.3) are quickly attracted towards a slow moving shock wave that moves to the equilibrium solution with exponentially slow velocity.

**Weighted Norms.** For the study of convergence we use the standard $L^2(-1,1)$-norm $\|\cdot\|$ and the Sobolev norm $\|\cdot\|_1$ defined by

$$\| u \|_1^2 := \varepsilon^2 \| u' \|^2 + \| u \|^2 = \int_{-1}^{1} \left\{ \varepsilon^2 |u'(x)|^2 + |u(x)|^2 \right\} dx \,. \tag{1.5}$$

Moreover, we shall consider weighted Sobolev norms $\|\cdot\|_h$ for given weight functions $h(x)^2$,

$$\| u \|_h^2 := \| u\, h \|_1^2 = \int_{-1}^{1} \left\{ \varepsilon^2 |(u(x)\, h(x))'|^2 + |u(x)\, h(x)|^2 \right\} dx \,. \tag{1.6}$$

Two norms $\|\cdot\|_{h_1}$ and $\|\cdot\|_{h_2}$ or two weight functions $h_1$ and $h_2$ are said to be equivalent if positive constants $c_1$ and $c_2$ exist such that

$$c_1 \| u \|_{h_1} \le \| u \|_{h_2} \le c_2 \| u \|_{h_1} \quad \text{for all } u\,, \quad \text{or} \quad c_1 h_1(x) \le h_2(x) \le c_2 h_1(x)\,;$$

we denote these equivalences by $\|\cdot\|_{h_1} \asymp \|\cdot\|_{h_2}$ or $h_1 \asymp h_2$, respectively.

# 2 Traveling waves on the line

In this section we establish existence and asymptotic properties of traveling wave solutions of equation (1.1) on the whole line, which satisfy the boundary conditions (1.2) at $x = \pm 1$ approximately, up to an exponentially small error during an exponentially large time interval. Specifically, we will prove existence of traveling wave solutions of (1.1) of the form

$$u(x,t) = \Phi(x - V(\varepsilon, x_0)t, \varepsilon) \quad \text{satisfying} \quad \Phi(\pm 1, \varepsilon) = \pm 1\,, \quad \Phi(x_0, \varepsilon) = 0\,, \tag{2.1}$$

for some $x_0 \in (-1\,,\,1)$. Such solutions of (1.1) move with speed $V$ and do not satisfy the boundary conditions (1.2) exactly (except if $t = 0$). Yet they are useful because their speed is exponentially small, and they describe very well the behaviour of a certain class of solutions of (1.1 − 1.2) during long time intervals. The function $\Phi(x,\varepsilon)$ is called the traveling wave profile; implicitly it depends on $x_0$ .

If (1.1) has a solution of the form (2.1), then the *profile* $x \mapsto \Phi(x,\varepsilon)$ has to satisfy the ODE

$$\varepsilon\Phi'' + (f(\Phi) + V)\Phi' = 0\,, \quad \Phi(\pm 1,\varepsilon) = \pm 1\,, \quad ('= d/dx)\,. \tag{2.2}$$

Scaling the independent variable around the zero $x_0$ of $\Phi$ by $x = x_0 + \varepsilon\eta$ and setting $\varphi(\eta,\varepsilon) := \Phi(x_0 + \varepsilon\eta,\varepsilon)$, we find that $\varphi$ and $V$ have to satisfy $('= d/d\eta)$

$$\begin{cases} \varphi'' + (f(\varphi) + V)\varphi' = 0\,, \\ \varphi(0,\varepsilon) = 0\,, \quad \varphi(\frac{1-x_0}{\varepsilon},\varepsilon) = 1\,, \quad \varphi(-\frac{1+x_0}{\varepsilon},\varepsilon) = -1\,. \end{cases} \tag{2.3}$$

Integrating the differential equation once we find

$$\frac{d\varphi}{d\eta} = C - V\varphi - F(\varphi)\,, \quad \text{where} \quad F(\varphi) := \int_{-1}^{\varphi} f(t)\,dt\,, \tag{2.4}$$

and where $C$ is a constant of integration. Using the condition $\varphi(0,\varepsilon) = 0$ , this equation is implicitly solved in terms of the function

$$G(\varphi; C, V) := \int_0^{\varphi} \frac{ds}{g(s)}\,, \quad \text{where} \quad g(s) := C - Vs - F(s)\,, \tag{2.5}$$

provided the denominator $g$ is non-zero. If the denominator is positive, $G$ is a monotone function. Hence, the solution of (2.3) is given by the inverse function $\varphi(\eta,\varepsilon) := G^{-1}(\eta; C, V)$, if we can find constants $C(\varepsilon, x_0)$ and $V(\varepsilon, x_0)$ such that $C - Vs - F(s) > 0$ for all $s \in [-1, 1]$ and

$$\int_0^1 \frac{ds}{C - Vs - F(s)} = \frac{1-x_0}{\varepsilon}\,, \quad \text{and} \quad \int_{-1}^0 \frac{ds}{C - Vs - F(s)} = \frac{1+x_0}{\varepsilon}\,. \tag{2.6}$$

It is clear that $C(0, x_0) = 0$, $V(0, x_0) = 0$, provided $x_0$ is bounded away from the boundaries $\pm 1$. This we shall assume throughout the paper. In other words, we consider profiles $\Phi(x,\varepsilon)$ with internal layers only.

**Proposition 2.1 (Existence and asymptotics of slow traveling waves)** *If the function $f$ satisfies (1.4), then for any $x_0 \in (-1, 1)$ and any $\varepsilon > 0$ unique solutions $C(\varepsilon, x_0)$ and $V(\varepsilon, x_0)$ of (2.6) exist, and hence also a unique solution $\Phi$ of (2.2), satisfying for $\varepsilon \to 0$ the asymptotics*

$$\begin{aligned} \Phi(x,\varepsilon) &= \pm 1 + O\left(\exp\left(f(\pm 1)\tfrac{x_0 - x}{\varepsilon}\right) + R_\varepsilon\right) \quad \textit{for} \quad x \gtrless x_0\,, \\ C(\varepsilon, x_0) &= \alpha \exp\left(-f(1)\tfrac{1-x_0}{\varepsilon}\right) + \beta \exp\left(f(-1)\tfrac{1+x_0}{\varepsilon}\right) + O(R_\varepsilon^2/\varepsilon), \\ V(\varepsilon, x_0) &= -\alpha \exp\left(-f(1)\tfrac{1-x_0}{\varepsilon}\right) + \beta \exp\left(f(-1)\tfrac{1+x_0}{\varepsilon}\right) + O(R_\varepsilon^2/\varepsilon), \end{aligned} \tag{2.7}$$

*where $\alpha$, $A$, $\beta$, $B$ and $R_\varepsilon$ are defined by*

$$\alpha := \tfrac{1}{2} f(1)\,\exp(Af(1))\,, \qquad A := \textstyle\int_0^1 [\frac{1}{F(1)-F(s)} - \frac{1}{f(1)(1-s)}]ds\,,$$
$$\beta := -\tfrac{1}{2} f(-1)\,\exp(-Bf(-1))\,, \qquad B := \textstyle\int_{-1}^0 [\frac{1}{F(-1)-F(s)} + \frac{1}{f(-1)(1+s)}]ds\,,$$
$$R_\varepsilon := \exp\left(-f(1)\tfrac{1-x_o}{\varepsilon}\right) + \exp\left(f(-1)\tfrac{1+x_o}{\varepsilon}\right)\,.$$

**Proof.** We begin with an analysis of the solution of the initial value problem (2.4) as a function of the parameters $C$ and $V$. So we consider the solution $\psi$ of

$$\psi' = g(\psi) := c - v\psi - F(\psi)\,, \quad \psi(0;c,v) = 0\,, \tag{2.8}$$

writing it either as $\psi(\eta;c,v)$ or as $\psi(\eta)$ for brevity, if there is no confusion concerning the parameters $c$ and $v$. Because of assumption (1.4) we have $F(s) < 0$ if $-1 < s < 1$. Since $F'(1) = f(1) > 0$ and $F'(-1) = f(-1) < 0$, the function $g(s) := c - vs - F(s)$ has well defined zeros $s_+$ and $s_-$ if $c$ and $v$ are in some small interval around 0; those zeros tend to $+1$ and $-1$ respectively if $c$ and $v$ tend to zero, and are such that $g(s)$ is strictly positive on $(s_-, s_+)$ and $g'(s_+) < 0$ and $g'(s_-) > 0$:

$$\begin{aligned} &g(s) := c - vs - F(s) > 0 \quad \text{for all } s \in (s_-, s_+)\,,\\ &g'(s_+) < 0 \quad \text{and} \quad g'(s_-) > 0\,,\\ &\left.\begin{matrix} s_- + 1 \\ s_+ - 1 \end{matrix}\right\} = O(|c| + |v|) \qquad (c\,,\, v \to 0)\,. \end{aligned} \tag{2.9}$$

So (2.8) has a unique strictly increasing solution $\psi$ on the whole line, implicitly defined by

$$\eta = \int_0^\psi \frac{ds}{g(s)} \quad \text{and satisfying} \quad \begin{cases} \lim\limits_{\eta\to\pm\infty} \psi(\eta) = s_\pm\,,\\ \lim\limits_{\eta\to\pm\infty} \psi'(\eta) = 0\,. \end{cases} \tag{2.10}$$

In order to derive more precise asymptotics of $\psi$, we define the integrals

$$\begin{aligned} a_+ &:= \int_0^{s_+} \left(\frac{1}{g(s)} - \frac{1}{g'(s_+)\,(s-s_+)}\right) ds\,,\\ a_- &:= \int_{s_-}^0 \left(\frac{1}{g(s)} - \frac{1}{g'(s_-)\,(s-s_-)}\right) ds\,, \end{aligned} \tag{2.11}$$

which are finite because the singularities at $s_\pm$ are removed. Clearly, if $v = 0$ and $c = 0$, then $s_\pm = \pm 1$, $a_+ = A$ and $a_- = B$. In the integral equation (2.10) we expand the numerator around $s_+$

$$\begin{aligned} \eta &= \int_0^\psi \frac{ds}{g(s)} = \int_0^\psi \frac{ds}{g'(s_+)\,(s-s_+)} + \int_0^\psi \left(\frac{1}{g(s)} - \frac{1}{g'(s_+)\,(s-s_+)}\right) ds\\ &= \frac{1}{g'(s_+)} \log(\frac{s_+ - \psi}{s_+}) + a_+ - O(s_+ - \psi) \qquad (\psi \to s_+) \quad \text{or} \quad (\eta \to \infty)\,, \end{aligned}$$

uniformly in $c$ and $v$. Exponentiation results in

$$s_+ - \psi(\eta) = s_+ \exp\{g'(s_+)(\eta - a_+) + O(s_+ - \psi(\eta)\} \quad (\eta \to \infty)\,. \tag{2.12}$$

Since $s_+ - \psi$ is uniformly bounded and since $g'(s_+) = -v - f(s_+) < 0$, this results in the asymptotic forms of $\psi$ and, using equation (2.8), also of $\psi'$ for $\eta \to \infty$

$$\begin{aligned} s_+ - \psi(\eta; c, v) &= s_+ \exp\{g'(s_+)(\eta - a_+)\} + O(\exp\{2\eta g'(s_+)\})\,, \\ \psi'(\eta; c, v) &= -s_+ \, g'(s_+) \exp\{g'(s_+)(\eta - a_+)\} + O(\exp\{2\eta g'(s_+)\})\,. \end{aligned} \tag{2.13}$$

Likewise we derive the asymptotic forms of $\psi$ and $\psi'$ for $\eta \to -\infty$,

$$\begin{aligned} s_- - \psi(\eta; c, v) &= s_- \exp\{g'(s_-)(\eta - a_-)\} + O(\exp\{2\eta g'(s_-)\})\,, \\ \psi'(\eta; c, v) &= -s_- \, g'(s_-) \exp\{g'(s_-)(\eta - a_-)\} + O(\exp\{2\eta g'(s_-)\})\,. \end{aligned} \tag{2.14}$$

We now have shown that, for any $c$ and $v$ in some small interval around 0, a monotone solution $\psi$ of (2.8) exists, ranging from $s_-$ at $-\infty$ to $s_+$ at $+\infty$. However, the question in (2.3) is to find $C$ and $V$ from the system of equations

$$\psi(\frac{1 - x_0}{\varepsilon}; C, V) = 1\,, \quad \psi(-\frac{1 + x_0}{\varepsilon}, C, V) = -1\,, \tag{2.15}$$

with $C(0, x_0) = 0$, $V(0, x_0) = 0$.

To this aim we approximate its Jacobian. We differentiate equation (2.8) with respect to its parameters, denoting the partial derivatives by $\psi_c := \partial\psi/\partial c$ and $\psi_v := \partial\psi/\partial v$ ($'$ denotes the derivative with respect to $\eta$):

$$\begin{aligned} \psi_c' &= -(v + f(\psi))\psi_c + 1, \quad \psi_c(0; c, v) = 0\,, \\ \psi_v' &= -(v + f(\psi))\psi_v - \psi, \quad \psi_v(0; c, v) = 0\,. \end{aligned}$$

Considered as ordinary differential equations for $\psi_c$ and $\psi_v$ as functions of $\eta$ in which the function $\psi$ is given, these equations are solved by

$$\begin{aligned} \psi_c(\eta) &= \int_0^\eta \exp\{-\textstyle\int_t^\eta (v + f(\psi(s)))\, ds\}\, dt \\ \psi_v(\eta) &= -\int_0^\eta \psi(t) \exp\{-\textstyle\int_t^\eta (v + f(\psi(s)))\, ds\} dt\,, \end{aligned}$$

and by the equation $\psi'' = -(v + f(\psi))\psi'$ they can be simplified to

$$\psi_c(\eta) = \psi'(\eta) \int_0^\eta \frac{ds}{\psi'(s)} \quad \text{and} \quad \psi_v(\eta) = -\psi'(\eta) \int_0^\eta \frac{\psi(s)ds}{\psi'(s)}\,. \tag{2.16}$$

Using the asymptotic formulae (2.13) and (2.14), we find

$$\lim_{\eta \to \pm\infty} \psi_c(\eta; c, v) = \frac{1}{f(s_\pm) + v} \quad \text{and} \quad \lim_{\eta \to \pm\infty} \psi_v(\eta; c, v) = \frac{-s_\pm}{f(s_\pm) + v}, \tag{2.17}$$

and from this it immediately follows that the Jacobian determinant of system (2.15) is bounded away from zero if $\varepsilon$ is sufficiently small. Hence $C(\varepsilon, x_0)$ and $V(\varepsilon, x_0)$ are uniquely determined by this equation.

The zeros $s_\pm$ of $g$ and the integrals $a_\pm$ defined in (2.11) are functions of $c$ and $v$. Let us now denote those quantities by the corresponding capital letters $S_\pm$ and $A_\pm$, if the solutions $C$ and $V$ of (2.15) are substituted in them. They are necessarily such that $C \pm V > 0$ and $S_- < -1 < 1 < S_+$. Moreover, $\varphi(\eta, \varepsilon) = \psi(\eta; C(\varepsilon, x_0), V(\varepsilon, x_0))$ is the solution of (2.3). The asymptotics of $S_+ - 1$ and $S_- + 1$ are now easily found by inserting the boundary conditions (2.15) into (2.13) and (2.14):

$$\begin{aligned} S_+ - 1 &= S_+ \exp\{g'(S_+)\,(\tfrac{1-x_0}{\varepsilon} - A_+)\} + O\left(\exp\{2g'(S_+)\,\tfrac{1-x_0}{\varepsilon}\}\right), \\ S_- + 1 &= S_- \exp\{-g'(S_-)\,(\tfrac{1+x_0}{\varepsilon} + A_-)\} + O\left(\exp\{-2g'(S_-)\,\tfrac{1+x_0}{\varepsilon}\}\right), \end{aligned} \tag{2.18}$$

and they are exponentially small. We can compute $C$ and $V$ from $g(S_\pm) = 0$,

$$\begin{aligned} C \,-\, V S_+ &= F(1 + (S_+ - 1)) \\ &= f(1)\, \exp\{g'(S_+)\,(\tfrac{1-x_0}{\varepsilon} - A_+ + O(S_+ - 1))\} + O((S_+ - 1)^2), \\ C \,-\, V S_- &= F(-1 + (S_- + 1)) \\ &= f(-1)\, \exp\{-g'(S_-)\,(\tfrac{1+x_0}{\varepsilon} - A_- + O(S_- + 1))\} + O((S_- + 1)^2). \end{aligned} \tag{2.19}$$

Adding and subtracting both equations, we see that $V$ and $C$ are exponentially small. With (2.18) this implies that the precise exponential order in (2.18) is $|S_\pm \mp 1| = O(R_\varepsilon)$. Inserting this in (2.19) we find

$$\begin{aligned} C \;&=\; \tfrac{1}{2} f(1)\, \exp\{-f(S_+)\,(\tfrac{1-x_0}{\varepsilon} - A_+))\} \\ &\quad - \tfrac{1}{2} f(-1)\, \exp\{f(S_-)\,(\tfrac{1+x_0}{\varepsilon} - A_-)\} + O(R_\varepsilon^2 + R_\varepsilon |V|/\varepsilon), \\ V \;&=\; -\,\tfrac{1}{2}\, f(1)\, \exp\{-f(S_+)\,(\tfrac{1-x_0}{\varepsilon} - A_+))\} \\ &\quad - \tfrac{1}{2} f(-1)\, \exp\{f(S_-)\,(\tfrac{1+x_0}{\varepsilon} - A_-)\} + O(R_\varepsilon^2 + R_\varepsilon |V|/\varepsilon). \end{aligned} \tag{2.20}$$

This implies that $V = O(R_\varepsilon)$ and that the error term in (2.20) can be simplified to $O(R_\varepsilon^2/\varepsilon)$. Finally, we show that $A_+ = A + O(R_\varepsilon)$ in the following way. Using the Taylor expansion of $g$ with the integral form for the remainder, we find

$$\begin{aligned} A_+ \;&=\; \int_0^{S_+} \left( \frac{1}{g(s)} - \frac{1}{g'(S_+)\,(s - S_+)} \right) ds \\ &=\; \int_0^{S_+} \frac{s - S_+}{g(s)\, f(S_+)} \int_0^1 (1-t)\, f'(S_+ + t(s - S_+))\, dt\, ds\,. \end{aligned}$$

Likewise we find for $A$, if we also shift the integration variable $s \to s + 1 - S_+$,

$$A = \int_{S_+ - 1}^{S_+} \frac{s - S_+}{(F(1) - F(s + 1 - S_+))\, f(1)} \int_0^1 (1-t)\, f'(1 + t(s - S_+))\, dt\, ds\,.$$

From these expressions we see that the integrands are uniformly bounded, that their difference is of order $O(R_\varepsilon)$ uniformly, and that the domain of integration of the first integral is $O(R_\varepsilon)$ larger than the domain of the second one. The difference $A_- - B$ is estimated likewise. Together with (2.20) this proves the assertion (2.7). $\diamondsuit$

Having established for any $x_0$ the existence of a wave profile $\Phi$ which goes through zero at $x_0$, we can show that this profile is a monotone function of $x_0$ and that also its velocity $V(\varepsilon, x_0)$ is monotone in $x_0$. Since we see from (2.7) that $V(\varepsilon, x_0)$ changes sign if $x_0$ moves from -1 to 1, this implies existence of a unique equilibrium solution $\Phi_e$ with velocity zero.

**Proposition 2.2** *The traveling wave solution $\Phi$ of* (2.1) *and its velocity $V$ are monotone functions of $x_0$ ,*

$$\frac{\partial \Phi}{\partial x_0} \le 0 \quad \textit{and} \quad \frac{\partial V}{\partial x_0} < 0\,. \tag{2.21}$$

**Proof.** Substituting $\Phi(x,\varepsilon) = \psi(\frac{x-x_0}{\varepsilon}; C(\varepsilon, x_0), V(\varepsilon, x_0))$ in the identity $G(\varphi(\eta); C, V) = \eta$ (see (2.5)), and differentiating with respect to $x_0$, we find

$$H(\psi) := \frac{1}{g(\psi)}\frac{\partial \Phi}{\partial x_0} = \frac{\partial C}{\partial x_0}\int_0^\psi \frac{ds}{g^2(s)} - \frac{\partial V}{\partial x_0}\int_0^\psi \frac{s\,ds}{g^2(s)} - \frac{1}{\varepsilon}. \tag{2.22}$$

Since $g(s) = C - Vs - F(s) > 0$, it suffices to show that $H$ is non-positive on $[-1, 1]$. Differentiating (2.6) with respect to $x_0$ , we find in particular:

$$\begin{aligned} H(1) &= \frac{\partial C}{\partial x_0}\int_0^1 \frac{ds}{g(s)^2} - \frac{\partial V}{\partial x_0}\int_0^1 \frac{s\,ds}{g(s)^2} - \frac{1}{\varepsilon} = 0\,, \\ H(-1) &= -\frac{\partial C}{\partial x_0}\int_{-1}^0 \frac{ds}{g(s)^2} - \frac{\partial V}{\partial x_0}\int_{-1}^0 \frac{(-s)\,ds}{g(s)^2} - \frac{1}{\varepsilon} = 0\,. \end{aligned} \tag{2.23}$$

Moreover, we can compute the partial derivatives $\partial C/\partial x_0$ and $\partial V/\partial x_0$ from this set of linear equations. Because all integrals in (2.23) are positive, $\partial V/\partial x_0$ is negative.

Finally, because $H(0) = -1/\varepsilon$ the function $H$ must have a minimum on $(-1, 1)$, and so its derivative $H'(t) = \left(\frac{\partial C}{\partial x_0} - t\,\frac{\partial V}{\partial x_0}\right) g^{-2}(t)$ must have a zero in this interval. Since there cannot be a second zero, $H$ cannot have a maximum on $(-1, 1)$ and so must be bounded from above by zero. $\diamondsuit$

**The equilibrium solution.** Since the velocity $V(x, \varepsilon)$ is a monotone function of $x_0$ , a unique "traveling" wave (2.1) with velocity zero exists. The corresponding solution is the equilibrium solution, which we denote by $\Phi_e(x, \varepsilon)$ or by the profile $\varphi_e(\eta, \varepsilon)$ in the stretched coordinate $\eta = (x - x_0)/\varepsilon$. We call the zero of this equilibrium solution the *equilibrium point* and we denote it by $x_e$. From (2.7) we find its position

$$x_e(\varepsilon) = \frac{f(1) + f(-1) + \varepsilon \log(\beta/\alpha)}{f(1) - f(-1)} + O(R_\varepsilon). \tag{2.24}$$

Hence the equilibrium solution has only an internal layer.

The profile $\varphi$ of the traveling wave depends on $\varepsilon$ very weakly. Its main term is the so-called shock layer profile (see [13]) $\varphi_s$ which we define as the solution of

$$\varphi'' + f(\varphi)\,\varphi' = 0\,, \qquad \lim_{\eta\to\pm\infty} \varphi(\eta) = \pm 1\,, \tag{2.25}$$

with normalization $\varphi_s(0) = 0$. Clearly $\varphi_s(\eta) = \psi(\eta; 0, 0)$. From (2.13) and (2.14) we find its asymptotic behaviour for large $|\eta|$:

$$\begin{aligned}
\varphi_s(\eta) &= \begin{cases} 1 - e^{f(1)(A-\eta)} + O(e^{-2\eta f(1)}) & \eta \to +\infty\,, \\ -1 + e^{f(-1)(B-\eta)} + O(e^{-2\eta f(-1)}) & \eta \to -\infty\,, \end{cases} \\
\varphi_s'(\eta) &= \begin{cases} f(1)\, e^{f(1)(A-\eta)} + O(e^{-2\eta f(1)}) & \eta \to +\infty\,, \\ -f(-1)\, e^{f(-1)(B-\eta)} + O(e^{-2\eta f(-1)}) & \eta \to -\infty\,. \end{cases}
\end{aligned} \tag{2.26}$$

This profile $\varphi_s$ does not depend on $\varepsilon$, and it is the main term in the traveling wave profile $\varphi(\eta, \varepsilon)$, up to an exponentially small order uniformly in $\eta$:

**Proposition 2.3** *Constants $K_1$, $K_2$ and $\varepsilon_0$ exist such that $\varphi$ and $\varphi_s$ satisfy uniformly for all $\eta \in \mathbb{R}$ and $\varepsilon \in (0, \varepsilon_0]$,*

$$|\varphi(\eta, \varepsilon) - \varphi_s(\eta)| \le K_1\, R_\varepsilon \quad \text{and} \quad |\varphi'(\eta, \varepsilon) - \varphi_s'(\eta)| \le K_2\, R_\varepsilon\,. \tag{2.27}$$

**Proof.** Since $\varphi_s(\eta) = \psi(\eta; 0, 0)$ and $\varphi(\eta, \varepsilon) = \psi(\eta; C(\varepsilon, x_0), V(\varepsilon, x_0))$, we find from the intermediate value theorem points $\zeta$ between $C$ and 0, and $\vartheta$ between $V$ and 0, such that

$$\psi(\eta; C, V) - \psi(\eta; 0, 0) = C\, \psi_c(\eta, \zeta, \vartheta) + V \psi_v(\eta, \zeta, \vartheta)\,.$$

Using in equation (2.16) the positivity of $\psi'$ and its asymptotics, as displayed in (2.13) and (2.14), we easily see that $\psi_c$ and $\psi_v$ are uniformly bounded on the whole real line. Moreover, $C$ and $V$ are of the order $O(R_\varepsilon)$. This proves the first estimate. The second estimate is reduced to the first one by the relations $\varphi' = C - V\varphi - F(\varphi)$ and $\varphi_s' = F(1) - F(\varphi_s)$. $\diamondsuit$

**Proposition 2.4** *Let $x_0 \in [-1+\delta, 1-\delta]$ for some small fixed $\delta > 0$. Then the equivalence*

$$\psi'(\eta, C(\varepsilon, x_0), V(\varepsilon, x_0)) \asymp \psi'(\eta, 0, 0) = \varphi_s'(\eta). \tag{2.28}$$

*is uniform with respect to $\eta, x_0, \varepsilon$ if $|\eta| \le \eta_0/\varepsilon, x_0 \in [-1+\delta, 1-\delta], \varepsilon \in (0, \varepsilon_0]$.*

**Proof.** Outside a compact interval, depending only on $f$, the estimates (2.28) follow from the asymptotics (2.13),(2.14), and on the same interval the equivalence $\psi'(\eta, C, V) \asymp 1$ is derived easily from (2.8). $\diamondsuit$

Using the evident equivalence $\psi'(\eta+u, 0, 0) \asymp \psi'(\eta, 0, 0)$, which is uniform with respect to $u$ in some compact interval $|u| \le u_0$, we get from proposition 2.4 the following corollary.

**Corollary 2.5** *The equivalence*

$$\psi'(\eta+u, C(\varepsilon, x_0), V(\varepsilon, x_0)) \asymp \psi'(\eta, C(\varepsilon, x_0), V(\varepsilon, x_0)). \tag{2.29}$$

*is uniform with respect to* $\eta, x_0, u, \varepsilon$ *if* $|\eta| \le \eta_0/\varepsilon, x_0 \in [-1+\delta, 1-\delta], |u| \le u_0$ .

For our convergence results we introduced weighted Sobolev norms (1.6). We use weight functions $h_w$ and $h_s$ related to the traveling wave profile $\varphi$ and to the shock layer profile $\varphi_s$ (the main term in the asymptotic approximation of $\varphi$),

$$h_w^{-2}(x) := \varphi'(\tfrac{x-x_0}{\varepsilon}, \varepsilon) = \varepsilon\, \Phi'(x, \varepsilon) \quad \text{and} \quad h_s^{-2}(x) := \varphi_s'(\tfrac{x-x_0}{\varepsilon})\,. \tag{2.30}$$

Apparently those weights depend on the shock layer location $x_0$ . In particular, for the weight at the equilibrium point $x_e(\varepsilon)$ we shall use the notation $h_e$ and $h_{se}$ ,

$$h_e^{-2}(x) := \varphi'(\tfrac{x-x_e}{\varepsilon}, \varepsilon) = \varepsilon\, \Phi_e'(x, \varepsilon) \quad \text{and} \quad h_{se}^{-2}(x) := \varphi_s'(\tfrac{x-x_e}{\varepsilon})\,. \tag{2.31}$$

With those weights we get families of $(\varepsilon, x_0)$-dependent weighted norms $\|\cdot\|_{h_w}$ and $\|\cdot\|_{h_e}$, for which we shall show equivalence uniformly with respect to $\varepsilon$. For convenience, we first show equivalence with a weighted norm defined in a different way. After that, we show equivalence of the weights $h_w$ and $h_s$ . The norms $h_e$ and $h_{se}$ with weights centered at the equilibrium point are special cases of $h_w$ and $h_s$, so the results apply to those too.

**Proposition 2.6** *The norms* $u \mapsto \|u\|_h$ *and* $u \mapsto \sqrt{\varepsilon^2\|u'\,h\|^2 + \|u\,h\|^2}$ *are equivalent if* $h$ *is one of the weight functions* $h_w$ *or* $h_s$ *defined in* (2.30)*; the constants in the equivalence depend on* $f$ *only.*

**Proof.** Let $h = h_w$. Then we find from equation (2.3) for $\varphi$ the relation

$$h' = \frac{f(\Phi)+V}{2\varepsilon} h \quad \text{and hence} \quad |u'h + uh'| = h|u' + \frac{f(\Phi)+V}{2\varepsilon} u|\,.$$

This implies

$$|u' + \frac{f(\Phi)+V}{2\varepsilon}\, u\,|^2 > (1-\delta)|u'\,|^2 - (1/\delta - 1)|f(\Phi)+V\,|^2|u\,|^2/4\varepsilon^2.$$

If $m > |f|$ is an upper bound for $|f|$ on $[-1,1]$, we choose $\delta$ so that $1 < 1/\delta < 1 + 2/(m^2 + O(\varepsilon))$. Integrating the inequality we then find a positive constant c such that

$$\|u\|_h^2 > c \int_{-1}^{1} h^2 \{|\varepsilon u'|^2 + |u|^2\} dx .$$

Because the inverse inequality is evident, the assertion is proved for $h = h_w$. The proof for $h = h_s$ is essentially the same. $\diamondsuit$

**Corollary 2.7** *The norms $\|\cdot\|_{h_w}$ and $\|\cdot\|_{h_s}$ are equivalent.*

**Proof.** Using the equivalence result of proposition 2.6 it is sufficient to find positive constants $c_1$ and $c_2$ such that $c_1 h_w \le h_s \le c_2 h_w$ . So we consider the quotient

$$\frac{h_w^2(x)}{h_s^2(x)} = \frac{\psi'(\eta,0,0)}{\psi'(\eta,C,V)} \quad \text{where} \quad \eta = \frac{x-x_0}{\varepsilon} \quad \text{and} \quad -\frac{1+x_0}{\varepsilon} \le \eta \le \frac{1-x_0}{\varepsilon} .$$

It remains to apply Proposition 2.4. $\diamondsuit$

**Remark 2.8** In these norms weighted by the derivative of the profile, the difference between the shock layer and the traveling wave is exponentially small. Defining $\Phi_s(x,\varepsilon) := \varphi_s(\frac{x-x_0}{\varepsilon})$ and using (2.27) (2.13) and (2.14) we find:

$$\| \Phi_s - \Phi \|_{h_s}^2 = O(R_\varepsilon^2) \int_{-1}^{1} \frac{dx}{\varphi_s'(\frac{x-x_0}{\varepsilon})} = O(\varepsilon R_\varepsilon) . \tag{2.32}$$

# 3 Local stability of the equilibrium solution

In this section we prove results concerning local stability of the equilibrium solution, using the contraction methods from [9]. We start with linearization around the equilibrium and more generally around a traveling wave profile.

## Linearization around a traveling wave profile

With all information on the traveling wave $\Phi$ available, we may consider variations $v$ around it. So we choose

$$v(x,t) = u(x,t) - \Phi(x,\varepsilon). \tag{3.1}$$

We remark that we do not consider variations around $\Phi(x - Vt, \varepsilon)$, because this introduces time-dependent inhomogeneous boundary conditions. Since $\Phi$ satisfies $\varepsilon\Phi'' + f(\Phi)\Phi' = -V\Phi'$, where $V$ stands for the speed $V(\varepsilon, x_0)$, the variation satisfies the equation

$$\begin{aligned} &v_t = -\mathcal{A}v - V\Phi' + g_1(v), \\ &v(x,0) = u_0(x) - \Phi(x,\varepsilon), \quad \text{and} \quad v(\pm 1, t) = 0 , \end{aligned} \tag{3.2}$$

where $\mathcal{A}v := -\varepsilon v_{xx} - f(\Phi)\,v_x - \Phi'\,f'(\Phi)\,v$ is the linearized operator and $g_1$ is the non-linear term,

$$g_1(v) := v^2\,(v_x + \Phi')\,g_2(v) + v\,v_x\,f'(\Phi)\,, \quad g_2(v) := \int_0^1 (1-s)\,f''(\Phi + sv)\,ds\,.$$

From our initial assumption that $f''$ is bounded, it follows that $g_2$ is uniformly bounded too.

The first variation is the linear operator $\mathcal{A}$ of second order, acting on the space $H^2 \cap H_0^1(-1,1)$ of functions on $[-1,1]$ that vanish at $\pm 1$ and have a square integrable second derivative. On this space the operator is selfadjoint, if we equip it with the weighted norm whose weight $\exp\{\frac{1}{\varepsilon}\int f(\Phi(x,\varepsilon))\,dx\}$ is derived from the coefficient of $v_x$. Since from equation (2.3) we have $f(\varphi) = -V - \varphi''/\varphi'$, we may fix the constant of integration in the weight by the choice

$$h^2(x) := \exp\{\tfrac{1}{\varepsilon}\textstyle\int f(\Phi(x,\varepsilon))\,dx\} = \exp\{-\textstyle\int \frac{\varphi''}{\varphi'} + \frac{V}{\varepsilon}\,dx\} = \dfrac{e^{-(x-x_0)V/\varepsilon}}{\varphi'(\frac{x-x_0}{\varepsilon},\varepsilon)}\,. \quad (3.3)$$

This weight $h$ differs only by an exponentially small amount from the weight $h_w$ defined in (2.31). For the analysis of its properties it is better to transform the spatial operator $\mathcal{A}$ to symmetric form with respect to the standard (unweighted) inner product and to stretch the time variable $t \to t/\varepsilon$. So we consider the substitution

$$w(x,t/\varepsilon) := v(x,t)\,h(x)\,. \quad (3.4)$$

Hence $w$ satisfies the equation

$$\begin{aligned} &w_t = \varepsilon^2 w_{xx} - qw + r(w) + g = -Aw + r(w) + g\,,\\ &w(x,0) = w_0(x), \quad w(\pm 1,t) = 0, \end{aligned} \quad (3.5)$$

where $A := -\varepsilon^2\partial_x^2 + q$ is a (standard) selfadjoint "Schrödinger" operator on $H^2 \cap H_0^1(-1,1)$ with "potential" $q$, where $r$ is the non-linear term and $g$ is an "inhomogeneous" term not depending on $w$,

$$\begin{aligned} q(x) &= \tfrac{1}{4}[f(\varphi(\tfrac{x-x_0}{\varepsilon},\varepsilon))]^2 - \tfrac{1}{2}f'(\varphi(\tfrac{x-x_0}{\varepsilon},\varepsilon))\,\varphi'(\tfrac{x-x_0}{\varepsilon},\varepsilon),\\ w_0(x) &= (u_0(x) - \Phi(x,\varepsilon))\,\exp(V\,\tfrac{x-x_0}{\varepsilon})\,[\varphi'(\tfrac{x-x_0}{\varepsilon},\varepsilon)]^{-1/2},\\ g(x) &= -\,\varepsilon V\Phi'\exp(\textstyle\int f(\Phi(x,\varepsilon))dx/2\varepsilon) =\\ &\qquad = -\,V[\varphi'(\tfrac{x-x_0}{\varepsilon},\varepsilon)]^{1/2}\exp(-V\,\tfrac{x-x_0}{2\varepsilon}).\\ r(w) &= g_2(h^{-1}w)\,\left\{\varepsilon\,w^2\,w_x\,h^{-2} - \tfrac{1}{2}\,w^3\,f(\Phi)\,h^{-2} + \varepsilon\,w^2\,h^{-1}\,\Phi'\right\}\\ &\qquad + h^{-1}\,f'(\Phi)\,(\varepsilon\,w\,w_x - \tfrac{1}{2}\,w^2\,f(\Phi)). \end{aligned} \quad (3.6)$$

Note that the inhomogeneous term satisfies

$$g(x) = -V(\varepsilon,x_0)[\varphi'(\frac{x-x_0}{\varepsilon},\varepsilon)]^{1/2}(1 + O(\varepsilon^N)),$$

and consequently,

$$\|g\| = |V(\varepsilon, x_0)|\sqrt{2\varepsilon}(1 + O(\varepsilon^N)), \tag{3.7}$$

where $\|\cdot\|$ stands for the $L^2(-1,1)$ norm. Using the embedding estimate

$$\|u\|_{L_\infty} \le \sqrt{\tfrac{2}{\varepsilon}}\|u\|_1, \ u \in H_0^1(-1,1), \tag{3.8}$$

we can bound the non-linear term $r$ for some constant $a_1 > 0$ by

$$\|r(w)\| < \frac{a_1}{\sqrt{\varepsilon}}\|w\|_1^2 + \frac{a_1}{\varepsilon}\|w\|_1^3 \tag{3.9}$$

and analogously the difference by

$$\|r(v) - r(w)\| < \frac{a_2}{\sqrt{\varepsilon}}\left(\|v\|_1 + \|w\|_1 + \frac{1}{\varepsilon}(\|v\|_1^3 + \|w\|_1^3)\right)\|v - w\|_1 \,. \tag{3.10}$$

As an example of those estimates, we consider one of the worst terms in (3.10). Since $h^{-1}$ is bounded by a constant independent of $\varepsilon$, and since $g_2$ and $g_2'$ are bounded on $\mathbb{R}$, see (1.4), we have (with C a generic positive constant that may differ on each occurrence)

$$\begin{aligned}
&\| h^{-2}\left(g_2(h^{-1}w)\,w^2\,w_x - g_2(h^{-1}u)\,u^2\,u_x\right)\| \le \\
&\quad \le C\|\,g_2(h^{-1}w) - g_2(h^{-1}u)\,\|_{L^\infty}\,\|w^2\|_{L^\infty}\,\|w_x\| \\
&\qquad + C\|g_2(h^{-1}u)\,\|_{L^\infty}\,\|w^2 - u^2\|_{L^\infty}\,\|w_x\| \\
&\qquad + C\|g_2(h^{-1}u)\,\|_{L^\infty}\,\|u^2\|_{L^\infty}\,\|w_x - u_x\| \\
&\quad \le C\|w - u\|_{L^\infty}\,\|w\|_{L^\infty}^2\,\|w_x\| + C\|u^2\|_{L^\infty}\,\|w_x - u_x\| \\
&\qquad + C\|w - u\|_{L^\infty}\,(\|w\|_{L^\infty} + \|u\|_{L^\infty})\|w_x\| \\
&\quad \le C\left\{\varepsilon^{-5/2}(\|w\|_1 + \|u\|_1)^3 + \varepsilon^{-2}(\|w\|_1 + \|u\|_1)^2\right\}\|w - u\|_1 \,.
\end{aligned} \tag{3.11}$$

With the differential operator $A = -\varepsilon^2\partial_x^2 + q$, we may rewrite (3.5) as the Cauchy problem,

$$w_t + Aw = r(w) + g, \ w(x,0) = w_0(x), \quad w \in C^1([0,T], L^2(-1,1)), \tag{3.12}$$

where $r(w)$ is the non-linear part and $g$ the inhomogeneous term. The operator $A$ is an ordinary differential operator of second order with separated boundary conditions on a bounded interval and a bounded potential, so its spectrum $\{\lambda_0(\varepsilon) < \lambda_1(\varepsilon) < \cdots\}$ consists of simple isolated eigenvalues only, and the corresponding set of orthonormal eigenfunctions $\omega_j(x)$ in $L^2(-1,1)$ is complete. Due to the special form of $q$, its bilinear form satisfies

$$(Au, u) = \int_{-1}^{1} |\,\varepsilon u'(x) + \tfrac{1}{2}f(\varphi(\tfrac{x-x_0}{\varepsilon}, \varepsilon))\,u(x)\,|^2 dx\,, \quad \text{for} \quad u \in \mathcal{D}(A), \tag{3.13}$$

implying that $A$ is a positive operator. Let $E_1$ be the orthogonal eigenprojection on $\{\omega_0\}$ and $E_2$ its orthogonal complement,

$$E_1 u = (u, \omega_0)\omega_0\,, \qquad E_2 = id - E_1\,. \tag{3.14}$$

The linear part of (3.12) may be solved via the eigenfunction expansion

$$e^{-tA}u = \sum_{j=0}^{\infty} e^{-t\lambda_j} u_j \omega_j, \qquad u_j = (u, \omega_j) \quad \text{and} \quad \|u\|^2 = \sum_{j=0}^{\infty} u_j^2\,. \tag{3.15}$$

This semigroup $e^{-tA}$ commutes with the projections $E_1$ and $E_2$,

$$e^{-At}E_j = e^{-A_j t}E_j\,, \quad \text{and} \quad e^{-A_1 t}E_1 u = e^{-\lambda_0 t}E_1 u\,, \tag{3.16}$$

where $A_j := AE_j$ $(j = 1\,,\,2)$. We solve the Cauchy problem (3.12) by the strict contraction theorem. To this end we rewrite (3.12) as the integral equation

$$w = Gw\,, \quad \text{where} \quad Gw(\cdot,t) = e^{-tA}w_0 + \int_0^t e^{-A(t-s)}(r(w(\cdot,t)) + g)ds. \tag{3.17}$$

From (2.7) and (2.4) it is clear that $q$ is bounded by some constant $q_0$ , $|q| \le q_0$ uniformly for all $\varepsilon \in (0, \varepsilon_0]$ . Hence, the bilinear form

$$(Au, u) = \|\, A^{\frac{1}{2}} u \,\|^2 = \int_{-1}^{1} (|\varepsilon u'|^2 + q|u|^2)dx\,, \quad u \in \mathcal{D}(A), \tag{3.18}$$

is comparable to the Sobolev norm (1.5),

$$\frac{1}{q_0+1}\,\|u\|_1^2 \le \|\, A^{\frac{1}{2}} u \,\|^2 + \|u\|^2 \le (q_0+1)\|u\|_1^2\,, \quad u \in \mathcal{D}(A), \tag{3.19}$$

uniformly for all $\varepsilon \in (0, \varepsilon_0]$. From the expansion (3.15) we infer that $A^\alpha e^{-tA}$ is a bounded operator on $L^2$ for any $t > 0$ and $\alpha \in \mathbb{R}$:

$$\begin{aligned} \|A^\alpha e^{-tA}u\|^2 &\le \textstyle\sum_{j=0}^{\infty} \lambda_j^{2\alpha} e^{-2t\lambda_j}|u_j|^2 \\ &\le t^{-2\alpha} \max_{s>0} s^{2\alpha}e^{-s} \textstyle\sum_{j=0}^{\infty} e^{-t\lambda_j}|u_j|^2 \\ &\le t^{-2\alpha}(2\alpha)^{2\alpha}e^{-2\alpha}e^{-t\lambda_0}\|u\|^2. \end{aligned} \tag{3.20}$$

Hence, we have for all $t > 0$ and $u \in L^2(-1,1)$

$$\|e^{-tA}u\|_1 \le (t^{-1/2}e^{-\lambda_0 t/2} + \sqrt{q_0+1}e^{-\lambda_0 t})\|u\|\,, \tag{3.21}$$

and analogously, for all $t > 0$ and $u \in H^1(-1,1)$,

$$\|e^{-tA}u\|_1 \le \sqrt{2(q_0+1)}e^{-\lambda_0 t}\|u\|_1\,. \tag{3.22}$$

Results for the operator $A$ can be translated back to the operator $\mathcal{A}$ on the weighted Sobolev space using the identity

$$\mathcal{A}u = \varepsilon^{-1}\, h^{-1}\, A(h\, u). \tag{3.23}$$

Evidently,

$$\mathcal{A}\,h^{-1}\,\omega_j = \varepsilon^{-1}\lambda_j(\varepsilon)\,h^{-1}\,\omega_j\,. \tag{3.24}$$

Finally, note that

$$\|u\|_{h_w}^2 \asymp \|h\mathcal{A}^{1/2}u\|^2 + \|hu\|^2, \tag{3.25}$$

due to the equivalence $h \asymp h_w$ .

## The smallest eigenvalues

In the estimates above, an important role is played by the two smallest eigenvalues of $A$. We derive their asymptotics for $\varepsilon \to 0$ (as always, from above) by the technique developed in [4]. We use the minimax characterization of eigenvalues of a selfadjoint differential operator $B := -d^2/dx^2 + q(x)$ with domain $\mathcal{D}(B) := H^2(I)\cap H_0^1(I)$ of functions on a bounded or unbounded interval $I \subset \mathbb{R}$. See [17], Theorem XIII.1, (p. 76), or [5], XIII.9.D2. If $B$ has isolated eigenvalues $\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$, ordered in increasing sense (and below the continuous spectrum if present), these satisfy

$$\lambda_k = \inf_{E\subset\mathcal{C}\,,\ \dim(E)\ge k+1}\ \max_{u\in E\,,\ \|u\|=1}\ (Bu\,,\,u)\,, \tag{3.26}$$

where $\mathcal{C} := C_0^\infty(I)$ is a core in the domain of the operator.

The eigenvalues of $A$ are invariant under the stretching $x = x_0 + \varepsilon\eta$; the unitary map $U:\ L^2(-1,1) \to L^2(I_\varepsilon)$ given by $U\,u(\eta) := \sqrt{\varepsilon}u(x_0+\varepsilon\eta)$ transforms the eigenvalue equation $Au = \lambda u$ on $[-1,1]$ to

$$A_\varepsilon u := -\frac{d^2u}{d\eta^2} + \widetilde{q}\,u = \lambda\,u\,,\quad \widetilde{q}(\eta) = \tfrac{1}{4}[f(\varphi(\eta,\varepsilon))]^2 - \tfrac{1}{2}f'(\varphi(\eta,\varepsilon))\,\varphi'(\eta,\varepsilon)\,, \tag{3.27}$$

where $u \in \mathcal{D}(A_\varepsilon) := H_0^1 \cap H^2(I_\varepsilon)$, $I_\varepsilon := (-\frac{1+x_0}{\varepsilon}\,,\ \frac{1-x_0}{\varepsilon})$, and where $\widetilde{q}$ depends on $\varepsilon$ only via the constants $C$ and $V$. (See (3.6).) As stated in (3.13), this operator is positive (semi-)definite. To compute asymptotic expressions for the eigenvalues of $A_\varepsilon$, it is convenient to define the approximate operator $B_\varepsilon$ on the same domain,

$$B_\varepsilon\,u := -\frac{d^2u}{d\eta^2} + q_s(\eta)\,u\,,\quad q_s := \tfrac{1}{4}[f(\varphi_s)]^2 - \tfrac{1}{2}f'(\varphi_s)\,\varphi_s' = \widetilde{q} + O(R_\varepsilon)\,, \tag{3.28}$$

where $\varphi$ is replaced by the shock profile $\varphi_s$ , which differs from $\varphi$ by an exponentially small amount, see (2.27). By analogy to (3.13), $B_\varepsilon$ is also a positive operator,

$$(B_\varepsilon u, u) = \int_{I_\varepsilon} |u'(\eta) + \tfrac{1}{2}f(\varphi_s(\eta))\,u(\eta)\,|^2\,d\eta \ge 0\,. \tag{3.29}$$

It is a second order ordinary differential operator with separated boundary conditions on a finite interval (for $\varepsilon > 0$) and its spectrum $\{\mu_0(\varepsilon) < \mu_1(\varepsilon) < \mu_2(\varepsilon) < \cdots\}$ consists solely of isolated eigenvalues of multiplicity one. However, in the limit $\varepsilon \to 0$ they may coalesce into the continuous spectrum. The formal limit of $B_\varepsilon$ as $\varepsilon \to 0$ is the operator $B_0$ on $H^2(\mathbb{R})$ with the usual norm

$\|u\|_2^2 = \int_{\mathbb{R}}(|u''|^2 + |u|^2)dx$ ; clearly the operator $B_0$ does not depend on $\varepsilon$. The bilinear form (3.29) suggests that a solution of $u' + \frac{1}{2}f(\varphi_s)u = 0$ may solve $B_\varepsilon u = 0$. We indeed find from (2.25) that

$$\chi_0(\eta) := \sqrt{\varphi_s'(\eta)} = \sqrt{\varphi_s'(0)}\, \exp\{-\tfrac{1}{2}\int_0^\eta f(\varphi_s(t))dt\} \tag{3.30}$$

satisfies this equation and $B_0\chi_0 = 0$. Moreover, this solution is square integrable on $\mathbb{R}$. Hence zero is an eigenvalue of $B_0$ and $\chi_0$ is the (exact) eigenfunction. However, $\chi_0(\eta)$ is non-zero for all finite values of $\eta$ and is not in the domain of $B_\varepsilon$ for $\varepsilon > 0$.

It is well-known that $B_0$ has a continuous spectrum whose bottom $m$ is the smaller of $\frac{1}{4}[f(1)]^2$ and $\frac{1}{4}[f(-1)]^2$. (See [9], p. 140.) Below this point, $B_0$ has a finite number of isolated eigenvalues $\mu_0(0) = 0 < \mu_1(0) < \cdots$, all of which are simple. Since a function in the core of $B_\delta$ can be extended by zero outside its support into an element of the core of $B_\varepsilon$ for any non-negative $\varepsilon < \delta$, the minimax property implies that each eigenvalue $\mu_k(\varepsilon)$ of $B_\varepsilon$ is decreasing as $\varepsilon$ decreases to 0, and $\mu_k(\varepsilon) \geq \mu_k(0)$. We shall show that $\mu_k(\varepsilon)$ converges to $\mu_k(0)$ or becomes incorporated into the continuous spectrum of $B_0$. Since $(B_\varepsilon u - A_\varepsilon u\,,\, u) = O(R_\varepsilon\|u\|^2)$, the $k$-th eigenvalues of $B_\varepsilon$ and $A_\varepsilon$ differ by an amount of order $O(R_\varepsilon)$ only. So we use $B_\varepsilon$ for the approximation of the eigenvalues of $A$:

**Lemma 3.1** *The zeroth and first eigenvalue of $A$ satisfy:*

$$\lambda_0(\varepsilon) = O(R_\varepsilon) \quad \textit{and} \quad \lambda_1(\varepsilon) = \mu_1(0) + O(\varepsilon^2)\,, \tag{3.31}$$

*where $\mu_1(0)$ is either the first true eigenvalue of the operator $B_0$ or the bottom of the continuous spectrum. Moreover, from below we have a better estimate $\lambda_1(\varepsilon) \geq \mu_1(0) + O(R_\varepsilon)$.*

**Proof.** We already know that $\lambda_0(\varepsilon) \geq 0$, $\lambda_1(\varepsilon) = \mu_1(\varepsilon) + O(R_\varepsilon)$ and $\mu_1(\varepsilon) \geq \mu_1(0) > 0$, where $\mu_1(0)$ is either the first eigenvalue of $B_0$ or the bottom of its continuous spectrum. Hence, we only have to construct upper bounds for $\mu_0$ and $\mu_1$. We begin with an upper bound for $\mu_0$.

The true eigenfunction $\chi_0 = \sqrt{\varphi_s'}$ of $B_0$ is not in the domain of $B_\varepsilon$ for $\varepsilon > 0$, because it is non-zero at the boundaries, albeit very small. From (2.26) we find

$$\chi_0(\tfrac{1-x_0}{\varepsilon}) = \sqrt{f(1)}\, \exp\big(\tfrac{1}{2}\, f(1)\,(A - \tfrac{1-x_0}{\varepsilon})\big)\, \big(1 + O(\exp(-f(1)\,\tfrac{1-x_0}{\varepsilon}))\big)$$

as $\varepsilon \to 0$, and at $-\frac{1+x_0}{\varepsilon}$ we have an analogous expression. We add to $\chi_0$ boundary layer corrections,

$$\widetilde{\chi}_0(\eta) := \chi_0(\eta) - \chi_0(\tfrac{1-x_0}{\varepsilon})\varrho(x_0 + \varepsilon\eta) - \chi_0(-\tfrac{1+x_0}{\varepsilon})\varrho(-x_0 - \varepsilon\eta)\,, \tag{3.32}$$

where $\varrho$ is a monotone $\mathcal{C}^\infty$ cut-off function satisfying $\varrho(x) = 0$ if $x \leq \frac{1+2|x_0|}{3}$ and $\varrho(x) = 1$ if $x \geq \frac{2+|x_0|}{3}$. This corrected function is in the domain of $B_\varepsilon$ and satisfies

$$\frac{(B_\varepsilon\, \widetilde{\chi}_0\,,\, \widetilde{\chi}_0)}{(\widetilde{\chi}_0\,,\, \widetilde{\chi}_0)} = O(R_\varepsilon)\,.$$

The minimax characterization (3.26) implies that this is an upper bound for $\mu_0(\varepsilon)$, and hence that $\lambda_0(\varepsilon)$ is of the same order.

For an upper bound for $\mu_1(\varepsilon)$ we have to distinguish between two cases. If $\mu_1(0) < m$ is a true eigenvalue of $B_0$ with eigenfunction $\chi_1$, this eigenfunction is a solution of the equation $-w'' + q_s w = \mu_1(0)w$, whose solutions for large $\eta$ have the asymptotic behaviour

$$w(\eta) = (\alpha_\pm \exp(\omega_\pm \eta) + \beta_\pm \exp(-\omega_\pm \eta))\,(1 + O(1/\eta)) \qquad (\eta \to \pm\infty),$$

where $\omega_\pm := \sqrt{\frac{1}{4} f(\pm 1)^2 - \mu_1(0)}$. Since $\chi_1$ is square integrable, it must have purely decaying exponentials towards both sides.

Hence, $\chi_1(\eta) = O(\exp(\mp\omega_\pm \eta))$ $(\eta \to \pm\infty)$ if it is normalized to order $O(1)$ in the center of the interval. Hence, choosing

$$\widetilde{\chi}_1(\eta) := \chi_1(\eta) + (\text{boundary layer corrections})$$

as in (3.32), we find an approximate eigenfunction in $\mathcal{D}(B_\varepsilon)$ that satisfies

$$\frac{(B_\varepsilon \widetilde{\chi}_1\,,\, \widetilde{\chi}_1)}{(\widetilde{\chi}_1\,,\, \widetilde{\chi}_1)} = \mu_1(0) + O\left(\exp(\tfrac{1-x_0}{\varepsilon}\,\omega_+) + \exp(-\tfrac{1+x_0}{\varepsilon}\,\omega_-)\right) \quad (\varepsilon \to 0). \quad (3.33)$$

Since $\chi_0$ and $\chi_1$ are orthogonal as functions on $\mathbb{R}$, the functions $\widetilde{\chi}_0$ and $\widetilde{\chi}_1$ are approximately orthogonal, hence the maximum of Rayleigh's quotient over the span of $\{\widetilde{\chi}_0\,,\, \widetilde{\chi}_1\}$ is of the same order as (3.33), such that this is an upper bound for the first eigenvalue.

If $B_0$ does not have other eigenvalues below $m$, then $\mu_1(\varepsilon) \geq m$ for all $\varepsilon > 0$. An upper bound is obtained by restricting the operator $B_\varepsilon$ to functions on an interval $\widetilde{I}_\varepsilon := (\frac{1-x_0}{2\varepsilon}\,,\, \frac{1-x_0}{\varepsilon})$. On such an interval, $\widetilde{q} = m + O(\exp(-\frac{f(1)}{4}\,\frac{1-x_0}{\varepsilon}))$ is almost constant. Again using (3.26), this implies that $\mu_1(\varepsilon)$ is bounded from above by the first eigenvalue of the operator $-d^2/dx^2 + m$ on $H_0^1 \cap H^2(\widetilde{I}_\varepsilon)$ but for an exponentially small term. This first eigenvalue can be computed easily and is of the order $m + O(\varepsilon^2)$. $\diamondsuit$

This lemma implies that the separation between the zeroth eigenvalue of $A$ and the rest of its spectrum is of order unity, such that the computation of precise asymptotics of $\lambda_0$ from an approximate eigenfunction is a well-conditioned problem.

**Lemma 3.2** *The zeroth eigenvalue of $A$ satisfies for every $N \in \mathbb{N}$:*

$$\lambda_0(\varepsilon) = \left[\alpha f(1) e^{-f(1)(1-x_0)/\varepsilon} - \beta f(-1) e^{f(-1)(1+x_0)/\varepsilon}\right](1 + O(\varepsilon^N)), \quad (3.34)$$

*where $\alpha$ and $\beta$ are defined in* (2.7).

**Remark 3.3** This smallest eigenvalue (or better: the asymptotic expression for it) is minimal in the neighborhood of the equilibrium point $x_e(\varepsilon)$. See (2.24).

**Proof.** We use the same technique as in [4]. We compute an approximate eigenfunction $w$ of unit norm $\|w\| = 1$ of the operator $A_\varepsilon$ and we show that

$$(A_\varepsilon w\,,\, w) = \nu_\varepsilon(1 + O(\varepsilon^N)) \quad \text{and} \quad \|A_\varepsilon w\|^2 = O(\varepsilon^N R_\varepsilon)\,. \tag{3.35}$$

The generalized Fourier expansion of $w$ in the true eigenfunctions of $A_\varepsilon$ is

$$w = \sum_{k=0}^{\infty} c_k \omega_k \quad \text{with} \quad \sum_{k=0}^{\infty} c_k^2 = \|w\|^2 = 1\,. \tag{3.36}$$

Since all eigenvalues of $A_\varepsilon$ except $\lambda_0$ are bounded from below by $\mu_1$, we find from (3.35)

$$1 - c_0^2 = \sum_{k=1}^{\infty} c_k^2 \le \mu_1^{-2} \sum_{k=1}^{\infty} \lambda_k^2\, c_k^2 \le \mu_1^{-2}\, \|A_\varepsilon w\|^2 = O(\varepsilon^N R_\varepsilon)\,,$$

implying that $c_0^2 = 1 + O(\varepsilon^N R_\varepsilon)$. The estimate for the inner product in (3.35) now implies that

$$(A_\varepsilon w\,,\, w) - \nu_\varepsilon = c_0^2 \lambda_0 - \nu_\varepsilon + \sum_{k=1}^{\infty} \lambda_k\, c_k^2 = O(\varepsilon^N(\nu_\varepsilon + R_\varepsilon))$$

and hence that $\lambda_0 = \nu_\varepsilon + O(\varepsilon^N(\nu_\varepsilon + R_\varepsilon))$. So it remains to construct a suitable approximate eigenfunction and to prove (3.35) for it. The function $\widetilde{\chi}_0$ defined in (3.32) is not precise enough. By analogy to (3.29), (3.3) and (3.13) we easily verify from (2.3) that

$$\widehat{\chi}_0(\eta) := \exp(\tfrac{1}{2} V\eta)\, \sqrt{\varphi'(\eta, \varepsilon)} = \sqrt{\varphi'(0, \varepsilon)}\, \exp\{-\tfrac{1}{2} \int_0^{\eta} f(\varphi(t, \varepsilon)) dt\} \tag{3.37}$$

is a solution of $u' + \frac{1}{2} f(\varphi) u = 0$ and satisfies $A_\varepsilon \widehat{\chi}_0 = 0$. Its norm satisfies

$$\begin{aligned} \|\widehat{\chi}_0\|^2 &= \int_{I_\varepsilon} \exp(V\eta)\, \varphi'(\eta, \varepsilon)\, d\eta \\ &= \Big[\exp(V\eta)\, \varphi(\eta, \varepsilon)\Big]_{-(1+x_0)/\varepsilon}^{(1-x_0)/\varepsilon} - V \int_{I_\varepsilon} \exp(V\eta)\, \varphi(\eta, \varepsilon)\, d\eta \\ &= 2(1 + O(R_\varepsilon))\,. \end{aligned} \tag{3.38}$$

As the tails are exponentially small but non-zero, we construct boundary layer terms at both endpoints by standard matched asymptotic expansions. Suitable boundary layer corrections at the right and left endpoints are

$$\begin{aligned} h(\eta) &:= \widehat{\chi}_0(\tfrac{1-x_0}{\varepsilon})\, \varrho(x_0 + \varepsilon\eta)\, \exp\Big(\tfrac{f(1)}{2}(\eta - \tfrac{1-x_0}{\varepsilon})\Big)\,, \\ k(\eta) &:= \widehat{\chi}_0(-\tfrac{1+x_0}{\varepsilon})\, \varrho(-x_0 - \varepsilon\eta)\, \exp\Big(\tfrac{f(-1)}{2}(\eta + \tfrac{1+x_0}{\varepsilon})\Big)\,, \end{aligned}$$

where $\varrho$ is the cut-off function defined in (3.32). The function $h$ satisfies:

$$\begin{aligned}
\|h\|^2 &= \widehat{\chi}_0(\tfrac{1-x_0}{\varepsilon})^2 \int_{I_\varepsilon} \exp\left(f(1)(\eta - \tfrac{1-x_0}{\varepsilon})\right)\ \varrho(x_0+\varepsilon\eta)^2\, d\eta \\
&\le \widehat{\chi}_0(\tfrac{1-x_0}{\varepsilon})^2/f(1) = O(R_\varepsilon)\,, \\
A_\varepsilon h &= \widehat{\chi}_0(\tfrac{1-x_0}{\varepsilon}) \exp\left(\tfrac{f(1)}{2}(\eta - \tfrac{1-x_0}{\varepsilon})\right) \times \\
&\qquad \left\{\varrho\ \left(\tfrac14 f(\varphi)^2 - \tfrac12 f'(\varphi)\varphi' - \tfrac14 f(1)^2\right) - \tfrac12 \varepsilon\varrho' f(1) - \varepsilon^2\varrho''\right\} \\
&= \widehat{\chi}_0(\tfrac{1-x_0}{\varepsilon}) \exp\left(\tfrac{f(1)}{2}(\eta - \tfrac{1-x_0}{\varepsilon})\right)\ O\left(R_\varepsilon + \exp\tfrac{f(1)(|x_0|-1)}{6\varepsilon}\right)\,,
\end{aligned}$$

such that $\|A_\varepsilon h\|^2 = O\left(R_\varepsilon\ \exp\frac{f(1)|x_0|-1)}{3\varepsilon}\right)$, and

$$\begin{aligned}
&h' + \tfrac12 f(\varphi)h \\
&= \widehat{\chi}_0(\tfrac{1-x_0}{\varepsilon}) \exp\left(\tfrac{f(1)}{2}(\eta - \tfrac{1-x_0}{\varepsilon})\right)\left(\tfrac12\varrho f(1) + \tfrac12 \varrho f(\varphi) + \varepsilon\varrho'\right) \\
&= f(1)\,\widehat{\chi}_0(\tfrac{1-x_0}{\varepsilon}) \exp\left(\tfrac{f(1)}{2}(\eta - \tfrac{1-x_0}{\varepsilon})\right)\left(1 + O\left(R_\varepsilon + \exp\tfrac{f(1)(|x_0|-1)}{6\varepsilon}\right)\right)\,,
\end{aligned}$$

and for $k$ we have analogous estimates. The term $\exp\frac{f(1)(|x_0|-1)}{6\varepsilon} = O(\varepsilon^N)$ (for any $n \in \mathbb{N}$) in the error terms is due to the choice of the cut-off function. Another choice for $\varrho$ may lead to a smaller term. By construction $w := \widehat{\chi}_0 - h - k$ satisfies:

$$\begin{aligned}
\|w\|^2 &= 2(1+O(R_\varepsilon))\,, \\
\|A_\varepsilon w\|^2 &= \|A_\varepsilon h\|^2 + \|A_\varepsilon k\|^2 = O\left(\varepsilon^N\, R_\varepsilon\right)\,, \\
(A_\varepsilon w, w) &= \|w' + \tfrac12 f(\varphi)w\|^2 \qquad (3.39)\\
&= \|h' + \tfrac12 f(\varphi)h\|^2 + \|k' + \tfrac12 f(\varphi)k\|^2 \\
&= \left(f(1)\,\widehat{\chi}_0(\tfrac{1-x_0}{\varepsilon})^2 - f(-1)\,\widehat{\chi}_0(-\tfrac{1+x_0}{\varepsilon})^2\right)\left(1+O(\varepsilon^N)\right) \\
&= \left(f(1)^2\, e^{-f(1)(1-x_0-\varepsilon A)/\varepsilon} + f(-1)^2\, e^{f(-1)(1+x_0-\varepsilon B)/\varepsilon}\right)(1+O(\varepsilon^N)),
\end{aligned}$$

where $A$ and $B$ are as defined in (2.7). Division of the last formula in (3.39) by the first one yields the desired estimate (3.34). $\diamondsuit$

## Contraction around the equilibrium solution

In the case of variations around the equilibrium solution $\Phi_e$ the velocity $V$ in (3.2) is zero and no inhomogeneous term is present in equation (3.12). Therefore in this subsection we consider only the homogeneous equation (3.12):

$$w_t + Aw = r(w),\ w(x,0) = w_0(x). \qquad (3.40)$$

First, using the contraction methods in [9], Theorem 5.1.1, it is easily seen that the equilibrium $w = 0$ is asymptotically stable:

**Lemma 3.4** *There exist positive constants $c_0$, $c_1$ depending on $f$ only, such that for all functions $w_0 \in H_0^1(-1,1)$ satisfying*

$$\|w_0\|_1 \le c_1 \varrho_1 \,, \quad 0 < \varrho_1 < c_0 \lambda_0(\varepsilon)\sqrt{\varepsilon}\,,$$

*the solution of* (3.40) *exists and satisfies*

$$\|w(\cdot,t)\|_1 \le \varrho_1 e^{-\lambda_0(\varepsilon)t/2}\,, \quad \textit{for all} \quad t>0\,. \tag{3.41}$$

Using the terminology from [9] and [8] we can say that the ball of small radius $\varrho_1$, centered at zero, is stable, that is, the trajectory starting in this ball approaches the equilibrium asymptotically and the rate of convergence is governed by the smallest eigenvalue. Since the gap between the smallest eigenvalue and the rest of the spectrum is of order unity, this suggests we consider in a ball, analogously to [9] and [8] (where an equilibrium of saddle point type is treated), a fast decaying stable submanifold $Y_\varepsilon$ of codimension one, tangent to the range of $E_2$ at zero, and at a distance $O(\varrho^2/\sqrt{\varepsilon})$ from this subspace, such that the trajectory starting from this submanifold is approaching the equilibrium faster and the rate of convergence is governed by the first eigenvalue $\lambda_1(\varepsilon)$.

The method is to show that the integral operator $G$ is a contraction in a suitable neighborhood. In the same way as in [9], page 113, we choose a neighborhood in which the semigroup $e^{-tA}$ is contracting sufficiently fast, namely, the range $\mathcal{R}(E_2)$ of $E_2$, and we show that the rest is drawn into this neighborhood by the non-linear part.

**Lemma 3.5** *There exist positive constants $c_0$, $c_1$, depending on $f$ only, such that for all functions $\omega \in \mathcal{R}(E_2)$ satisfying*

$$\|\omega\|_1 \le c_1 \varrho\,, \quad 0 < \varrho < c_0\sqrt{\varepsilon}\,,$$

*a constant $\kappa(\omega) = O(\varrho^2/\sqrt{\varepsilon})$ exists such that the solution of* (3.40) *with* $w(\cdot,0) = \omega + \kappa(\omega)\omega_0$ *satisfies*

$$\|w(\cdot,t)\|_1 \le \varrho e^{-\lambda_1(\varepsilon)t/2}\,, \quad \textit{for all} \quad t>0\,. \tag{3.42}$$

**Remark 3.6** The submanifold $Y_\varepsilon$ is explicitly given by the formula:

$$Y_\varepsilon := \{\omega + \kappa(\omega)\omega_0 : (\omega,\omega_0) = 0,\ \|\omega\|_1 \le c_1\varrho\}\,,$$

where the function $\kappa$ is Lipschitz in $\omega$.

**Proof.** We use a contraction argument to solve the integral equation (3.17), with $g \equiv 0$ in this special case, in the set

$$S_\varrho := \left\{ w \in C([0,\infty); H_0^1(-1,1)) \;\middle|\; \|w(\cdot,t)\|_1 \le \varrho e^{-\beta t},\ \ t>0 \right\} \tag{3.43}$$

for suitable $\varrho > 0$ and suitable (fixed) $\beta \in (\lambda_0, \lambda_1)$. Suppose that such a solution $w$ of (3.17) exists and satisfies $w(\cdot,0) = \omega + \kappa\omega_0$ and $(\omega\,,\,\omega_0) = 0$. Then we find from (3.16)

$$\begin{aligned} e^{\lambda_0 t}E_1 w(\cdot,t) &= e^{\lambda_0 t - At}\kappa\omega_0 + \int_0^t e^{\lambda_0 t - A(t-s)}E_1 r(w(\cdot,s))\,ds \\ &= \kappa(\omega)\omega_0 + \int_0^t e^{\lambda_0 s}E_1 r(w(\cdot,s))\,ds\,. \end{aligned}$$

Since $\|e^{\lambda_0 t}E_1 w(\cdot,t)\|_1 \le 2\varrho e^{-(\beta-\lambda_0)t} \to 0$ as $t\to\infty$ by the assumption on $w$, the last integral is convergent in $\|\cdot\|_1$−norm as $t\to\infty$. Taking this limit, we find that $w$ satisfies the equation

$$\int_0^\infty e^{\lambda_0 s}E_1 r(w(\cdot,s))\,ds = -\kappa(\omega)\omega_0\,. \tag{3.44}$$

Using this equality and the assumption $E_2\omega = \omega$ , we may rewrite the operator $G$ (see 3.17) in the form for which we can prove contraction,

$$Gw(\cdot,t) = e^{-A_2 t}\omega + \int_0^t e^{A_2(s-t)}E_2 r(w(\cdot,s))\,ds - \int_t^\infty e^{\lambda_0(s-t)}E_1 r(w(\cdot,s))\,ds\,. \tag{3.45}$$

For estimates of (3.45) we may use, instead of (3.21) and (3.22), the better estimates:

$$\begin{aligned} \|e^{-A_2 t}u\|_1 &\le \sqrt{q_0+1}(t^{-1/2}e^{-5\lambda_1 t/8} + e^{-\lambda_1 t})\|u\|, \quad t>0\,, \\ \|e^{-A_2 t}u\|_1 &\le \sqrt{2(q_0+1)}\,e^{-\lambda_1 t}\|u\|_1\,. \end{aligned} \tag{3.46}$$

Since $\|\omega_0\|_1 \le 2$ for $0<\varepsilon<\varepsilon_0$, it follows that $\|E_1 u\|_1 < 2\|u\|$. Furthermore, from (3.9) we get

$$\|r(w(\cdot,t))\| < a_1\,(\frac{\varrho}{\sqrt{\varepsilon}} + \frac{\varrho^2}{\varepsilon})\,\|w(\cdot,t)\|_1\,e^{-\beta t}\,, \quad \text{for all} \quad w\in S_\varrho\,. \tag{3.47}$$

Defining in $S_\varrho$ the norm $\|w\|_S = \sup_{t\ge 0} e^{\beta t}\|w(\cdot,t)\|_1$ , we get

$$\begin{aligned} &\int_0^t (t-s)^{-1/2}e^{5\lambda_1(s-t)/8}\|r(w(\cdot,s))\|\,ds \le \\ &\qquad \le \frac{c\varrho\|w\|_S}{\sqrt{\varepsilon}}\int_0^t (t-s)^{-1/2}e^{5\lambda_1(s-t)/8}e^{-\beta s}\,ds \le \frac{c_2\varrho\,e^{-\beta t}\|w\|_S}{\sqrt{\varepsilon}}. \end{aligned}$$

Analogously we find

$$\begin{aligned} \int_0^t e^{\lambda_1(s-t)}\|r(w(s))\|\,ds &\le \frac{c_2 e^{-\beta t}\varrho\|w\|_S}{\sqrt{\varepsilon}} \\ \int_t^\infty e^{\lambda_0(s-t)}\|r(w(s))\|\,ds &\le \frac{c_2 e^{-\beta t}\varrho\|w\|_S}{\sqrt{\varepsilon}}\,. \end{aligned}$$

Hence, if $\lambda_0 < \beta < 5\lambda_1/8$ , we find

$$\|Gw(\cdot,t)\|_1 \le \sqrt{2(q_0+1)}e^{-\lambda_1 t}\|\omega\|_1 + 3c_2\varrho\|w\|_S e^{-\beta t}/\sqrt{\varepsilon}\,.$$

Choosing $\varrho$ so that $c_2\varrho < \sqrt{\varepsilon}/6$ and taking $\|\omega\|_1 \le c_1\varrho$ we obtain $\|Gw\|_S < \varrho$.

Likewise, from (3.10) we get

$$\|r(v(\cdot,t)) - r(w(\cdot,t))\| \le \frac{c_3\varrho\, e^{-\beta t}}{\sqrt{\varepsilon}}\|v(\cdot,t) - w(\cdot,t)\|_1\,, \quad v,\ w \in S_\varrho\,. \tag{3.48}$$

If $c_3\varrho < \sqrt{\varepsilon}/2$, we find contraction $\|Gv - Gw\|_S < \frac{1}{2}\|v - w\|_S$. This shows that the equation $w = Gw$ has a unique solution $w \in S_\varrho$ for every $\omega \in \mathcal{R}(E_2)$ provided $\|\omega\|_1 \le c_1\varrho$. Since the integral in the left-hand side of (3.44) is convergent, this defines a unique function $\kappa(\omega)$ that clearly satisfies $\kappa(\omega) = O(\varrho^2/\sqrt{\varepsilon})$. Thus the lemma is proved. $\diamondsuit$

Using the results of Lemmas 3.4–3.5, we can improve slightly Lemma 3.5, choosing the initial data $w_0$ in the same ball of radius $\varrho$ and at a distance at most $\varrho_1$ from the fast decaying stable manifold $Y_\varepsilon$:

**Lemma 3.7** *There exist positive constants $c_0$, $c_1$, $c_2$, $c_3$, depending on $f$ only, such that for all functions $\omega \in \mathcal{R}(E_2)$ and all $z_0 \in \mathcal{R}(E_1)$ satisfying*

$$\|\omega\|_1 \le c_1\varrho\,, \quad 0 < \varrho < c_0\sqrt{\varepsilon}\,, \quad \textit{and} \quad \|z_0\|_1 \le c_2\varrho_1\,, \quad 0 < \varrho_1 < c_3\lambda_0\sqrt{\varepsilon}\,,$$

*exists a constant, $\kappa(\omega, z_0) = O(\varrho^2/\sqrt{\varepsilon})$, such that the solution of* (3.40) *starting at $w(\cdot,0) = \omega + \kappa(\omega,z_0)\omega_0 + z_0$ satisfies the estimate*

$$\|w(\cdot,t)\|_1 \le \varrho e^{-\lambda_1(\varepsilon)t/2} + \varrho_1 e^{-\lambda_0(\varepsilon)t/2}\,, \quad \textit{for all} \quad t > 0\,. \tag{3.49}$$

**Proof.** We split the solution in the parts $y$ and $z$, $w = y + z$, satisfying

$$z_t + Az = r(z), \quad z(x,0) = z_0\,, \tag{3.50}$$

and

$$y_t + Ay = r(y+z) - r(z), \quad y(x,0) = y_0 := \omega + \kappa(\omega,z_0)\omega_0\,. \tag{3.51}$$

By Lemma 3.4 we know that the solution of the problem (3.50) satisfies

$$\|z(\cdot,t)\|_1 \le \varrho_1 e^{-\lambda_0(\varepsilon)t/2}. \tag{3.52}$$

For problem (3.51) we may repeat the proof of Lemma 3.5, if we keep in mind that its right hand side can be written as a linear term $y\,r'(z)$ that is exponentially small as a consequence of (3.52), and a non-linear term $r(y+z) - r(z) - y\,r'(z)$ that satisfies the same properties as the non-linear term in Lemma 3.5. So we find the estimate

$$\|y(\cdot,t)\|_1 \le \varrho e^{-\lambda_1(\varepsilon)t/2}. \tag{3.53}$$

The lemma follows from (3.52–3.53). $\diamondsuit$

**Remark 3.8** The function $\kappa(\omega, z_0)$ is Lipschitz and $\kappa(\omega,0) = \kappa(\omega)$. Hence the function $\kappa(\omega, z_0)$ generates a family of submanifolds $Y_\varepsilon(z_0)$ with the same properties as $Y_\varepsilon$.

Now we can state our results about local stability of the equilibrium solution. Namely, as a consequence of Lemmas 3.4, 3.5, and 3.7, we have respectively the following theorems.

**Theorem 3.9** *The equilibrium solution $\Phi_e$ is asymptotically stable: There exist positive constants $C_0$, $C_1$ depending only on $f$, and $\varepsilon_0 > 0$, such that if $u$ is the solution of the problem* (1.1)–(1.3) *and*

$$\|u_0 - \Phi_e(x,\varepsilon)\|_{h_e} \le C_1 \varrho_1 \,, \quad 0 < \varrho_1 < C_0 \lambda_0(\varepsilon)\sqrt{\varepsilon}\,,$$

*then*

$$\|u(\cdot,t) - \Phi_e(x,\varepsilon)\|_{h_e} \le \varrho_1 e^{-\lambda_0(\varepsilon)t/2\varepsilon}\,, \quad \forall\ t > 0 \quad \text{and} \quad 0 < \varepsilon < \varepsilon_0. \tag{3.54}$$

**Proof.** Clearly this theorem follows from Lemma 3.4. We only have to translate by (3.1), (3.4) and (3.24) the result from $w$-variables to $u$-variables; the relation is:

$$w(x,t/\varepsilon) = (u(x,t) - \Phi_e(x,\varepsilon))\, h(x), \tag{3.55}$$

where $h(x) = [\varepsilon \Phi_e'(x,\varepsilon)]^{-1/2} = h_e(x)$. $\diamondsuit$

In order to translate the result of Lemma 3.5, we have first to transform the submanifold $Y_\varepsilon$. Using (3.55) and (3.24), we see that the submanifold $U_\varepsilon$ in the original $u$-variables is given by the formula

$$U_\varepsilon = \{u_0 |\, (u_0 - \Phi_e) h_e \in Y_\varepsilon\}. \tag{3.56}$$

**Theorem 3.10** *The solution $u$ of problem* (1.1)–(1.3) *starting at $u_0 \in U_\varepsilon$ satisfies for all $t > 0$ and $0 < \varepsilon < \varepsilon_0$ the estimate*

$$\|u(\cdot,t) - \Phi_e(x,\varepsilon)\|_{h_e} \le \varrho e^{-\lambda_1(\varepsilon)t/2\varepsilon}\,. \tag{3.57}$$

Finally, from Lemma 3.7 we get the following slight improvement of Theorem 3.10, allowing the initial data $u_0$ to be taken in a small neighborhood of the fast decaying stable manifold $U_\varepsilon$.

**Theorem 3.11** *There exist positive constants $c_0, c_1, c_2, c_3$, depending on $f$ only, such that for all functions $\omega \in \mathcal{R}(E_2)$ and all $z_0 \in \mathcal{R}(E_1)$ satisfying*

$$\|\omega\|_1 \le c_1 \varrho\,, \quad 0 < \varrho < c_0\sqrt{\varepsilon}\,, \quad \text{and} \quad \|z_0\|_1 \le c_2 \varrho_1\,, \quad 0 < \varrho_1 < c_3 \lambda_0 \sqrt{\varepsilon}\,,$$

*a constant $\kappa(\omega, z_0) = O(\varrho^2/\sqrt{\varepsilon})$ exists such that the solution $u$ of the problem* (1.1)–(1.3) *with initial condition $u_0 = \Phi_e + (\omega + \kappa(\omega,z_0)\omega_0 + z_0)/h_e$ satisfies the estimate*

$$\|u(\cdot,t) - \Phi_e\|_{h_e} \le \varrho e^{-\lambda_1(\varepsilon)t/2\varepsilon} + \varrho_1 e^{-\lambda_0(\varepsilon)t/2\varepsilon}\,, \quad \text{for all} \quad t > 0\,. \tag{3.58}$$

**Remark 3.12** We remark that the error in the estimates (3.54), (3.57), and (3.58) could be measured equally well in the norm $\|\cdot\|_{h_{es}}$ based on the shock layer profile.

# 4 Global stability of the equilibrium solution

As the conditions of Theorem 4.4, Ch. VI in [12] are satisfied, the problem (1.1)–(1.3) has a unique classical solution for all $t > 0$. In Theorem 3.9 we have shown that the solution $u$ of (1.1)–(1.3) converges to the equilibrium solution if $u$ starts in a tiny neighborhood of this equilibrium. Using the strong maximum principle for linear parabolic operators ([7]) and techniques of Bernstein and Filippov, we can relax this and show uniform convergence of $u$ and its derivative $u_x$ to the equilibrium state as $t$ tends to infinity for all continuous (and compatible) initial data.

We use the maximum principle in the following form. If $a$ and $c$ are continuous functions in the strip $Q_s := (-1,1) \times (0,s)$, then

$$(\varepsilon\partial_x^2 + a\partial_x - \partial_t)w - cw \geq 0 \quad \text{and} \quad c \geq 0 \quad \text{implies} \quad w(x,t) \leq \sup_{\Gamma_s} w, \tag{4.1}$$

where $\Gamma_s := \{(x,0) \,|\, -1 \leq x \leq 1\} \cup \{(\pm 1,t) \,|\, 0 \leq t \leq s\}$ is the part of the boundary before time $s$. We can extend it to non-linear operators as follows:

**Lemma 4.1** *Let $L$ be the non-linear operator*

$$Lu = \varepsilon\partial_x^2 u + f(u)\partial_x u - \partial_t u\,.$$

*If $Lu$ and $Lv$ are continuous in $Q_s$, and if at least one of the derivatives $\partial_x u$ or $\partial_x v$ is bounded on $\overline{Q}_s$, then*

$$Lu \geq Lv \quad \text{in} \quad Q_s \quad \text{and} \quad u \leq v \quad \text{on} \quad \Gamma_s \quad \textit{imply} \quad u \leq v \quad \text{on} \quad Q_s. \tag{4.2}$$

**Proof.** Consider the case where $\partial_x v$ is bounded, and introduce the function $w := (u-v)e^{-\alpha t}$ for large positive $\alpha$. This function is non-positive on $\Gamma_s$. Since

$$\begin{aligned}(\varepsilon\partial_x^2 + f(u)\partial_x - \partial_t)w &= e^{-\alpha t}(Lu - Lv) + \alpha\, w + e^{-\alpha t}(f(v) - f(u))\,\partial_x v \\ &\geq (\alpha + O(1))w\end{aligned}$$

and $\alpha + O(1) \geq 0$ for large positive $\alpha$ depending on the bound for $\partial_x v$ and the maximal Lipschitz constant of $f$, the maximum principle implies $w \leq 0$ in $Q_s$.

**Corollary 4.2** *Let $u_1$ and $u_2$ be solutions of problem (1.1)–(1.2) whose initial condition satisfies $u_1(x,0) \leq u_2(x,0)$. Then $u_1(x,t) \leq u_2(x,t)$ for all $t \geq 0$.*

As in [1], if the time derivative is non-negative at the initial time $t = 0$ and at the boundary points $x = \pm 1$, then the solution is monotone with respect to $t$.

**Lemma 4.3** *Let the function $u(x,t)$ satisfy the equation $Lu = 0$, the initial condition $u(x,0) = u_0(x)$, where $Lu_0(x) \leq 0$, and the boundary conditions $u(\pm 1,t) = u_\pm(t)$, where $\partial_t u_\pm(t) \leq 0$ and $u_+(0) = u_0(1)$, $u_-(0) = u_0(-1)$. Then $\partial_t u(x,t) \leq 0$.*

**Proof.** Since $u(x,t) \leq u(x,0)$ on $\Gamma_s$, the comparison lemma implies $u(x,t) \leq u(x,0)$. Now $v(x,t) := u(x,t+h)$ satisfies $Lv = 0$ and $v \leq u$ on $\Gamma_s$ for all $h > 0$. Hence the comparison lemma implies $u(x,t+h) \leq u(x,t)$. $\diamondsuit$

To show that the solution of (1.1)–(1.3) has a bounded derivative, we squeeze it between suitable barrier functions. By Bernstein's method we show:

**Lemma 4.4** *Let $u$ be a solution of $Lu = 0$ satisfying for some $a \geq 1$ and $t_0 > 0$ the boundary data*

$$u(x,0) = a\,, \quad \textit{and} \quad u(\pm 1,t) = \begin{cases} \pm 1 & \textit{if} \quad t \geq t_0\,, \\ a & \textit{if} \quad 0 \leq t \leq \frac{1}{2}t_0\,, \end{cases} \tag{4.3}$$

*where $u(\pm 1,t) \in C^3([0,\infty))$ and $\partial_t u(\pm 1,t) \leq 0$. Then $u$ and $u_x$ are bounded on $\overline{Q}_\infty$.*

**Proof.** It is easily seen that the function

$$\psi(x,t) = \tfrac{1}{2}(1-x)u(-1,t) + \tfrac{1}{2}(1+x)u(1,t)$$

is in $C^3(\overline{Q}_\infty)$ and satisfies the same conditions on $\Gamma_\infty$ as $u$ does. Hence, Theorem 6.1, Ch. V in [12] guarantees that $u$ is the unique classical solution and $u \in C^1(\overline{Q}_\infty)$, for which we only have to find the bounds. From the previous lemma it follows that $u$ is non-increasing in time, and result 4.2 implies that $u$ is bounded from below by $-1$, hence $u$ is bounded and $\varepsilon \partial_x^2 u + f(u)\partial_x u \leq 0$. Integrating this inequality we find

$$\varepsilon \partial_x u(1,t) + F(u(1,t)) \leq \varepsilon \partial_x u(x,t) + F(u(x,t)) \leq \varepsilon \partial_x u(-1,t) + F(u(-1,t))\,,$$

so it suffices to show that $\partial_x u(\pm 1,t)$ is bounded. Let $v := u - \psi$, then $v = 0$ on $\Gamma_\infty$ and

$$v_t - \varepsilon v_{xx} = h(x,t) := f(v+\psi)(v_x + \psi_x) - \psi_t\,.$$

Since $u$ is bounded, a positive constant $C$ exists such that $|h(x,t)| \leq C(1+|v_x|)$. Consider $z := e^{kv} - 1 + \lambda e^{-x}$. For sufficiently large constants $k$ (depending on $\varepsilon$) and $\lambda$ (depending on $k$) it satisfies

$$z_t - \varepsilon z_{xx} = ke^{kv}(v_t - \varepsilon v_{xx} - \varepsilon v_x^2) - \lambda e^{-x} \leq ke^{kv}(C + C|v_x| - \varepsilon v_x^2) - \lambda e^{-x} \leq 0\,.$$

The maximum principle implies that $z$ is bounded from above by its maximum at $\Gamma_\infty$. Since

$$z(-1,t) = \lambda e, \quad z(1,t) = \lambda/e, \quad \text{and} \quad z(x,0) \leq \lambda e$$

we have $z(x,t) \leq z(-1,t)$, and as a consequence $z_x(-1,t) \leq 0$. This implies that $v_x$, and hence also $u_x$, is bounded from above uniformly with respect to $t$ at $x = -1$. Analogously we show that it is bounded from above at $x = 1$ and from below at $x = \pm 1$ uniformly with respect to $t$. $\diamondsuit$

**Lemma 4.5** *Let $u$ be the solution of* (1.1)–(1.3). *Then $u_x(x,t)$ is uniformly bounded for $t \geq t_0 > 0$, $|x| \leq 1$ and $x \mapsto u(x,t)$ is in $H^2(-1,1)$ for all $t > 0$.*

**Proof.** Assume $a_1 \le u_0(x) \le a_2$. Clearly this implies $a_1 \le -1$ and $a_2 \ge 1$. Let the functions $u_1$ and $u_2$ be solutions of $Lu_j = 0$ with initial values $u_j(x,0) = a_j$ and monotone boundary values $u(\pm 1, t)$ in $C^3([0,\infty))$ satisfying for some $t_0 > 0$

$$Lu_j = 0\,, \quad \text{and} \quad \begin{cases} u_j(\pm 1,t) = u_j(x,0) = a_j & \text{if} \quad t \le \frac{1}{2}t_0\,, \\ u_j(\pm 1,t) = \pm 1 & \text{if} \quad t \ge t_0\,. \end{cases} \tag{4.4}$$

According to Lemma 4.4, $u_1$ is increasing and $u_2$ is decreasing, and both have a uniformly bounded $x$-derivative. Moreover, from result 4.1 we infer that

$$u_1(x,t) \le u(x,t) \le u_2(x,t). \tag{4.5}$$

Since all three are equal at $x = \pm 1$, $u(\pm 1,t) = u_j(\pm 1,t) = \pm 1$ for $t \ge t_0$, we find for all $t \ge t_0$ the the inequalities

$$\partial_x u_1(-1,t) \le \liminf_{x\to -1} \partial_x u(x,t) \le \limsup_{x\to -1} \partial_x u(x,t) \le \partial_x u_2(-1,t)\,, \tag{4.6}$$

and the analogous estimate at $x = +1$. Thus (4.6) implies that the function $x \mapsto u_x(x,t)$ is bounded in $(-1,1)$, uniformly for all $t \in [t_0, \infty)$. In particular, $x \mapsto u(x,t)$ is in $H^1(-1,1)$ for $t \ge t_0$. Now we consider $u(x,t)$ as a solution to the linear problem

$$w_t = \varepsilon w_{xx} + f(u)w_x\,, \quad w(x,t_0) = u(x,t_0) \quad w(\pm 1,t) = \pm 1\,.$$

According to Theorem 9.1, Ch. IV in [12], the $x$-derivative $u_x$ is an element of $H^1(-1,1) \times H^1[t_0,\infty)$, hence it is continuous on $[-1,1] \times [t_0\,,\,\infty)$. It remains to prove that it is uniformly bounded on this strip. To this end we use Filippov's method (cf. [12] Ch. VI Lemma 5.1) and consider the function $v$ defined by

$$\varepsilon v(x,t) := \varepsilon u_x(x,t) + F(u(x,t)). \tag{4.7}$$

It satisfies the differential equation,

$$v_t = \varepsilon v_{xx} + f(u)v_x\,, \tag{4.8}$$

its initial value $v(\cdot,t_0)$ is continuous on $[-1,1]$, and its boundary values $v(\pm 1,t) = u_x(\pm 1,t)$ are bounded and continuous for $t \ge t_0$. Therefore the maximum principle implies that $v$, and hence $u_x$ too, are uniformly bounded on $[-1,1] \times [t_0\,,\,\infty)$.

**Lemma 4.6** *Let $u$ be the solution of* (1.1)–(1.3). *Then $x \mapsto u(x,t)$ is in $H^2(-1,1)$ and $x \mapsto u_x(x,t)$ is of Hölder class $C^{1/2}[-1,1]$, uniformly for all $t \ge t_0 + 2$, with $t_0$ as above.*

**Proof.** From the previous lemma we already know that $u(\cdot,t) \in H^2(-1,1)$ for every $t > 0$, so only the uniformity is a problem. It is sufficient to prove the lemma for the solution $w(x,t)$ of the equivalent integral equation (3.17) with $g \equiv 0$ in $L^2(-1,1)$:

$$w(\cdot,t) = e^{A(t_0-t)}w(\cdot,t_0) + \int_{t_0}^{t} e^{A(s-t)}r(w(\cdot,s))ds\,,$$

where $\|w(\cdot,t)\|_1$ and, by (3.9), also $\|r(w(\cdot,t))\|$ are uniformly bounded for $t > t_0$. Because of the equivalences

$$\|u\|_1 \asymp \|A^{1/2}u\| + \|u\| \quad \text{and} \quad \|\varepsilon^2 u_{xx}\| + \|u\| \asymp \|Au\| + \|u\|\,,$$

it suffices to show that $\|Aw\|$ is uniformly bounded with respect to $t$. First we establish the Hölder-type estimate

$$\|A^{1/2}(w(\cdot,t+h) - w(\cdot,t))\| \le h^{\delta} \max_{s} (\|w(\cdot,s)\| + \tfrac{2}{1-2\delta}\|r(w(\cdot,s))\|), \tag{4.9}$$

uniformly for all $h > 0$, $\delta \in (0,\frac{1}{2}]$ and $t \ge t_0 + 1$. The integral equation implies

$$\begin{aligned} &A^{1/2}(w(\cdot,t+h) - w(\cdot,t)) = \\ &\qquad = (e^{-Ah}-1)A^{1/2}w(\cdot,t) + \int_{t}^{t+h} A^{1/2}e^{A(s-t-h)}r(w(\cdot,s))\,ds. \end{aligned} \tag{4.10}$$

Using (3.20), the second term of (4.10) is estimated by

$$\begin{aligned} \|\int_{t}^{t+h} A^{1/2}e^{A(s-t-h)}r(w(\cdot,s))\,ds\,\| &\le \int_{t}^{t+h} (t+h-s)^{-\frac{1}{2}}\|r(w(\cdot,s))\|\,ds \\ &\le \tfrac{1}{2}h^{1/2} \max_{s} \|r(w(\cdot,s))\|\,. \end{aligned}$$

In a way analogous to (3.20) we estimate in the first term of (4.10) the difference operator by

$$\|A^{-\alpha}(1-e^{-Ah})u\| \le h^{\alpha}\|u\| \max_{s\ge 0} s^{-\alpha}(1-e^{-s}) \le h^{\alpha}\|u\|\,, \quad \text{if} \quad 0 \le \alpha \le 1\,,$$

and we apply the integral equation and (3.20) again to find for any $\delta \in [0,\frac{1}{2})$,

$$\begin{aligned} &(e^{-Ah}-1)A^{1/2}w(\cdot,t) = \\ &\quad = (e^{-Ah}-1)A^{1/2}e^{A(\tau-t)}w(\cdot,\tau) + \int_{\tau}^{t} (e^{-Ah}-1)A^{1/2}e^{A(s-t)}r(w(\cdot,s))ds \\ &\quad \le h^{1/2}(t-\tau)^{-1}\|w(\cdot,\tau)\| + h^{\delta}\int_{\tau}^{t} (t-s)^{-\frac{1}{2}-\delta}\|r(w(\cdot,s))\|\,ds\,. \end{aligned}$$

If we choose $\tau = t-1 \ge t_0$, this proves (4.9). In order to prove the bound for $\|Aw(\cdot,t)\|$, we use again the integral equation

$$\begin{aligned} Aw(\cdot,t) \;=\; &Ae^{A(\tau-t)}w(x,\tau) + \int_{\tau}^{t} Ae^{A(s-t)}r(w(\cdot,t))\,ds \\ &+ \int_{\tau}^{t} Ae^{A(s-t)}(r(w(\cdot,s)) - r(w(\cdot,t)))ds\,. \end{aligned}$$

The norm of the first term in the right hand side is bounded by $(t-\tau)^{-1}\|w(\cdot,\tau)\|$. In the second term we may integrate explicitly:

$$\|\int_\tau^t Ae^{A(s-t)}r(w(\cdot,t))ds\| = \|(1-e^{A(\tau-t)})r(w(\cdot,t))\| \le \|r(w(\cdot,t))\|\,.$$

Since (3.10) and (4.9) imply the estimate

$$\|r(w(\cdot,s))-r(w(\cdot,t))\| \le C\,\|w(\cdot,s)-w(\cdot,t)\|_1 \le C_\delta|t-s|^\delta\,,$$

we may estimate the norm of the third term by

$$\int_\tau^t Ae^{A(s-t)}(r(w(\cdot,s))-r(w(\cdot,t)))ds \le C_\delta\int_\tau^t (t-s)^{\delta-1}ds \;= C_\delta(t-\tau)^\delta/\delta\,.$$

With the choice $\tau = t-1$ we find that $\|Aw(\cdot,t)\|$ is uniformly bounded for $t \ge t_0+2$. Standard embedding implies that the function $x \mapsto w(x,t)$ is of Hölder class $C^{3/2}[-1,1]$ uniformly if $t-2 \ge t_0 > 0$.

**Remark 4.7** The bound on $\|Aw(\cdot,t)\|$ depends on $\varepsilon$. We did not try to find an optimal one. However, from the estimates used we easily find a rather pessimistic bound of the order $O(\varepsilon^{-5/2})$. If the smoothness of $f$ allows, we may repeat this proof for higher order derivatives of $u$.

**Lemma 4.8** *Let $u$ be the solution of* (1.1)–(1.3)*. Then $u(x,t) \to \Phi_e(x,\varepsilon)$ as $t\to\infty$, uniformly in $x\in[-1,1]$.*

**Proof.** According to (4.5) $u$ is squeezed between $u_1$ and $u_2$, so it is sufficient to prove that both $u_1$ and $u_2$ converge to $\Phi_e$ as $t\to\infty$ uniformly for $x\in[-1,1]$.

Consider the lower bound $u_1$. It is bounded from above for all $t$ by $u_2$ and $t\mapsto u_1(x,t)$ is non-decreasing by Lemma (4.3). Hence it converges pointwise to a limit $\tau(x)$ for every $x\in[-1,1]$. Since the Lemmas 4.5 and 4.6 also apply to $u_1$, the $x$-derivative $\partial_x u_1(x,t)$ is uniformly bounded in $Q_\infty$ and $x\mapsto\partial_x u_1(\cdot,t)$ is of Hölder class $C^{1/2}[-1,1]$, uniformly for $t\ge t_0$. Using the Arzela-Ascoli theorem twice, we conclude that $\tau\in C^1[-1,1]$ and hence

$$u_1(x,t)\to\tau(x) \quad \text{as} \quad t\to\infty, \qquad \text{uniformly in} \qquad x\in[-1,1].$$

In particular, $f(\tau(x))\in C^1[-1,1]$ and $f(u_1(t,x))\to f(\tau(x))$ uniformly in $x\in[-1,1]$. Therefore Theorem 2, Ch. 6 in [7] implies that $u_1(x,t)$ converges for $t\to\infty$ uniformly to the unique solution of

$$\varepsilon\nu''(x)+f(\tau(x))\nu'(x)=0, \qquad \nu(\pm1)=\pm1.$$

Hence $\tau$ satisfies (2.4) with $V=0$, and we conclude $\tau(x)=\Phi_e(x)$.
In the same way, $u_2$ converges to $\Phi_e(x)$ from above. $\diamondsuit$

**Theorem 4.9** *Let $u(x,t)$ be the solution to the problem* (1.1)–(1.3)*, where $u_0$ is continuous. Then*

$$u(x,t)\to\Phi_e(x,\varepsilon) \quad \textit{and} \quad u_x(x,t)\to\Phi_e'(x,\varepsilon) \quad \textit{as} \quad t\to\infty, \tag{4.11}$$

*uniformly in $x\in[-1,1]$ (and uniformly with respect to $u_0$ in a bounded set in $C[-1,1]$).*

**Proof.** We notice that the Arzela-Ascoli theorem and Lemma 4.8 give a sequence $t_n \to \infty$ such that $u_x(x,t_n) \to \Phi_e'(x,\varepsilon)$ uniformly with respect to $x \in [-1,1]$. Since $\Phi_e'(1,\varepsilon) = C - F(\Phi_e(1,\varepsilon)) = C = \Phi_e'(-1,\varepsilon)$ by (1.4) and (2.4), we find in particular that $u_x(\pm 1, t_n) \to C$ as $t_n \to \infty$.

Now we can apply Theorem 2, Ch. 6 in [7] to the linear problem (4.8) and conclude that $v(x,t) \to \omega(x)$ as $t \to \infty$ uniformly in $x \in [-1,1]$, where $\omega$ satisfies the boundary value problem:

$$\varepsilon\omega'' + f(\Phi_e)\omega' = 0, \quad \omega(\pm 1) = C\,.$$

Evidently, $\omega(x) = C$. This and the definition of $v(x,t)$ in (4.7) imply

$$\varepsilon u_x(x,t) \to \varepsilon C - F(\Phi_e(x,\varepsilon)) = \varepsilon \Phi_e'(x,\varepsilon) \quad \text{as} \quad t \to \infty,$$

uniformly in $x \in [-1,1]$. ♢

Combining Theorem 3.9 and Theorem 4.9, we obtain the rate of convergence in (4.11).

**Corollary 4.10** *Under the conditions of Theorem 4.9 we have for some* $t_\varepsilon$,

$$\|u(x,t) - \Phi_e(x,\varepsilon)\|_{h_e} \le \varrho_1 e^{-\lambda_0(t-t_\varepsilon)/2\varepsilon} \quad \text{for all} \quad t \ge t_\varepsilon \tag{4.12}$$

*and in particular,*

$$|u(x,t) - \Phi_e(x,\varepsilon)| h_e(x) \le C\lambda_0 e^{-\lambda_0(t-t_\varepsilon)/2\varepsilon} \quad \text{for all} \quad t \ge t_\varepsilon\,.$$

# 5 Metastability of the slow motion

In this section our goal is to explain the behaviour of the solution when it is still far away from the equilibrium state. We consider only the case when the initial data is near a traveling wave, and prove that the solution moves in a small neighborhood of the traveling wave with exponentially slow speed during an exponentially long (but finite) time interval $(0, T_\varepsilon)$.

In the case of variations around the traveling wave profile $\Phi \ne \Phi_e$, the velocity $V$ in (3.2) is not zero and the inhomogeneous term is present in equation (3.12). Therefore in this subsection we consider only the inhomogeneous equation (3.12):

$$w_t + Aw = r(w) + g,\ w(x,0) = w_0(x). \tag{5.1}$$

Our method is to solve the inhomogeneous integral equation (3.17) and to show that the Sobolev norm of the solution $\|w(\cdot,t)\|_1$ is small enough during an exponentially large time interval $(0,T)$.

To prove the analogue to Lemma 3.5, we first consider the problem (5.1) with zero initial data, which corresponds to the evolution starting at a traveling wave. By Theorem 4.9 we know that the limit of $z(x,t/\varepsilon) := [u(x,t) - \Phi(x,\varepsilon)]\, h(x)$ is the function $[\Phi_e(x) - \Phi(x,\varepsilon)]\, h(x)$, so in general we can expect that the norm $\|z(\cdot,t)\|_1$ will be small only in some finite (but exponentially long) time interval $(0,T)$.

## Evolution starting at a traveling wave profile

Here we consider for given $x_0$ the evolution problem (1.1)–(1.3) starting with the particular initial condition

$$u(x,0) = \Phi(x,\varepsilon) = \psi(\frac{x-x_0}{\varepsilon}; C, V). \tag{5.2}$$

We shall assume that $x_0 < x_e$ and hence $V > 0$. The formulae for $V < 0$ are analogous. First, from the comparison Lemma 4.1 we find immediately:

**Corollary 5.1** *If $x_0 < x_e$ , the solution of* (1.1)–(1.2)–(5.2) *is squeezed between $\Phi(x,\varepsilon)$ and $\Phi(x - Vt, \varepsilon)$ and bounded from below by the equilibrium solution,*

$$\Phi(x,\varepsilon) \geq u(x,t) \geq \Phi(x - Vt, \varepsilon) \quad \textit{and} \quad u(x,t) \geq \Phi_e(x,\varepsilon) \tag{5.3}$$

*for all $t > 0$ and $V > 0$. If $x_0 > x_e$ and, hence, $V < 0$, the inequalities are reversed.*

From Lemma 4.3 we find the monotonicity:

**Corollary 5.2** *The solution $u(x,t)$ of (1.1)–(1.2)–(5.2) is monotone in $t$ :*

$$\begin{aligned} u_t &= (\varepsilon u_x + F(u))_x \leq 0 \quad if \quad V > 0 \\ u_t &= (\varepsilon u_x + F(u))_x \geq 0 \quad if \quad V < 0\,. \end{aligned} \tag{5.4}$$

A bound on the derivative $u_x$ can be derived as follows. In Corollary 5.1 we have shown the inclusion

$$\Phi_e(x,\varepsilon) \leq u(x,t) \leq \Phi(x,\varepsilon) \quad \text{for all} \quad x \quad \text{and} \quad t \quad \text{if} \quad V > 0$$

with equality if $x = \pm 1$ . Hence, if $V > 0$, we have for all $t$

$$\Phi'(1,\varepsilon) \leq u_x(1,t) \leq \Phi_e'(1,\varepsilon)\,, \quad \Phi_e'(-1,\varepsilon) \leq u_x(-1,t) \leq \Phi'(-1,\varepsilon)\,. \tag{5.5}$$

For $V < 0$ the inequality is reversed. Using the monotonicity (5.4) we find the inequality

$$\varepsilon u_x(1,t) \leq \varepsilon u_x(x,t) + F(u) \leq \varepsilon u_x(-1,t) \tag{5.6}$$

and when we eliminate $C$ from the identity $\varepsilon\Phi' + F(\Phi) = -V\Phi + C$ using the values at $\pm 1$ we find

$$\varepsilon\Phi'(x,\varepsilon) + F(\Phi) = \varepsilon\Phi'(\pm 1,\varepsilon) - V(\Phi(x,\varepsilon) - \Phi(\pm 1,\varepsilon))\,.$$

Subtracting both formulae and using (5.5) we get the inequalities

$$\begin{aligned} &\varepsilon(u_x(x,t) - \Phi'(x,\varepsilon)) + F(u(x,t)) - F(\Phi(x,\varepsilon)) \\ &\qquad \begin{cases} \leq \varepsilon(u_x(-1,t) - \Phi'(-1,\varepsilon)) + V(\Phi(x,\varepsilon) - \Phi(-1,\varepsilon))\,, \\ \geq \varepsilon(u_x(1,t) - \Phi'(1,\varepsilon)) + V(\Phi(x,\varepsilon) - \Phi(1,\varepsilon))\,. \end{cases} \end{aligned} \tag{5.7}$$

This estimate, together with (5.5), implies

**Lemma 5.3** *There is a constant $c > 0$, depending only on $f$, such that the solution $u$ of (1.1)–(1.2)–(5.2) satisfies*

$$\varepsilon|u_x(x,t) - \Phi'(x,\varepsilon)| \le c|u(x,t) - \Phi(x,\varepsilon)| + 2|V|. \tag{5.8}$$

Now we can formulate our first result about metastability. It concerns only the special solution starting at a traveling wave profile. We shall see that the solution stays in a small neighborhood of the traveling wave and has almost the same form during an exponentially long time interval.

**Corollary 5.4** *If $|V|t \le \varepsilon$, the solution $u$ of* (1.1)–(1.2)–(5.2) *satisfies the pointwise estimate*

$$\varepsilon|u_x(x,t) - \Phi'(x,\varepsilon)| + |u(x,t) - \Phi(x,\varepsilon)| \le c|V|t\Phi'(x,\varepsilon) + 2|V|, \tag{5.9}$$

*and the estimate in the weighted Sobolev norm,*

$$\|u(\cdot,t) - \Phi\|_{h_w} \le c|V|t/\sqrt{\varepsilon} + c|V|\sqrt{\varepsilon/R_\varepsilon}\,. \tag{5.10}$$

**Proof.** From Corollary 5.1 we have the pointwise estimate

$$|u(x,t) - \Phi(x,\varepsilon)| \le |\Phi(x - Vt,\varepsilon) - \Phi(x,\varepsilon)|, \tag{5.11}$$

and Corollary 2.5 implies

$$|\Phi(x - Vt,\varepsilon) - \Phi(x,\varepsilon)| \le C(\delta)|V|t\Phi'(x,\varepsilon) \quad \text{if} \quad |V|t \le \varepsilon, \tag{5.12}$$

Evidently (5.9) follows from (5.11)–(5.12) and Lemma 5.3. Further, since

$$h_w(x) = [\varepsilon\Phi'(x,\varepsilon)]^{-1/2} \text{ and } h_w(x)/2 \le h(x) \le 2h_w(x) \text{ for } 0 < \varepsilon \le \varepsilon_0(f,\delta),$$

if $x_0 \in [-1+\delta, 1-\delta]$, we get from (5.11) and (5.12) the estimate

$$\|(u(\cdot,t) - \Phi)h_w\| \le c_\delta|V|t/\sqrt{\varepsilon} \quad \text{if} \quad |V|t \le \varepsilon\,. \tag{5.13}$$

Using the bound $\|h_w\| \le c\sqrt{\varepsilon/R_\varepsilon}$ which is uniform with respect to $x_0$, we obtain from Lemma 5.3 the estimate

$$\|\varepsilon(u_x(\cdot,t) - \Phi')h_w\| \le C(\delta)|V|t/\sqrt{\varepsilon} + C(\delta)|V|\sqrt{\varepsilon/R_\varepsilon}$$

for $|V|t \le \varepsilon$. Together with (5.13) this implies (5.10). $\diamondsuit$

**Remark 5.5** We can further improve estimate (5.3) on a finite time interval. If in the case $V > 0$ we choose $w$ in the proof of Lemma 4.1 as

$$w(x,t) := (u(x,t) - \Phi(x + Vt,\varepsilon))e^{-\alpha t}\,, \quad \text{we have} \quad (\varepsilon\partial_x^2 + f(u)\partial_x - \partial_t)w \ge 0\,,$$

provided $\alpha$ is large enough. Hence, $w$ is bounded from above by its value on $\Gamma_t$,

$$\begin{aligned} 0 &\leq u(x,t) - \Phi(x - Vt, \varepsilon) \\ &\leq e^{\alpha t} \max_{0\leq s\leq t} e^{-\alpha s} \max\{1 - \Phi(1 - Vs, \varepsilon), -1 - \Phi(-1 - Vs, \varepsilon)\} \\ &\approx Ve^{\alpha t} \max_{0\leq s\leq t} se^{-\alpha s} \max\{\Phi'(1, \varepsilon), \Phi'(-1, \varepsilon)\}(1 + O(R_\varepsilon)) \\ &\approx Vt \max\{\Phi'(1, \varepsilon), \Phi'(-1, \varepsilon)\}(1 + O(R_\varepsilon)) \end{aligned} \tag{5.14}$$

if $0 \leq t \leq 1/\alpha$ and and $V > 0$.

For $V < 0$ the signs are reversed. Since $\alpha = O(1/\varepsilon)$, this sharper estimate implies only that the onset of the evolution is just the shift of the wave with velocity $V(\varepsilon, x_0)$ towards the equilibrium position.

## Contraction around a traveling wave profile

First, using contraction methods as in [9], Theorem 5.1.1, it is easily seen that we have the following analogue to Lemma 3.4:

**Lemma 5.6** *There exist positive constants $c_1$, $c_2$ depending on $f$ only, such that for all functions $w_0 \in H_0^1(-1, 1)$ satisfying*

$$\|w_0\|_1 \leq c_1\sigma_1\,, \qquad \sigma_1 \asymp \sqrt{\varepsilon|V(\varepsilon, x_0)|}$$

*and for all $0 < t < c_2/\sqrt{|V(\varepsilon, x_0)|}$ the solution of (5.1) satisfies*

$$\|w(\cdot, t)\|_1 \leq \sigma_1\,. \tag{5.15}$$

**Proof.** We solve the integral equation (3.17), showing that $G$ is a contraction in a ball of radius $\sigma_1$. To this end we use (3.9), (3.7), and (3.10) to get the estimates:

$$\|Gw(\cdot, t)\|_1 \leq C\|w_0\|_1 + C\sigma_1^2 T/\sqrt{\varepsilon} + C\sqrt{\varepsilon}|V(\varepsilon, x_0)|T,$$

for $0 < t < T$, and

$$\sup_{0<t<T} \|Gv(\cdot, t) - Gw(\cdot, t)\|_1 \leq CT\sigma_1/\sqrt{\varepsilon} \sup_{0<t<T} \|v(\cdot, t) - w(\cdot, t)\|_1,$$

provided $T > 1$ and $\sigma_1 < \sqrt{\varepsilon}$. Now we choose $T$ and $\sigma_1$ so that:

$$T = c_2/\sqrt{|V|},\ Cc_2\sigma_1 < \sqrt{\varepsilon|V|}/3,\ Cc_2\sqrt{\varepsilon|V|} < \sigma_1/3,$$

for some small $c_2 > 0$. Therefore if $\|w_0\|_1 \leq c_1\sigma_1$ for some small $c_1 > 0$, then

$$\|Gw(\cdot, t)\|_1 \leq \sigma_1$$

for all $0 < t < T$ and

$$\sup_{0<t<T} \|Gv(\cdot, t) - Gw(\cdot, t)\|_1 \leq 1/3 \sup_{0<t<T} \|v(\cdot, t) - w(\cdot, t)\|_1.$$

**Remark 5.7** Another possible choice for $T$ and $\sigma_1$ is:

$$T = c_2/\sqrt{\lambda_0} \quad \text{and} \quad \sigma_1 \asymp \sqrt{\varepsilon\lambda_0}\,.$$

Now we will prove an analogue of Lemma 3.5, using the same technique. However, we first have to single out the inhomogeneous part of the equation and show that it is kept small during an exponentially long time by the nonlinearity. Of course, the initial data $w_0$ have to be taken from the fast decaying stable manifold $Y_\varepsilon$ (cf. remark 3.6).

**Lemma 5.8** *Let $-1+\delta \le x_0 \le 1-\delta$ for some $\delta > 0$. There exist positive constants $C_0$, $C_1$, depending only on $f$ and positive constants $C(\delta), \varepsilon_0 = \varepsilon_0(f,\delta)$ such that if $0 < \varepsilon \le \varepsilon_0$ and*

$$w_0 \in Y_\varepsilon,\ \|w_0\|_1 \le C_1\varrho\,, \quad 0 < \varrho \le C_0\sqrt{\varepsilon}\,, \quad \textit{and} \quad 0 \le t \le \frac{\gamma}{|V(\varepsilon,x_0)|}, \tag{5.16}$$

*where $\gamma$ may be taken arbitrarily in $(0,1)$, then the solution of (5.1) satisfies*

$$\|w(\cdot,t)\|_1 \le \varrho\, e^{-\lambda_1(\varepsilon)t/2} + C(\delta)\, t|V(\varepsilon,x_0)|\sqrt{\varepsilon} + C(\delta)\,|V(\varepsilon,x_0)|\sqrt{\varepsilon/R_\varepsilon}. \tag{5.17}$$

PROOF: We separate the influence of the inhomogeneous term on the solution from the influence of the initial condition by splitting $w = y + z$ and considering first the solution of the inhomogeneous equation with zero initial conditions

$$z_t + Az = r(z) + g\,, \quad z(x,0) = 0 \tag{5.18}$$

and then the remainder:

$$y_t + Ay = \widetilde{r}(y) := r(y+z) - r(z)\,, \quad y(x,0) = w_0(x)\,. \tag{5.19}$$

Equation (5.18) is related to problem (1.1)–(1.2)–(5.2), considered in subsection 5.a, transformed by (3.4) and (3.1):

$$u_t = \varepsilon u_{xx} + f(u)u_x\,, \quad u(x,0) = \Phi(x,\varepsilon), \tag{5.20}$$

where

$$z(x,t/\varepsilon) = [u(x,t) - \Phi(x,\varepsilon)]\,h(x)\,,$$

in particular,

$$z_x(x,t/\varepsilon) = [u_x(x,t) - \Phi'(x,\varepsilon)]\,h(x) + z(x,t/\varepsilon)\,f(\Phi(x,\varepsilon))/2\varepsilon \tag{5.21}$$

From (5.10) we have the estimate

$$\|z(\cdot,t)\|_1 \le C(\delta)|V|t\sqrt{\varepsilon} + C(\delta)|V|\sqrt{\varepsilon/R_\varepsilon} \quad \text{if} \quad |V|t \le 1, \tag{5.22}$$

where $V := V(\varepsilon,x_0)$.

Next we have to solve equation (5.19) with $w_0 \in Y_\varepsilon$. We apply the technique of Lemma 3.5, proving contraction in a ball $S_\varrho$ as defined in (3.43). To this end we extend the function $z$ from (5.19) to the whole time axis by assuming $z(\cdot,t)=z(\cdot,T)$ for all $t>T$, where $T=\beta_0/|V|$ for small $\beta_0>0$. Now we may repeat the proof of Lemma 3.5 by replacing the estimates (3.47) and (3.48) by the corresponding estimates for $\widetilde{r}$. From (3.10) we find a constant $a_3$, depending only on $f$, such that

$$\|\widetilde{r}(y)\| = \|r(y+z)-r(z)\| \le \frac{a_3}{\sqrt{\varepsilon}}\,(\varrho+\|z\|_1)\,\|y\|_1\,, \tag{5.23}$$

if $y\in S_\varrho$ and $0<\varrho\le C\sqrt{\varepsilon}$. The term $\|z\|_1$ can be bounded by $c\sqrt{\varepsilon}$ with small $c>0$, depending on $\beta_0$ and $\delta$. Likewise, we find

$$\begin{aligned} \|\widetilde{r}(u(\cdot,t))-\widetilde{r}(v(\cdot,t))\| &= \|r(u(\cdot,t)+z(\cdot,t))-r(v(\cdot,t)+z(\cdot,t))\| \\ &\le \frac{a_3}{\sqrt{\varepsilon}}\,(\varrho+\|z(\cdot,t)\|_1)\,\|u(\cdot,t)-v(\cdot,t)\|_1\,, \end{aligned} \tag{5.24}$$

if $u$ and $v\in S_\varrho$ and $0<\varrho\le C\sqrt{\varepsilon}$. As before, these inequalities imply the existence of a constant $C_0$, depending only on $f$, such that $G$ is a contraction inside $S_\varrho$ if $0<\varrho\le C_0\sqrt{\varepsilon}$. ♢

Finally, we can prove a variant of Lemma 5.8, allowing the initial data $w_0$ to run in a small strip around the manifold $Y_\varepsilon$ (see remark 3.6), arguing analogously to the proof of Lemma 3.7.

**Lemma 5.9** *There exist positive constants $c_0$, $c_1$, $c_2$, depending on $f$ only, such that for all functions $\omega\in\mathcal{R}(E_2)$ and all $z_0\in\mathcal{R}(E_1)$ satisfying*

$$\|\omega\|_1\le c_1\varrho\,,\quad 0<\varrho<c_0\sqrt{\varepsilon}\,,\quad \textit{and}\quad \|z_0\|_1\le c_2\sigma_1\,,\quad \sigma_1\asymp\sqrt{\varepsilon|V(\varepsilon,x_0)|}\,,$$

*a constant $\kappa(\omega,z_0)=O(\varrho^2/\sqrt{\varepsilon})$ exists such that the solution of* (5.1) *starting at $w(\cdot,0)=\omega+\kappa(\omega,z_0)\omega_0+z_0$ satisfies*

$$\|w(\cdot,t)\|_1\le\varrho e^{-\lambda_1(\varepsilon)t/2}+\sigma_1,\quad \textit{for all}\quad 0<t<c_3/\sqrt{|V(\varepsilon,x_0)|}. \tag{5.25}$$

Proof. We split the solution in parts $y$ and $z$, $w=y+z$, satisfying

$$z_t+Az=r(z)+g,\ z(x,0)=z_0, \tag{5.26}$$

and

$$y_t+Ay=r(y+z)-r(z),\ y(x,o)=y_0:=\omega+\kappa(\omega,z_0)\omega_0, \tag{5.27}$$

By Lemma 5.6 we know that the solution of the problem (5.26) satisfies

$$\|z(\cdot,t)\|_1\le\sigma_1\quad \text{for all}\quad 0<t<c_4/\sqrt{|V(\varepsilon,x_0)|}\,. \tag{5.28}$$

Now repeating the proof of Lemma 3.5, using (5.28) as in the proof of Lemma 5.8 while solving (5.19), we find the estimate

$$\|y(\cdot,t)\|_1\le\varrho e^{-\lambda_1(\varepsilon)t/2}\quad \text{for all}\quad 0<t<c_4/\sqrt{|V(\varepsilon,x_0)|}. \tag{5.29}$$

The lemma follows from (5.28)–(5.29). ♢

## Evolution starting near a traveling wave profile

Now we can state our results about metastability of the slow motion. This means that the solution starting near a traveling wave profile stays in its small neighborhood during an exponentially long time interval, the speed of movement being exponentially small for small $\varepsilon$. Namely, as a consequence of Lemmas 5.6, 5.8, and 5.9, respectively, we have the following theorems.

**Theorem 5.10** *There exist positive constants* $C_1$, $C_2$, *depending only on* $f$, *such that for all functions* $u_0 \in H^1(-1,1)$ *satisfying*

$$\|u_0 - \Phi\|_{h_w} \le C_1 \sigma_1 \,, \qquad \sigma_1 \asymp \sqrt{\varepsilon |V(\varepsilon, x_0)|}$$

*and for all* $0 < t < \varepsilon C_2 / \sqrt{|V(\varepsilon, x_0)|}$ *the solution of* (1.1)–(1.3) *satisfies*

$$\|u(\cdot,t) - \Phi\|_{h_w} \le \sigma_1 . \tag{5.30}$$

**Proof.** This theorem follows from Lemma 5.6. We only have to translate the result from $w-$coordinates to $u-$coordinates using the relation:

$$w(x, t/\varepsilon) = (u(x,t) - \Phi(x,\varepsilon))\, h(x) \,, \tag{5.31}$$

where $h \asymp h_w$. $\diamondsuit$

In order to translate the result of Lemma 5.8, we have first to translate the submanifold $Y_\varepsilon$. Using (5.31) we see that the new submanifold, $U_\varepsilon$, is given by the formula

$$U_\varepsilon = \{u_0 | (u_0 - \Phi)\, h \in Y_\varepsilon\}. \tag{5.32}$$

**Theorem 5.11** *Assume* $-1+\delta \le x_0 \le 1-\delta$ *for some* $\delta > 0$. *There exist positive constants* $C_0$, $C_1$, *depending only on* $f$, *and positive constants* $C(\delta), \varepsilon_0 = \varepsilon_0(\delta)$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$, *for all initial values* $u_0 \in U_\varepsilon$ *and for all* $t$ *satisfying*

$$\|u_0 - \Phi\|_{h_w} \le C_1 \varrho \,, \quad 0 < \varrho \le C_0 \sqrt{\varepsilon} \,, \quad \textit{and} \quad 0 \le t \le \frac{\varepsilon}{|V(\varepsilon, x_0)|} \,,$$

*the solution* $u$ *of* (1.1)–(1.3) *satisfies*

$$\|u(\cdot,t) - \Phi\|_{h_w} \le \varrho\, e^{-\lambda_1(\varepsilon) t / 2\varepsilon} + C(\delta)\, |V(\varepsilon, x_0)| \left( \frac{t}{\sqrt{\varepsilon}} + \sqrt{\frac{\varepsilon}{R_\varepsilon}} \right) . \tag{5.33}$$

Finally, from Lemma 5.9 we get the following slight improvement of Theorem 5.11, allowing the initial data $u_0$ to be taken in a small strip around the fast decaying stable manifold $U_\varepsilon$.

**Theorem 5.12** *Assume* $-1+\delta \le x_0 \le 1-\delta$ *for some* $\delta > 0$. *There exist positive constants* $c_0$, $c_1$, $c_2$, *depending only on* $f$, *such that for all functions* $\omega \in \mathcal{R}(E_2)$ *and all* $z_0 \in \mathcal{R}(E_1)$ *satisfying*

$$\|\omega\|_1 \le c_1 \varrho \,, \quad 0 < \varrho < c_0 \sqrt{\varepsilon} \,, \quad \textit{and} \quad \|z_0\|_1 \le c_2 \sigma_1 \,, \quad \sigma_1 \asymp \sqrt{\varepsilon |V(\varepsilon, x_0)|} \,,$$

*the solution of* (1.1)–(1.3) *with initial condition* $u_0 = \Phi + (\omega + \kappa(\omega, z_0)\omega_0 + z_0)/h$ *satisfies the estimate*

$$\|u(\cdot,t) - \Phi\|_{h_w} \le \varrho e^{-\lambda_1(\varepsilon) t / 2\varepsilon} + \sigma_1, \quad \text{for all} \quad 0 < t < c_4 \varepsilon / \sqrt{|V(\varepsilon, x_0)|}. \tag{5.34}$$

## 6 Conclusions

In this paper we explained the behaviour of the solution at infinity where it approaches the equilibrium. Concerning the previous stages, we were able to explain that behaviour only for the solutions starting near a traveling wave profile. We proved that such a solution moves in a small neighborhood of the traveling wave profile with exponentially small speed during an exponentially long time interval. In a future paper we are going to explain the behaviour of the solution, starting from more general data, during the whole time interval $(0, T)$ for any $T > 0$. We expect that during an interval $(0, T_1)$ multiple interfaces are created comparatively rapidly, which during the next time interval $(T_1, T_2)$ coalesce with a tempered speed so that finally only one internal layer is left. After that moment, the solution is near a traveling wave profile, and, during the next time interval $(T_2, T)$, it moves with an exponentially small speed that decreases very slowly until the solution reaches a neighborhood of the equilibrium with an almost zero speed.

## References


[1] D.G. Aronson & H.F. Weinberger, *Nonlinear diffusion in popular genetics, combustion and nerve impulse propagation*, Lecture Notes in Mathematics, **446**, pp. 5-49, 1975, Springer, New York.

[2] A. Bronsard & D. Hilhorst, *On the slow dynamics for the Cahn-Hilliard equation in one space-dimension*, Proc. R. Soc. Lond. A, **439**, pp. 669–682, 1992.

[3] J. Carr & R.L. Pego, *Metastable Patterns in Solutions of* $u_t = \varepsilon^2 u_{xx} - f(u)$ Comm. Pure Appl. Math, XLII, pp. 523–576, 1989.

[4] P.P.N. de Groen, *The nature of resonance in a singular perturbation problem of turning point type*, SIAM J. Math. Anal., **11**, pp. 1 – 22, 1980.

[5] N. Dunford & J.T. Schwartz, *Linear Operators, part II, Spectral theory*, Interscience Publ., New York, 1963.

[6] A. Friedman, *Asymptotic behaviour of solution of parabolic equations of any order*, Acta Math., **106**, pp. 1–43, 1961.

[7] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Englewood Cliffs, N.J., 1964.

[8] J. Hale, *Ordinary Differential Equations*, Wiley-Interscience, New-York, 1969.

[9] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics **840**, Springer-Verlag, Berlin, 1981.

[10] D. Hilhorst, *On some nonlinear problems arising in the physics of ionised gases*, Chapter III: *A nonlinear evolution problem arising in the physics of ionised gases*, pp. 43–80, Mathematisch Centrum, Amsterdam, 1981.

[11] G. Kreiss & H. Kreiss, *Convergence to steady state of solutions of Burgers' equation*, Appl. Numer. Math. **2**, pp.161–179, 1986.

[12] O.A. Ladyzhenskaja, V.A. Solonnikov, N.N. Uraltzeva, *Linear and Quasi-linear Equations of Parabolic Type*, Nauka, Moscow, 1967 (Russian); English translation: AMS, Providence RI, 1968.

[13] J.G.L. Laforgue & R.E. O'Malley, *Shock Layer Movement for Burgers' Equation*, SIAM J. Appl. Math. **55**, pp. 332–347, 1995.

[14] J.G.L. Laforgue & R.E. O'Malley, *On the Motion of Viscous Shocks and the Supersensitivity of Their Steady-State Limits*, Meth. Appl. Anal., **1**, pp. 456–487, 1994.

[15] A. Lyberopoulos, *Asymptotic oscillations of solutions of scalar conservation laws with convexity under the action of linear excitation*, Quart. Appl. Math, **48**, pp. 755–765, 1991.

[16] C. Mascia & C. Sinestrari, *The perturbed Riemann problem for a balance law*, Advances in Differential Equations, Vol **2**, pp. 779-810, 1997.

[17] M. Reed & B. Simon, *Methods of modern mathematical physics* **4***, Analysis of operators* , Academic Press, London, 1978. - 396 p. - ISBN 0125850042

[18] L.G. Reyna & M.J. Ward, *On the Exponentially Slow Motion of a Viscous Shock*, Commun. on Pure and Applied Mathem. XLVIII, pp. 79-120, 1995.

[19] J.- F. Mallordy & J.- M. Roquejoffre, *A Parabolic Equation of the KPP Type in Higher Dimensions*, SIAM J. Math. Anal., **26**, pp. 1–20, 1995.

[20] M.J. Ward, *Metastable Patterns, Layer Collapses, and Coarsening for a One-Dimensional Ginzburg-Landau Equation*, Studies in Applied Mathematics, **91**, pp. 51–93, 1994.

[21] M.J. Ward, *Eliminating Indeterminacy in Singularly Perturbed Boundary Value Problems with Translation Invariant Potentials*, Studies in Applied Mathematics, **87**, pp. 95–134, 1992.

P. P. N. de Groen
Vrije Universiteit Brussel
Department of Mathematics
Pleinlaan 2,
B–1050, Brussels, Belgium
E-mail: pdegroen@vub.ac.be

G. E. Karadzhov
Bulgarian Academy of Sciences
Institute of Mathematics
and Informatics
Sofia, Bulgaria
E-mail: geremika@math.bas.bg