Electronic Journal of Differential Equations, Vol. **1999**(1999) No. 24, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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EXISTENCE RESULTS FOR BOUNDARY PROBLEMS FOR UNIFORMLY ELLIPTIC AND PARABOLIC FULLY NONLINEAR EQUATIONS

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Abstract

We study existence of continuous weak (viscosity) solutions of Dirichlet and Cauchy-Dirichlet problems for fully nonlinear uniformly elliptic and parabolic equations. Two types of results are obtained in contexts where uniqueness of solutions fails or is unknown. For equations with merely measurable coefficients we prove solvability of the problem, while in the continuous case we construct maximal and minimal solutions. Necessary barriers on external cones are also constructed.

0. INTRODUCTION

The main results of this note concern existence of continuous solutions of the Dirichlet problem for fully nonlinear elliptic equations as well as parabolic variants. To illustrate the issues, we consider the Isaacs' equation

$$\sup_\alpha \inf_\beta \Big(-\sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x) u_{x_i,x_j}(x) + \sum_{j=1}^n b_j^{\alpha,\beta}(x) u_{x_j}(x) + c^{\alpha,\beta}(x)u(x) - f^{\alpha,\beta}(x)\Big) = 0 \tag{0.1}$$

in a bounded open domain $\Omega \subset \mathbb{R}^n$ coupled with the Dirichlet condition

$$u(x) = \psi(x) \quad \text{for} \quad x \in \partial\Omega \tag{0.2}$$

in two situations. In both cases the indices α, β can range over countable sets while the symmetric matrices $A^{\alpha,\beta} = \big(a_{i,j}^{\alpha,\beta}\big)$, the vectors $b^{\alpha,\beta} = (b_1^{\alpha,\beta}, \dots, b_n^{\alpha,\beta})$, and the functions $c^{\alpha,\beta}$ satisfy

$$\lambda I \le A^{\alpha,\beta}(x) \le \Lambda I \tag{0.3}$$

for some positive constants $0 < \lambda \le \Lambda$, and

$$|b^{\alpha,\beta}(x)| \le \gamma, \qquad 0 \le c^{\alpha,\beta}(x) \le \gamma \tag{0.4}$$

for some constant γ, both uniformly in α, β. In the first situation, the functions $A^{\alpha,\beta}$, $b^{\alpha,\beta}$, $c^{\alpha,\beta}$ and $f^{\alpha,\beta}$ are equicontinuous and equibounded on $\overline{\Omega}$; we will call this the *continuous coefficient case.* The continuous coefficient case stands in contrast to the *measurable coefficient case* which assumes (0.3), (0.4), the mere measurability of the data $A^{\alpha,\beta}, b^{\alpha,\beta}, c^{\alpha,\beta}, f^{\alpha,\beta}$, and the technical but essential condition $\inf_\alpha \sup_\beta f^{\alpha,\beta} \in$

1991 *Subject Classification:* 35J25, 35J60, 35J65, 35K20, 35K55, 35K60, 49L25.
Key words and phrases: Uniformly elliptic and parabolic equations, viscosity solutions, good solutions, exterior cone condition, barrier functions.

Submitted May 20, 1999. Published July 1, 1999.

$L^n(\Omega)$. In both cases, the domain Ω satisfies a uniform exterior cone condition while $\psi \in C(\partial\Omega)$.

As far as we know, no existence result in the literature covers the Dirichlet problem for (0.1) for the continuous coefficient case in the generality stated. When speaking of solutions in this note, we mean continuous viscosity solutions. For background color, we note that the continuous coefficient linear equation

$$-\sum_{i,j=1}^{n} a_{i,j}(x)u_{x_i,x_j}(x) + \sum_{j=1}^{n} b_j(x)u_{x_j}(x) + c(x)u(x) - f(x) = 0 \qquad (0.5)$$

is a special case of (0.1). Of course, the Dirichlet problem for (0.5) has a unique strong (here meaning $W^{2,n}_{\mathrm{loc}}$ pointwise a.e.) solution, which by [4], Theorem 2.10 and Proposition 2.9, is a unique viscosity solution. However, it is an interesting artifact of the history of the subject that there seems to be no quotable direct statement of the existence of viscosity solutions of the Dirichlet problem in the literature covering this case. However, there are "good solutions" - see e.g. Cerutti, Fabes and Manselli [6] - and "good solutions" are standard viscosity solutions in the continuous coefficient linear case. The current note handles the general Isaacs' equations in a similar manner, and puts the matter in some perspective. Moreover, we treat the parabolic analogue as well. Finally, existence is proved for the measurable coefficient case in both the elliptic and parabolic settings.

We will use two viscosity solution contexts. The continuous coefficient case lies within the "classical" viscosity solutions theory outlined in [8]. The measurable coefficient case requires infrastructure from Caffarelli, Crandall, Kocan and Święch [4], and Crandall, Fok, Kocan and Święch [7]. The body of the paper is organized to accommodate the reader who is not interested in the more technical measurable coefficient case at this time.

The outline of our method is standard. In both cases, a fully nonlinear equation $F = 0$ including the Isaacs' equation as a special case is treated. The equation $F = 0$ is approximated by better equations $F^\epsilon = 0$ for which the Dirichlet problem is uniquely solvable, the solutions of the approximate problems are uniformly bounded and equicontinuous, and the approximations were set up so that available results guarantee that the original problem is solved by uniform limits of solutions of the approximate problems. This last step uses the appropriate result from viscosity solutions theory, which varies between the two cases.

In the continuous coefficient case, we exploit the additional structure to bracket the original equation by approximations $F_\epsilon \le F \le F^\epsilon$ which are monotone in the parameter ϵ. This automatically constructs maximal and minimal solutions of the original Dirichlet problem. The approximation process is interesting and replaces linear equations by nonlinear equations. In the measurable coefficient case (which of course includes the continuous coefficient case), approximation is by simple mollification of the equation in the independent variables. We remark that as this paper goes to print R. Jensen and A. Święch have a paper in preparation which establishes the existence of maximal and minimal solutions in the measurable case using different arguments than employed by us in the continuous case.

Part of our motivation arises from the desire to quote these results elsewhere. Part of our motivation is that others should have a quotable source for these results.

But there is a bit more to the matter than that. For example, if the viscosity solution framework is to provide a basic existence platform for uniformly elliptic equations (in particular, we mean to exclude the use of second derivative estimates on solutions of the equation under discussion, as these are not available in general), as perhaps it should, then certainly it is nice to have the linear equation appear as special case of a general nonlinear result for (0.1) with little cost for the added generality.

The existence of continuous solutions is historically linked to uniqueness via Ishii's implementation of Perron's method (e.g., [8], Section 4). The issue of uniqueness of viscosity solutions of Isaacs' equation fans out in two directions. If additional restrictions are put on the coefficients beyond those of the continuous coefficient case, one can establish uniqueness and then existence by standard viscosity solutions theory (see Ishii and Lions [15] and [8]) even for some degenerate equations ((0.3) need not hold). If the equation is uniformly elliptic and either convex or concave in the Hessian matrix – as is the case for (0.1) when $a^{\alpha,\beta}$ is independent of β – the existence of strong solutions in the continuous coefficient case is a consequence of Caffarelli's estimates [2] (see Caffarelli and Cabré [3]) and some further arguments, see [4]. Święch [28], Theorem 3.1, provides a sufficiently general statement. In the presence of strong solutions, viscosity solutions are unique as noted above. For other available results on the existence of continuous viscosity solutions we refer to [8], [15] and Trudinger [31].

Existence for the problems studied here is decoupled from uniqueness and higher regularity by means of simple approximations and compactness arguments. Uniqueness in general remains an interesting issue for the continuous coefficient case of Isaacs' equation and fails even for the linear equation in the measurable coefficient case according to Nadirashvili [24], see also Safonov [27]. Further comments on uniqueness issues can be found in Section 1.

Our results in the measurable coefficient case may be regarded as a fully nonlinear generalization of the well-known "good solution" existence theory for the linear problem (0.5), (0.2); see Cerrutti, Escauriaza and Fabes [5]. Existence for the linear problem in the good solution framework is demonstrated by smoothing the coefficients of the problem, using the estimates of Krylov and Safonov [19], [14] to obtain compactness and defining by fiat the limit of strong solutions of approximate problems to be a good solution. We follow this outline, but do not have regular solutions available at any stage of approximation. Thus we rely on viscosity solution theory to provide solutions of approximate problems and on a suitable intrinsic notion of solution of the equation itself when passing to the limit.

Section 1 contains some preliminaries and the statement of the result on maximal and minimal solutions in the continuous coefficient case. Section 2 contains the proof of existence together with its parabolic analogue. An ingredient used in the proof of Section 2 is the existence of subsolutions and supersolutions. These are constructed in Section 3. We present explicit constructions here since a quotable reference is needed and the literature concerning this issue under the exterior cone condition is a little hazy (see Remark 3.3 and various comments made in Section 3). Section 4 treats the measurable coefficient case.

1. PRELIMINARIES

The equation (0.1) can be written as $F(x, u(x), Du(x), D^2u(x)) = 0$, where

$F(x,r,p,X)$ is defined for $x \in \overline{\Omega}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $X \in \mathcal{S}(n)$ (the set of symmetric $n \times n$ real matrices) by

$$F(x,r,p,X) = \sup_{\alpha} \inf_{\beta} \left(-\operatorname{Trace}(A^{\alpha,\beta}(x)X) + \langle b^{\alpha,\beta}(x), p\rangle + c^{\alpha,\beta}(x)r - f^{\alpha,\beta}(x) \right). \tag{1.1}$$

Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner-product.

Properties of F guaranteed by (0.3) and (0.4) are recalled next. Let X^+, X^- denote the positive and negative parts of $X \in \mathcal{S}(n)$; e.g. $X = X^+ - X^-$ and $\operatorname{Trace}(X^+)$ is the sum of the positive eigenvalues of X. Let

$$\mathcal{P}^+(X) = -\lambda \operatorname{Trace}(X^+) + \Lambda \operatorname{Trace}(X^-), \quad \mathcal{P}^-(X) = -\Lambda \operatorname{Trace}(X^+) + \lambda \operatorname{Trace}(X^-)$$

be the "Pucci extremal operators". It is standard and straightforward to show that F satisfies the following conditions:

$$\mathcal{P}^-(X-Y) - \gamma|p-q| \le F(x,r,p,X) - F(x,r,q,Y) \le \mathcal{P}^+(X-Y) + \gamma|p-q| \tag{1.2}$$

for $x \in \overline{\Omega}$, $r \in \mathbb{R}$, $p, q \in \mathbb{R}^n$ and $X, Y \in \mathcal{S}(n)$, and

$$F(x,r,p,X) \quad \text{is nondecreasing in} \quad r. \tag{1.3}$$

To prove (1.2), one first treats a single linear operator and then observes that the inequalities are preserved under sup-infs. For convenience, we adopt the shorthand:

$$F \in \mathcal{SC} \iff (1.2) \ \& \ (1.3) \quad \text{hold.}$$

The parameters λ, Λ, γ are fixed throughout the discussion. $\mathcal{SC}$ corresponds to "structure conditions".

The first inequality of (1.2) with $p = q$ shows that $F(x,r,p,X)$ is nonincreasing in X, which together with the monotonicity in r is the meaning of "F is proper" in the language of [8]. When F is proper and continuous, [8] outlines the basic theory of merely continuous viscosity solutions of $F = 0$.

Our main result in the continuous coefficient elliptic case concerns the Dirichlet problem (DP) below. In (DP) and everywhere else, Ω is assumed to be a bounded open domain in $\mathbb{R}^n$, $n \ge 2$.

$$F(x,u,Du,D^2u) = 0 \quad \text{in} \quad \Omega, \quad u = \psi \quad \text{on} \quad \partial\Omega. \tag{DP}$$

Terminology used in the statement is explained following it.

Theorem 1.1. *Let Ω satisfy a uniform exterior cone condition, $F \in \mathcal{SC}$ and $\psi \in C(\partial\Omega)$. Assume also that F is continuous. Then there are C-viscosity solutions $\overline{u}$, $\underline{u} \in C(\overline{\Omega})$ of (DP) such that any other C-viscosity solution u of (DP) satisfies $\underline{u} \le u \le \overline{u}$.*

We note again that neither the theory of [8] nor additional results of [15] applicable to the case $F \in \mathcal{SC}$ provide uniqueness of solutions of the Dirichlet problem for continuous $F \in \mathcal{SC}$; the question seems to be open. This is one reason the existence issue has not been treated to date. It remains possible that uniqueness holds and

$\underline{u} = u = \overline{u}$. Further comments on the current situation regarding uniqueness can be found at the end of this section.

The "uniform exterior cone condition" is recalled in Section 3, during the construction of appropriate subsolutions and supersolutions. One of the resources provided by this work is a quotable source for these subsolutions and supersolutions under this general assumption on the boundary.

It remains to explain the term a "C-viscosity solution of (DP)". There are two parts to this: the equation and the boundary condition. The boundary condition is interpreted in the strict sense:

$$\overline{u}(x) = \underline{u}(x) = \psi(x) \quad \text{for} \quad x \in \partial\Omega.$$

Regarding the equation "C-viscosity" means what "viscosity" means in [8]. That is, u is a C-viscosity solution of $F \le 0$ (equivalently, a C-viscosity subsolution of $F = 0$) if $u \in \mathrm{USC}(\Omega)$ – the space of all upper semicontinuous functions on Ω – and for every $\varphi \in C^2(\Omega)$ and local maximum point $\hat{x} \in \Omega$ of $u - \varphi$, one has $F\left(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})\right) \le 0$. The notion of a C-viscosity supersolution (equivalently, a solution of $F \ge 0$) arises by replacing "upper semicontinuous" by "lower semicontinuous", "max" by "min" and reversing the inequality to

$$F\left(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})\right) \ge 0\,.$$

C-viscosity solutions are functions which are simultaneously a C-viscosity subsolution and a C-viscosity supersolution. The appendage of the modifier "C" here results from the fact that "L^p-viscosity" notions are used in Section 4, and it will be necessary to refer to both concepts there. The parabolic case is incorporated in the obvious way (or see [8]).

At this juncture, we revisit the uniqueness issue. In the course of proof of Theorem 1.1, continuous C-viscosity subsolutions and supersolutions of (DP) satisfying the boundary condition will be constructed. This demonstration uses $F \in \mathcal{SC}$. If we also know that a subsolution u and a supersolution v of $F = 0$ in Ω satisfying $u, -v \in \mathrm{USC}(\overline{\Omega})$ and $u \le v$ on $\partial\Omega$ must satisfy $u \le v$ in Ω, then it is standard that there is a unique solution of (DP). When this strong comparison result holds, we say *comparison holds.*

To assert that comparison holds via the theorems of [8] and [15] one must impose structure on the continuity of $F(x, r, p, X)$ in x. In the current case, for F given by (1.1), if the $A^{\alpha,\beta}$, etc., are uniformly continuous in x uniformly in α, β, then

$$|F(x, r, p, X) - F(y, r, p, X)| \le \omega(|x - y|)(1 + |r| + |p| + \|X\|) \tag{1.4}$$

for some continuous $\omega\colon [0, \infty) \to [0, \infty)$ satisfying $\omega(0) = 0$ ($\|X\|$ is any matrix norm of X); ω is obtained from the uniform moduli of continuity of the coefficients of the linear operators.

While proving uniqueness it is enough however to establish comparison for two solutions, say u and v. It follows from the results of Caffarelli [2], that $F \in \mathcal{SC}$ and (1.4) are enough to guarantee that C-viscosity solutions of $F = 0$ are $C^{1,\alpha}_{\mathrm{loc}}$ for all $0 < \alpha < \bar{\alpha}$, where $\bar{\alpha} = \bar{\alpha}(\lambda, \Lambda, n) \in (0, 1)$. (See also Trudinger [29] for an early result in this direction and Święch [28], Theorem 2.1, for a more general statement.) In particular, $u, v \in C^{1,\alpha}_{\rm loc}$, and then uniqueness can be proved under restrictions on ω as follows. First, by Section V.1 of [15], without loss of generality we can assume that u solves $F \le -\kappa$ for some $\kappa > 0$. Then we can combine the proof of Proposition III.1 (2) of [15] with the arguments from Section 5.A of [8] (which show how to relax the standard comparison assumptions when functions are $C^{1,\alpha}$) to prove comparison if ω in (1.4) satisfies $\omega(r) \le Cr^\theta$ for some $\theta > (1-\bar\alpha)/(2-\bar\alpha)$. This approach makes full use of $\mathcal{SC}$. If $\theta > \frac{1}{2}$ then $F \in \mathcal{SC}$ can be somewhat weakened, and uniqueness of solutions still holds, see [15]. For other results in this direction see Jensen [16] and Trudinger [31].

For the continuous coefficient linear case (0.5) existence of $W^{2,n}_{\rm loc}(\Omega)$ solutions is known. These solutions are also unique C-viscosity solutions – this is a special case of Theorem 2.10 of [4] aided by Proposition 2.9 of the same work. For the elliptic case, the most general results on existence of $W^{2,n}_{\rm loc}(\Omega)$ solutions for $F \in \mathcal{SC}$ and convex or concave in the Hessian are found in [28]; the parabolic analogue is in [10] (which makes use of the results herein). These latter results rely on foundational estimates of Caffarelli [3], Wang [32] and the contributions of Escauriaza [11].

Finally, in the good solution framework for linear equations, the results and commentary of Safonov [25] and Cerutti, Fabes and Manselli [6] indicate what is known on the positive side. As mentioned in the introduction, Nadirashvili [24] shows nonuniqueness in general. See also Safonov [27].

2. EXISTENCE PROOF FOR CONTINUOUS F

Throughout this section, the terms subsolution, supersolution and solution mean, respectively, C-viscosity subsolution, C-viscosity supersolution and C-viscosity solution (see above).

We are ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. The idea for the proof, once conceived, makes the rest simple. The point is to find approximating equations $F_\epsilon = 0$ and $F^\epsilon = 0$ with better dependence on x and satisfying $F_\epsilon \le F \le F^\epsilon$. Even if F is linear, there is in general no linear approximation with the properties we need. Put

$$F_\epsilon(x,r,p,X) = \min_{y\in\overline{\Omega}} \left(F(y,r,p,X) + \tfrac{1}{\epsilon}|x-y|\right). \tag{2.1}$$

Since for each fixed $y \in \overline{\Omega}$

$$F(y,r,p,X) + \tfrac{1}{\epsilon}|x-y|$$

belongs to $\mathcal{SC}$ and has the same parameters γ, λ, Λ and continuity in r as F, F_ϵ shares these properties. The striking thing is that F_ϵ is Lipschitz continuous in x with constant $1/\epsilon$ uniformly in r, p, X while preserving the rest of the structure. The operation of "inf-convolution" used here is standard, but this use of it is unusual. By the definition $F_\epsilon \le F$ (choose $y = x$ in (2.1)). Next, F is continuous on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n)$, so for $R > 0$ there exists ω_R: $[0,\infty) \to [0,\infty)$ such that $\omega_R(0+) = 0$ and

$$|F(x,r,p,X) - F(y,r,p,X)| \le \omega_R(|x-y|) \quad \text{for} \quad |r| + |p| + \|X\| \le R \tag{2.2}$$

when $x, y \in \overline{\Omega}$. Then for $|r| + |p| + \|X\| \le R$

$$\begin{aligned} F(x,r,p,X) \le& F(y,r,p,X) + \omega_R(|x-y|) \\ &\le F(y,r,p,X) + \tfrac{1}{\epsilon}|x-y| + \omega_R(|x-y|) - \tfrac{1}{\epsilon}|x-y|, \end{aligned}$$

so we have

$$F_\epsilon(x,r,p,X) \le F(x,r,p,X) \le F_\epsilon(x,r,p,X) + \delta_R(\epsilon),$$

where

$$\delta_R(\epsilon) = \sup_{0<s\le \operatorname{diam}(\Omega)} \left(\omega_R(s) - \tfrac{s}{\epsilon}\right) \to 0 \quad \text{as} \quad \epsilon \downarrow 0. \tag{2.3}$$

In particular,

$$F_\epsilon(x,r,p,X) \to F(x,r,p,X) \quad \text{uniformly for} \quad x \in \overline{\Omega} \quad \text{and bounded} \quad r,p,X \tag{2.4}$$

as $\epsilon \downarrow 0$. Similarly, we define

$$F^\epsilon(x,r,p,X) = \sup_{y\in\overline{\Omega}} \left(F(y,r,p,X) - \tfrac{1}{\epsilon}|x-y|\right), \tag{2.5}$$

which has all the same structure properties as F_ϵ and satisfies (2.4) in place of F_ϵ. Moreover for $0 < \hat{\epsilon} < \epsilon$

$$F_\epsilon \le F_{\hat{\epsilon}} \le F \le F^{\hat{\epsilon}} \le F^\epsilon.$$

We next claim that the Dirichlet problem (DP) with F_ϵ in place of F has a unique solution $u_\epsilon \in C(\overline{\Omega})$. To prove this in the standard way (see Section 4 of [8]), we need to know that comparison holds. Since F_ϵ is *globally Lipschitz continuous in x with constant* $1/\epsilon$ and proper, if $X \le Y$ then

$$F_\epsilon(y,r,p,Y) - F_\epsilon(x,r,p,X) \le F_\epsilon(y,r,p,Y) - F_\epsilon(x,r,p,Y) \le \tfrac{1}{\epsilon}|x-y|$$

whenever $x, y \in \overline{\Omega}$, $r \in R$, $p \in \mathbb{R}^n$; in particular, (3.14) of [8] trivially holds. Moreover, any subsolution may be perturbed to a strict subsolution via [15], Section V, so comparison holds, see Section 5.C in [8]. Suitable supersolutions and subsolutions achieving boundary values in a continuous fashion are constructed in Section 3, so we may invoke Perron's method to obtain the result. The same analysis produces a solution u^ϵ of the Dirichlet problem for $F^\epsilon = 0$.

Since $F_\epsilon \le F \le F^\epsilon$, if u is any solution of the Dirichlet problem for $F = 0$, it is also a subsolution for F_ϵ and a supersolution for F^ϵ and so $u^\epsilon \le u \le u_\epsilon$ by comparison for the approximate equations. In the same way, if $0 < \hat{\epsilon} \le \epsilon$, then $u^\epsilon \le u^{\hat{\epsilon}} \le u_{\hat{\epsilon}} \le u_\epsilon$, so these families of functions attain the boundary values in an equicontinuous manner. According to e.g. Trudinger [30] (see also Caffarelli [2], Caffarelli and Cabré [3], Fok [13] and [28]), they are also locally equi-Hölder continuous in Ω, so there are uniform limits

$$\lim_{\epsilon\downarrow 0} u^\epsilon = \underline{u} \le \overline{u} = \lim_{\epsilon\downarrow 0} u_\epsilon.$$

By the standard stability result for viscosity solutions (see Section 6 of [8]) and (2.4) for F_ϵ and F^ϵ, $\underline{u}, \overline{u}$ are solutions of $F = 0$ and thus they are minimal and maximal solutions of the Dirichlet problem for $F = 0$. $\square$

We turn to the parabolic case. In this situation, we have $0 < T$, set $Q = \Omega \times (0,T]$, and use

$$\partial_p Q = \partial\Omega \times (0,T] \cup \overline{\Omega} \times \{0\}$$

to denote the parabolic boundary of Q.

We consider the Cauchy-Dirichlet problem

$$u_t + F(x,t,u,Du,D^2u) = 0 \quad \text{in} \quad Q \quad \text{and} \quad u = \psi \quad \text{on} \quad \partial_p Q, \tag{CDP}$$

where $\psi \in C(\partial_p Q)$. Now $F \in \mathcal{SC}$ will mean that for each $t \in [0,T]$ the mapping $(x,r,p,X) \mapsto F(x,t,r,p,X)$ belongs to $\mathcal{SC}$ in the sense of Section 1.

The analogue of Theorem 1.1 is:

Theorem 2.1. *Let Ω satisfy a uniform exterior cone condition, $F \in \mathcal{SC}$ be continuous on $\overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n)$ and $\psi \in C(\partial_p Q)$. Then there are C-viscosity solutions $\overline{u}$, $\underline{u} \in C(\overline{Q})$ of (CDP) such that if u is another C-viscosity solution, then $\underline{u} \le u \le \overline{u}$.*

Precisely the same outline as succeeds for the elliptic case proves Theorem 2.1. The subsolutions and supersolutions required are provided in the next section. Concerning the interior Hölder continuity of viscosity solutions of parabolic equations see Wang [32] and Section 5 of [10] (see Krylov [18] and Lieberman [20] for classical results). See [8] Section 8 for other standard adaptations to prove existence in the parabolic case.

The compactness of the approximations u_ϵ, u^ϵ in $C(\overline{\Omega})$ (or $C(\overline{Q})$) is used above. The monotonicity of these families was invoked to control continuity up to the boundary. However, this does not reveal the full compactness available in these circumstances. In Section 4 more is needed, and the appropriate general compactness result is given.

3. CONSTRUCTION OF SUBSOLUTIONS AND SUPERSOLUTIONS

Elliptic Case.

We turn to the construction of subsolutions and supersolutions. There is only one task here, not two, as the next remark recalls:

Remark 3.1. We note the following standard reduction: u is a supersolution of an equation $F(x,u,Du,D^2u) = 0$ if and only if $v = -u$ is a subsolution of $\tilde{F}(x,v,Dv,D^2v) = 0$, where $\tilde{F}(x,r,p,X) = -F(x,-r,-p,-X)$; moreover, noting that if $F(p,X) = \mathcal{P}^-(X) - \gamma|p|$, then $\tilde{F}(p,X) = \mathcal{P}^+(X) + \gamma|p|$, one sees $\tilde{F} \in \mathcal{SC}$ if and only if $F \in \mathcal{SC}$. Thus it suffices to construct either supersolutions or subsolutions.

We will construct a supersolution $U \in C(\overline{\Omega})$ of (DP) for F_ϵ such that $U(x) = \psi(x)$ on $\partial\Omega$ and $U \ge R = \inf_{\partial\Omega} \psi$. Moreover, we will track the continuity properties of U. To begin, we use $F_\epsilon \in \mathcal{SC}$ so that by (1.2)

$$F_\epsilon(x,r,p,X) \ge \mathcal{P}^-(X) - \gamma|p| + F_\epsilon(x,r,0,0)$$

and therefore a supersolution of $\mathcal{P}^-(D^2U) - \gamma|DU| + F_\epsilon(x, U, 0, 0) = 0$ in Ω is also a supersolution for F_ϵ. If $U \geq R$, then $-F_\epsilon(x, U, 0, 0) \leq -F_\epsilon(x, R, 0, 0)$. It follows that if $M \geq -F_\epsilon(x, R, 0, 0)$ for all $x \in \Omega$, and $U \geq R$ solves

$$\mathcal{P}^-(D^2U) - \gamma|DU| \geq M, \tag{3.1}$$

then U is a supersolution for F_ϵ.

On the other hand it is true in general that if U is a C-viscosity solution of (3.1) with $M \geq 0$ and $U \geq \psi$ on $\partial\Omega$, then the Alexandrov–Bakelman–Pucci maximum principle for viscosity supersolutions (see, e.g., [3], [30] or [4], Proposition 2.12) implies that $u \geq \inf_{\partial\Omega} \psi$. Thus if $M \geq \sup_{x\in\Omega} (-F_\epsilon(x, \inf_{\partial\Omega} \psi, 0, 0))$ and $M \geq 0$, a solution of (3.1) satisfying $U = \psi$ on $\partial\Omega$ is a supersolution of the Dirichlet problem for F_ϵ.

Let $M > 0$. The results of Miller [22], [23] provide the existence of local barriers for the Dirichlet problem for (3.1) under a uniform exterior cone condition. We do not know a place in the literature where global barriers (supersolutions) are constructed, although the case $\gamma = 0$ is treated in Michael [21] and the proof there may be modified to handle the general case – see Remark 3.4. We present another option, using the flexibility of viscosity solutions.

First we recall the nature of the barriers on exterior cones for extremal elliptic operators constructed by Miller [22], [23]. For $n \geq 2$ and $\beta \in (0, \pi)$ let

$$T_\beta = \{x \in \mathbb{R}^n\colon\ x_n \geq (\cos\beta)|x|\}$$

be the closed circular cone of aperture β with axis in the direction of $-e_n$. Consider barriers of the form

$$w(x) = r^b f(\theta), \tag{3.2}$$

where $r = |x|$ and $\theta = \arccos(x_n/|x|)$. It is shown in [23], Theorem 3 and Section 7, that for every $\beta \in (0, \pi)$ there exist $b \in (0, 1)$ and $f \in C^2([0, \pi))$, depending only on $\lambda, \Lambda, n, \beta, \gamma$, such that $f'(0) = 0$ and $f > 1$ on $[0, \beta]$, so that w given by (3.2) is continuous on T_β and C^2 on $\mathbb{R}^n \setminus \{\text{closed negative } x_n \text{ axis}\}$ and

$$w > r^b \quad \text{on } T_\beta \setminus \{0\}, \quad w(0) = 0, \tag{3.3}$$

and, crucially,

$$\mathcal{P}^-(D^2w) - \gamma|Dw| \geq r^{b-2} \quad \text{on } \operatorname{int}(T_\beta). \tag{3.4}$$

Now let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, satisfying a uniform exterior cone condition. This means that there exist $\beta \in (0, \pi)$ and $r_0 > 0$ so that for every $z \in \partial\Omega$ there is a rotation $\Theta = \Theta(z)$ such that

$$\overline{\Omega} \cap B_{r_0}(z) \subset z + \Theta T_\beta. \tag{3.5}$$

Here, $B_{r_0}(z)$ denotes the open ball in $\mathbb{R}^n$ of radius r_0 centered at z. The “local” nature of the barriers below is due to the possibility that $\Omega \subset z + \Theta T_\beta$ does not hold. Setting $w_z(x) = w(\Theta^{-1}(x - z))$, from (3.3) we have

$$w_z(z) = 0, \quad w_z(x) \geq |z - x|^b \quad \text{on } \overline{\Omega} \cap B_{r_0}(z). \tag{3.6}$$

In particular, we have

$$w_z \geq (r_0)^b \quad \text{on} \quad \{x \in \overline{\Omega}\colon \quad |x-z| = r_0\}. \tag{3.7}$$

Using (3.4) and (3.5) we arrive at

$$\mathcal{P}^-(D^2 w_z(x)) - \gamma |Dw_z(x)| \geq |x-z|^{b-2} \quad \text{for} \quad x \in \Omega \cap B_{r_0}(z). \tag{3.8}$$

We need to extend these local barriers to global ones. Choose any point $y \notin \overline{\Omega}$ and $2r_1 <$ distance $(y, \partial\Omega)$. For $\sigma > 0$ put

$$G(x) = \left(\tfrac{1}{r_1^\sigma} - \tfrac{1}{|x-y|^\sigma}\right). \tag{3.9}$$

Clearly

$$G(x) \geq \frac{2^\sigma - 1}{(2r_1)^\sigma} > 0 \quad \text{on} \quad \overline{\Omega} \tag{3.10}$$

and a standard computation ([14]) shows that

$$\mathcal{P}^-(D^2 G(x)) - \gamma |DG(x)| \geq \frac{\sigma}{|x-y|^{\sigma+2}} \left((\sigma+1)\lambda - \Lambda(n-1) - \gamma|x-y|\right) > 0 \tag{3.11}$$

on $\overline{\Omega}$ for large σ (depending only on $n, \lambda, \Lambda, \gamma$ and diam (Ω)). Replacing G by aG for a suitable $a > 0$ we can achieve all of:

$$G > 0 \quad \text{and} \quad \mathcal{P}^-(D^2 G) - \gamma |DG| \geq \kappa \quad \text{on} \quad \overline{\Omega} \tag{3.12}$$

for some $\kappa > 0$ and

$$G(x) < \tfrac{1}{2}(r_0)^b \quad \text{for} \quad x \in \overline{\Omega}. \tag{3.13}$$

Then the function

$$W_z(x) = \begin{cases} G(x) & \text{for } x \in \overline{\Omega},\ |x-z| > r_0, \\ \min\left(G(x), w_z(x)\right) & \text{for } x \in \overline{\Omega},\ |x-z| \leq r_0 \end{cases} \tag{3.14}$$

agrees with w_z in a neighborhood of $z \in \partial\Omega$ relative to $\overline{\Omega}$ (in view of (3.6) and (3.12)), agrees with G on $\overline{\Omega} \setminus B_{r_2}(z)$ for some $0 < r_2 < r_0$ (due to (3.7), (3.13)), and is a solution of $\mathcal{P}^-(D^2 W_z) - \gamma |DW_z| \geq \kappa_1$ in Ω provided that both w_z and G satisfy the same relation in $B_{r_0}(z) \cap \Omega$ and G does in all of Ω. Hence, in view of (3.12) and (3.8), we may multiply W_z by a constant and have all of the following properties of the resulting function (still called W_z):

$$W_z \in C(\overline{\Omega}), \quad W_z(z) = 0, \quad W_z > 0 \quad \text{on} \quad \overline{\Omega} \setminus \{z\} \tag{3.15}$$

and

$$\mathcal{P}^-(D^2 W_z) - \gamma |DW_z| \geq 1 \quad \text{on} \quad \Omega. \tag{3.16}$$

The task of satisfying the boundary condition remains. Let $\psi \in C(\partial\Omega)$ and

$$|\psi(x) - \psi(z)| \leq \rho(|x-z|) \quad \text{for} \quad x, z \in \partial\Omega, \tag{3.17}$$

where $\rho(0+) = 0$, so ρ is a modulus of continuity for ψ. In addition to $\mathcal{P}^-(D^2U) - \gamma|DU| \ge M$, the supersolution U we construct will satisfy

$$|U(x) - \psi(z)| \le \omega(|x-z|) \quad \text{for} \quad x \in \Omega,\ z \in \partial\Omega, \tag{3.18}$$

where $\omega(0+) = 0$; that is the boundary values are assumed uniformly. Moreover, ω will depend only on the parameters of the cone condition, $\lambda, \Lambda, n, \gamma$ and the diameter of Ω (which already determine the character of each W_z), and M and ρ. For each $\kappa > 0$ and $z \in \partial\Omega$ put

$$W_{\kappa,z}(x) = \psi(z) + \kappa + M_\kappa W_z(x),$$

where $M_\kappa \ge M$ (guaranteeing $\mathcal{P}^-(D^2W_{\kappa,z}) - \gamma|DW_{\kappa,z}| \ge M$ by (3.16)) is chosen so that

$$\psi(z) + \kappa + M_\kappa W_z(x) \ge \psi(x) \quad \text{for} \quad x \in \partial\Omega.$$

In view of (3.17), it suffices to take

$$M_\kappa \ge \sup_{x\in\partial\Omega, x\ne z} \frac{(\rho(|x-z|) - \kappa)^+}{W_z(x)};$$

this may evidently be done uniformly in $z \in \partial\Omega$. Finally we put

$$W(x) = \inf_{z\in\partial\Omega, \kappa>0} W_{\kappa,z}(x).$$

By construction $W \ge R$. Since for all κ

$$W(x) - \psi(z) \le W_{\kappa,z}(x) - \psi(z) = \kappa + M_\kappa W_z(x)$$

and $W_z(x)$ is uniformly continuous in x uniformly in z, we conclude that for all $z \in \partial\Omega$ and $x \in \Omega$

$$W(x) - \psi(z) \le \omega(|z-x|) \tag{3.19}$$

for some ω satisfying $\omega(0+) = 0$. We now use Remark 3.1 – the supersolutions of (3.1) we have constructed imply the existence of corresponding subsolutions of $F_\epsilon \le 0$ (or $\mathcal{P}^-(D^2U) - \gamma|DU| \le -M$ for an appropriate M) with boundary values below ψ, call them $Y_{\kappa,z}$ and the supremum Y. By the analogue of (3.19) for Y we have

$$-\omega(|x-z|) \le Y(x) - \psi(z). \tag{3.20}$$

According to [8], $U = W_*$, the lower semicontinuous envelope of W, is a supersolution of (3.1), and consequently of $F_\epsilon = 0$. Similarly, $V = Y^*$, the upper semicontinuous envelope of Y, is a subsolution of $F_\epsilon = 0$. Since $V = U = \psi$ on $\partial\Omega$, comparison gives $V \le U$ and using this together with (3.19) and (3.20) yields

$$|V(x) - \psi(z)|,\ |U(x) - \psi(z)| \le \omega(|z-x|)$$

and we are done.

For convenient reference, we summarize the results of this construction in terms of extremal equations:

Proposition 3.2. *Let Ω satisfy a uniform exterior cone condition, $\psi \in C(\partial\Omega)$ and $M \in \mathbb{R}$. Then the problems*

$$\mathcal{P}^-(D^2u) - \gamma|Du| = M \quad in \quad \Omega, \quad u = \psi \quad on \quad \partial\Omega \tag{3.21}$$

and

$$\mathcal{P}^+(D^2v) + \gamma|Dv| = -M \quad in \quad \Omega, \quad v = \psi \quad on \quad \partial\Omega \tag{3.22}$$

have unique C-viscosity solutions $u, v \in C(\overline{\Omega})$ satisfying $u = v = \psi$ on $\partial\Omega$. Moreover, there is a modulus ω depending only on the parameters of the cone condition, $\lambda, \Lambda, n, \gamma$, $\mathrm{diam}\,(\Omega)$, M and the modulus of continuity of ψ such that

$$|u(x) - \psi(z)|,\ |v(x) - \psi(z)| \le \omega(|x - z|) \quad for \quad z \in \partial\Omega,\ x \in \Omega.$$

Regarding the statement, recall Remark 3.1; moreover, given the subsolutions and supersolutions exhibited above, we may assert the existence of solutions, not only subsolutions and supersolutions.

Remark 3.3. Bellman equations (3.22) and (3.21) are concave/convex in the Hessian matrix and as such can be studied by classical methods, see Krylov [18]. In particular, Safonov [26], Theorem 1.1, proves that under the assumptions of Proposition 3.2, the problems (3.22) and (3.21) have classical $C^{2,\alpha}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ solutions. From uniqueness it follows that C-viscosity solutions u, v of Proposition 3.2 coincide with Safonov's; in particular $u, v \in C^2(\Omega)$. However, we feel that the construction presented here is useful. The problems (3.22) and (3.21) were solved here – albeit in a weaker sense – without invoking the apparatus of hard $C^{2,\alpha}$ estimates for nonlinear equations. Our objective is to solve equations $F_\epsilon = 0$ that are not expected to have classical solutions. $F_\epsilon = 0$ is solved by Perron's method, and for this purpose the information summarized in Proposition 3.2 is sufficient. Corollary 3.10 of [4] extends the existence to cover the situation when M is replaced by $f \in L^p(\Omega)$ for suitable p, a fact used in Section 4.

As for the parabolic result, Proposition 3.5 below, we were not able to locate a quotable result in literature asserting classical solvability of (3.25) and (3.26). However, Krylov in [18], Theorem 6.4.3, proves an analogous result in the case of Ω satisfying an exterior sphere condition, and this generalizes to handle the cone condition. Krylov's method consists of approximating Ω from the inside by more regular domains; a delicate argument involving barrier functions and Hölder estimates is used to pass to the limit. The result of Krylov shows that C-viscosity solutions of Proposition 3.5 are $C^{2,1}$ (and more). This follows easily by solving (CDP) classically on regular subdomains of $\Omega \times (0, T]$ using u (v, respectively) as the boundary data, and invoking uniqueness.

Remark 3.4. Our construction of barriers relies on the barrier w from (3.2) for a canonical unbounded cone T_β at the origin, taken from [22], [23]. We used uniform exterior cone condition to obtain a local barrier w_z at every $z \in \partial\Omega$; w_z is obtained by composing w with an appropriate isometry making T_β into an exterior cone at z. Then w_z was extended to a global barrier W_z (3.14) by means of a fixed function G in (3.9).

Another possibility would be to first construct a barrier W_0 on the exterior of a canonical bounded cone $C_{\beta,\delta}$ at the origin:

$$C_{\beta,\delta} = \{x \in \mathbb{R}^n \colon\ |x| \le \delta,\ x_n \le (\cos\beta)|x|\}\,.$$

This can be accomplished by a similar procedure as above. Namely, taking $y = -\frac{\delta}{2}e_n$, sufficiently small $r_1 > 0$ (determined only by δ, β) and G as in (3.9), for σ large, depending only on $n, \lambda, \Lambda, \gamma$ and $\operatorname{diam}(\Omega)$ known in advance, one can guarantee

$$G > 0 \quad \text{and} \quad \mathcal{P}^-(D^2G) - \gamma|DG| > 0 \quad \text{on} \quad B_{\operatorname{diam}(\Omega)}(0) \setminus C_{\beta,\delta}.$$

As in (3.14), a multiple of this G combined with w would produce a desired barrier W_0. Now one can use uniform exterior cone condition to find for every $z \in \partial\Omega$ a rotation $\Theta = \Theta(z)$ such that

$$\overline{\Omega} \cap (z + \Theta C_{\beta,\delta}) = \{z\},$$

and then $W_z(x) = W_0\left(\Theta^{-1}(x - z)\right)$ defines a global barrier at z.

Michael [21] considers barriers

$$\varphi(x) = 1 - e^{-Kw(x)},$$

where w is Miller's barrier (3.2) and $K > 0$. [21] gives explicit recipes for K, b and f so that φ becomes a barrier for $\mathcal{P}^-(D^2\varphi) \ge r^{b-2}$ on the exterior of $C_{\beta,\delta}$. This construction can be easily modified to handle first order terms to obtain $\mathcal{P}^-(D^2\varphi) - \gamma|D\varphi| \ge 1$ on $B_{\operatorname{diam}(\Omega)}(0) \setminus C_{\beta,\delta}$. This canonical barrier φ can be used instead of W_0 constructed above; note that φ is C^2 unlike W_0.

Finally, if $n = 2$ the exterior cone condition can be replaced by a weaker condition, see [22], [23] and Section 2.8 of [14].

Parabolic Case.

The work done above renders the parabolic case simple. To construct supersolutions, we reduce as before to the problem

$$U_t + \mathcal{P}^-(D^2U) - \gamma|DU| \ge M.$$

Reviewing the preceding construction, we see that all we need will follow if we produce a function $W_{z,\tau}$ for each point (z, τ) of the (parabolic) boundary $\partial\Omega \times (0, T] \cup \overline{\Omega} \times \{0\}$ of Q satisfying the analogues of (3.15) and (3.16):

$$W_{z,\tau} \in C(\overline{Q}), \quad W_{z,\tau}(z,\tau) = 0, \quad W_{z,\tau} > 0 \quad \text{on} \quad \overline{Q} \setminus \{(z,\tau)\} \tag{3.23}$$

and

$$(W_{z,\tau})_t + \mathcal{P}^-(D^2W_{z,\tau}) - \gamma|DW_{z,\tau}| \ge 1 \quad \text{on} \quad Q. \tag{3.24}$$

For $\tau > 0$ and $z \in \partial\Omega$ we set

$$W_{z,\tau}(x,t) = \tfrac{1}{2T}(t-\tau)^2 + 2W_z(x),$$

where W_z was constructed above and satisfies (3.15) and (3.16). It is clear that (3.23) holds. Moreover, by (3.16),

$$\begin{aligned}(W_{z,\tau})_t &+ \mathcal{P}^-(D^2 W_{z,\tau}) - \gamma|DW_{z,\tau}| \\ &= \tfrac{1}{T}(t-\tau) + 2\left(\mathcal{P}^-(D^2 W_z) - \gamma|DW_z|\right) \ge -1+2 = 1\end{aligned}$$

and we have (3.24).

For $\tau = 0$ and $z \in \overline{\Omega}$ we set

$$W_{z,0}(x,t) = At + \tfrac{1}{2}|x-z|^2;$$

again if $A > 0$ we clearly have (3.23). Finally,

$$\begin{aligned}&(W_{z,0})_t + \mathcal{P}^-(D^2 W_{z,0}) - \gamma|DW_{z,0}| = \\ &\quad A + \mathcal{P}^-(I) - \gamma|x-z| = A - n\Lambda - \gamma|x-z| \ge A - n\Lambda - \gamma \operatorname{diam}(\Omega).\end{aligned}$$

Thus we have (3.24) if $A = n\Lambda + \gamma \operatorname{diam}(\Omega) + 1$. The rest of the analysis follows that of the elliptic case step by step.

Here is the parabolic version of Proposition 3.2. We use the notation introduced in Section 2.

Proposition 3.5. *Let Ω satisfy a uniform exterior cone condition, $\psi \in C(\partial_p Q)$ and $M \in \mathbb{R}$. Then the problems*

$$u_t + \mathcal{P}^-(D^2 u) - \gamma|Du| = M \quad \text{in} \quad Q, \quad u = \psi \quad \text{on} \quad \partial_p Q \tag{3.25}$$

and

$$v_t + \mathcal{P}^+(D^2 v) + \gamma|Dv| = -M \quad \text{in} \quad Q, \quad v = \psi \quad \text{on} \quad \partial_p Q \tag{3.26}$$

have unique C-viscosity solutions $u, v \in C(\overline{Q})$ satisfying $u = v = \psi$ on $\partial_p Q$. Moreover, there is a modulus ω depending only on the parameters of the cone condition, $\lambda, \Lambda, n, \gamma, T$, $\operatorname{diam}(\Omega)$, M and the modulus of continuity of ψ such that

$$|u(x,t)-\psi(z,\tau)|,\ |v(x,t)-\psi(z,\tau)| \le \omega(|x-z|+|t-\tau|) \quad \text{for} \quad (z,\tau) \in \partial_p Q,\ (x,t) \in Q.$$

4. L^p THEORY: GENERAL EXISTENCE

The requirements (1.2) and (1.3) constituting the basic structure conditions require no continuity of $F(x,r,p,X)$ in x and very little in r. In this section, we assume that F is merely measurable in x (or (x,t) in the parabolic case) while satisfying the structure conditions for almost every x (or (x,t)). Due to this generality we have to impose a requirement on the r dependence: for $R > 0$ there exists $\omega_R\colon [0,\infty) \to [0,\infty)$ such that $\omega_R(0+) = 0$ and

$$|F(x,r,p,X) - F(x,s,p,X)| \le \omega_R(|r-s|) \tag{4.1}$$

for almost all $x \in \Omega$ and $|r| + |s| + |p| + \|X\| \le R$. (Obviously, if F is continuous then (4.1) automatically holds.) In the parabolic case, x is replaced by $(x,t) \in Q$.

Of course, the notions of C-viscosity subsolutions, etc., are no longer appropriate in this measurable situation, and there is now a well-developed theory using corresponding "L^p-viscosity" notions ([4], [9], [28], [7], [10]) which is built up from the fundamental regularity results of Caffarelli [2] (see [3]) as further developed by Escuariaza [11] and Wang [32]. In the linear case other notions of weak solutions were proposed, see Cerrutti, Escauriaza and Fabes [5] and Jensen [17]; relationships between various notions of solutions are studied in [17] and [9].

$\mathbf{L^p}$-Viscosity Notions

In contrast to the C-viscosity notions recalled at the end of Section 1, L^p-viscosity notions use "test functions" $\varphi \in W^{2,p}_{\mathrm{loc}}(\Omega)$ (functions whose distributional second derivatives are in $L^p_{\mathrm{loc}}(\Omega)$) in the elliptic case and $\varphi \in W^{2,1,p}_{\mathrm{loc}}(Q)$ (functions whose distributional first derivatives and second order spatial derivatives are in $L^p_{\mathrm{loc}}(Q)$) in the parabolic case. In addition, all subsolutions, etc., are required to be continuous. For example, a continuous function u on Ω is an L^p-viscosity subsolution of $F(x,u,Du,D^2u)=0$ if for every $\varphi \in W^{2,p}_{\mathrm{loc}}(\Omega)$ and local maximum $\hat{x}$ of $u-\varphi$ one has

$$\operatorname*{ess\,lim\,inf}_{x\to\hat{x}} F(x,u(x),D\varphi(x),D^2\varphi(x)) \le 0;$$

equivalently, if for some $\epsilon > 0$

$$F(x,u(x),D\varphi(x),D^2\varphi(x)) \ge \epsilon \quad \text{a.e.}$$

in some neighborhood of $\hat{x}$, then $\hat{x}$ is not a local maximum of $u-\varphi$. The corresponding notions of L^p-viscosity supersolutions, L^p-viscosity solutions, and the parabolic versions are what then one expects.

Consulting the literature mentioned above, one finds that there is an equation dependent appropriate range of p determined by the parameters $\lambda, \Lambda, n, \gamma \mathrm{diam}(\Omega)$ (or $\gamma\mathrm{diam}(Q)$ for parabolic equations). One always has $n/2 < p$ in the elliptic case and $(n+2)/2 < p$ in the parabolic case. The range also extends below n in the elliptic case and below $n+1$ in the parabolic case. The choices $p = n$ and $p = n+1$ are the least possible which are appropriate for all choices of $\lambda, \Lambda, \gamma, \Omega$. In statements below we restrict our attention to the "universal" choices $p = n$ and $p = n+1$ for simplicity. However, certain explicit arguments as well as proofs of quoted results use the fact that extended ranges exist.

Note that if $q < p$, then the L^q-viscosity notions imply the corresponding L^p-viscosity notions, as there are more test functions to check in the L^q-viscosity case. Thus L^n-viscosity solutions are automatically L^p-viscosity solutions for $n < p$. Similarly, if F happens to be continuous, L^n-viscosity notions imply C-viscosity notions. It is a substantial result that the converse is true in this form: if F is continuous and u is a continuous C-viscosity subsolution, etc., then it is an L^n-viscosity subsolution, etc. In the elliptic case this is proved in Proposition 2.9 of [4], for the parabolic case see [10].

General Existence of $\mathbf{L^n}$-Viscosity Solutions

We consider the Dirichlet problem for $G(x,u,Du,D^2u)=0$, where $G \in \mathcal{SC}$ is merely measurable in x:

$$G(x,u,Du,D^2u) = 0 \quad \text{in} \quad \Omega, \quad u = \psi \quad \text{on} \quad \partial\Omega. \tag{4.2}$$

As usual, $\psi \in C(\partial\Omega)$. However, it will be convenient to rewrite (4.2) as

$$F(x,u,Du,D^2u) = f(x) \quad \text{in} \quad \Omega, \quad u = \psi \quad \text{on} \quad \partial\Omega, \tag{4.3}$$

where

$$F(x,r,p,X) = G(x,r,p,X) - G(x,0,0,0) \quad \text{and} \quad f(x) = -G(x,0,0,0),$$

so that

$$F(x,0,0,0) \equiv 0. \tag{4.4}$$

Without loss of generality we hereafter assume that (4.4) holds. Note that then, via $F \in \mathcal{SC}$, (4.1) and (4.4)

$$F(x,r,p,X) \le F(x,r,0,0) + \mathcal{P}^+(X) + \gamma|p| \le \beta(r) + \mathcal{P}^+(X) + \gamma|p|,$$

where $\beta(r) = \omega_{|r|}(|r|)$. The analogous estimate from below combines with this to yield

$$|F(x,r,p,X)| \le \Lambda\|X\| + \gamma|p| + \beta(r), \tag{4.5}$$

where we used the trace norm $\|X\| = \mathrm{Trace}(X^+)+\mathrm{Trace}(X^-)$ and invoked $|\mathcal{P}^\pm(X)| \le \Lambda\|X\|$. In particular, $F(x,r,p,X)$ is bounded and measurable in x for fixed r,p,X. This guarantees that integrals occurring below are well defined. We have the following theorem:

Theorem 4.1. *Let $F \in \mathcal{SC}$ satisfy (4.1) and (4.4), let $f \in L^n(\Omega)$, $\psi \in C(\partial\Omega)$ and let Ω satisfy a uniform exterior cone condition. Then (4.3) has an L^n-viscosity solution.*

As a tool in the proof we will use:

Proposition 4.2. *Let Ω satisfy a uniform exterior cone condition and $\mathcal{C} \subset C(\partial\Omega)$ be compact, $R > 0$ and $B_R = \{f \in L^n(\Omega)\colon \|f\|_{L^n(\Omega)} \le R\}$. Then the set of all functions $u \in C(\overline{\Omega})$ such that there exists $\psi \in \mathcal{C}$ and $f \in B_R$ for which u is an L^n-viscosity solution of both*

$$\mathcal{P}^-(D^2u) - \gamma|Du| \le f \quad \textit{and} \quad -f \le \mathcal{P}^+(D^2u) + \gamma|Du| \tag{4.6}$$

in Ω and $u = \psi$ on $\partial\Omega$ is precompact in $C(\overline{\Omega})$.

Proof. According to [4] Corollary 3.10, if $(\varphi, g) \in C(\partial\Omega) \times L^n(\Omega)$, there exist a unique $U = U(\varphi,g) \in C(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ such that

$$\mathcal{P}^-(D^2U) - \gamma|DU| = g \quad \text{a.e. in} \quad \Omega \quad \text{and} \quad u = \varphi \quad \text{on} \quad \partial\Omega.$$

We require several facts. First, there exist $p \in (n/2, n)$ depending on $\lambda, \Lambda, n, \gamma\,\mathrm{diam}\,(\Omega)$ (see [13], [11], [4], [28], [1]) and C such that

$$U(\varphi,g) \le \sup_{\partial\Omega} \varphi + C\left(\int_\Omega (g^+)^p\right)^{\frac{1}{p}} \quad \text{for} \quad (\varphi,g) \in C(\partial\Omega) \times L^n(\Omega). \tag{4.7}$$

Next the mapping

$$C(\partial\Omega)\times L^n(\Omega)\ni(\varphi,g)\to U(\varphi,g)\quad\text{is sublinear and order preserving.}\tag{4.8}$$

Finally, if $u\in C(\overline{\Omega})$ is an L^n-viscosity solution of

$$\mathcal{P}^-(D^2u)-\gamma|Du|\le f\quad\text{in}\quad\Omega\quad\text{and}\quad u=\psi\quad\text{on}\quad\partial\Omega,\tag{4.9}$$

where $f\in L^n(\Omega)$, then

$$u\le U(\psi,f).\tag{4.10}$$

We review the genesis of these results in Remark 4.4 below.

The second inequality of (4.6) may be restated as $w=-u$ is an L^n-viscosity solution of $\mathcal{P}^-(D^2w)-\gamma|Dw|\le f$, so if (4.6) holds (4.10) implies $-u\le U(-\psi,f)$ or $-U(-\psi,f)\le u$. All told, (4.6) and $u=\psi$ on $\partial\Omega$ yield

$$-U(-\psi,f)\le u\le U(\psi,f).\tag{4.11}$$

From (4.11) and (4.7) it follows that u remains bounded in $C(\overline{\Omega})$ if (ψ,f) remains bounded in $C(\partial\Omega)\times L^n(\Omega)$ (or even in $C(\partial\Omega)\times L^p(\Omega)$). We now use (4.11) to show that u assumes the boundary values ψ in an equicontinuous manner. In this regard, let $f_M=\max(\min(f,M),-M)$ be the standard truncation of f for $M>0$. We note that for $f\in B_R$

$$\|f-f_M\|_{L^p(\Omega)}\le(\text{measure}(\{|f|>M\}))^{\frac{n-p}{np}}R\le R\left(\frac{R}{M}\right)^{\frac{n-p}{p}},$$

which tends to 0 as $M\to\infty$ uniformly in $f\in B_R$. Using the properties (4.8), (4.7) of U we thus have

$$\begin{aligned}U(\psi,f)&\le U(\psi,f_M)+U(0,f-f_M)\le U(\psi,M)+C\|f-f_M\|_{L^p(\Omega)}\\&\le U(\psi,M)+CR\left(\frac{R}{M}\right)^{\frac{n-p}{p}}.\end{aligned}$$

According to Proposition 3.2, $U(\psi,M)$ assumes the boundary values ψ in a manner controlled by the modulus of continuity of ψ for fixed M. The “error term” on the right above can be made as small as desired by choosing M sufficiently large, and $u\le U(\psi,f)$ thus guarantees an estimate $u(x)-\psi(y)\le\rho(|x-y|)$ for $x\in\overline{\Omega},y\in\partial\Omega$, where $\rho(0+)=0$. Similarly, $-U(-\psi,f)\le u$ provides control of $u-\psi$ at the boundary from below.

Finally, once u is bounded, (4.6) guarantees equi-Hölder continuity of u on compact subsets of Ω so long as f remains bounded in $L^n(\Omega)$ (see, for example, [13] for a sufficiently general statement and Remark 4.7 below). The result follows. ☐

Remark 4.3. Proposition 4.2 can be reformulated by saying that if u satisfies (4.6) and $u=\psi$ on $\partial\Omega$ then u has a modulus of continuity on $\overline{\Omega}$ that only depends on the parameters of the cone condition, λ,Λ,n,γ, diam$\,(\Omega)$, R and the modulus of continuity of ψ.

Remark 4.4. The inequality (4.7) generalizes the original work of Fabes and Stroock [12] and is proved in [13] in the spirit of this work, but it could also be deduced from Cabré [1]; its relevance in this arena was first shown by Escauriaza [11]. In fact, the existence of $U(\varphi, g)$ for $\gamma = 0$ was proved in [11] relying on (4.7) with $\gamma = 0$. The properties (4.8) are a consequence of the positive homogeneity and superlinearity of $(p, X) \to \mathcal{P}^-(X) - \gamma|p|$ and (4.7) ($p = n$ suffices). For example, the superadditivity implies that $W = U(\varphi, g) - U(\hat{\varphi}, \hat{g})$ solves

$$\mathcal{P}^-(D^2W) - \gamma|DW| \le g - \hat{g} \le 0$$

if $g \le \hat{g}$ and an application of (4.7) ($p = n$ suffices) then proves the order preserving property. The relation (4.10) given (4.9) follows upon observing that $v = u - U(\psi, f)$ is an L^n-viscosity solution of $\mathcal{P}^-(D^2v) - \gamma|Dv| \le 0$ and the Alexandrov-Bakelman-Pucci maximum principle for viscosity solutions proved in [2] ($\gamma = 0$), [30], [4]. Finally, Proposition 4.2 itself appears in [3], Theorem 4.14, in the situation where $\gamma = 0$, Ω is a ball, and all functions f appearing in (4.6) are continuous. This proof could be adapted, with effort, to the current case. The current proof uses the work already done in Section 3.

Proof of Theorem 4.1. First we assume that $F(x, r, p, X)$ is defined for all (r, p, X) for all $x \in \mathbb{R}^n$ and satisfies the structure conditions (1.2), (1.3) and (4.1) for all $x \in \mathbb{R}^n$. To achieve this, if necessary extend $F(x, r, p, X)$ to be $\mathcal{P}^-(X) - \gamma|p|$ (or $\mathcal{P}^+(X) + \gamma|p|$) for those x's where it was not originally defined. Now mollify F in x:

$$F_\epsilon(x, r, p, X) = \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \eta\left(\frac{x - y}{\epsilon}\right) F(y, r, p, X)\, dy,$$

where $\eta \in C_0^\infty(\mathbb{R}^n)$ satisfies $\eta \ge 0$ and $\int_{\mathbb{R}^n} \eta(x)\, dx = 1$. The structure conditions are preserved under this sort of averaging, so $F_\epsilon \in \mathcal{SC}$.

Clearly F_ϵ satisfies (4.1), (4.4) and (4.5) as well as F. Moreover, the bound (4.5) on $|F|$ gives us

$$|F_\epsilon(x, r, p, X) - F_\epsilon(y, r, p, X)| \le \tfrac{C}{\epsilon}|x - y| \left(\Lambda\|X\| + \gamma|p| + \beta(r)\right)$$

for some C. Fix $f \in L^n(\Omega)$ and let $f_j \in C(\overline{\Omega})$ satisfy

$$\|f_j - f\|_{L^n(\Omega)} \to 0 \quad \text{as} \quad j \to \infty.$$

Since $F_\epsilon \in \mathcal{SC}$ is continuous, according to Theorem 1.1 the problem

$$F_\epsilon(x, u, Du, D^2u) = f_j \quad \text{in} \quad \Omega \quad \text{and} \quad u = \psi \quad \text{on} \quad \partial\Omega \tag{4.12}$$

has a C-viscosity solution (and hence L^n-viscosity solution) $u = u_{\epsilon,j}$. Clearly $u = u_{\epsilon,j}$ also solves

$$\mathcal{P}^-(D^2u) - \gamma|Du| + F_\epsilon(x, u, 0, 0) \le f_j \quad \text{and} \quad f_j \le \mathcal{P}^+(D^2u) + \gamma|Du| + F_\epsilon(x, u, 0, 0).$$

Since $F_\epsilon(x, u, 0, 0) \ge F_\epsilon(x, 0, 0, 0) = 0$ if $u \ge 0$, the first relation above and the maximum principle for viscosity solutions implies

$$u \le \sup_{\partial\Omega} u^+ + C\|f_j\|_{L^n(\Omega)} \le \sup_{\partial\Omega} \psi^+ + C \sup_j \|f_j\|_{L^n(\Omega)}$$

and we conclude that the $u_{\epsilon,j}$ are bounded above independently of ϵ, j. Likewise, the $u_{\epsilon,j}$ are bounded below independently of ϵ, j, and hence the family is uniformly bounded. Using this information and (4.5) for F_ϵ, there exists a constant K such that $|F_\epsilon(x, u_{\epsilon,j}, 0, 0)| \leq K$ and the $u_{\epsilon,j}$ satisfy

$$\mathcal{P}^-(D^2u_{\epsilon,j}) - \gamma|Du_{\epsilon,j}| \leq g_j \quad \text{and} \quad -g_j \leq \mathcal{P}^+(D^2u_{\epsilon,j}) + \gamma|Du_{\epsilon,j}|,$$

where $g_j = |f_j| + K$.

Therefore, using Proposition 4.2, there exists $\epsilon_m \downarrow 0$, $j_m \to \infty$ such that $u_m = u_{\epsilon_m, j_m}$ converges uniformly on $\overline{\Omega}$ to a limit u. By Theorem 3.8 of [4] this u is an L^n-viscosity solution of (4.3); indeed, what we need to check to use this result is only that for $\varphi \in W^{2,n}_{\mathrm{loc}}(\Omega)$ we have

$$F_{\epsilon_m}(x, u_m(x), D\varphi(x), D^2\varphi(x)) \to F(x, u(x), D\varphi(x), D^2\varphi(x)) \tag{4.13}$$

in $L^n_{\mathrm{loc}}(\Omega)$. However, $F_\epsilon(x, r, p, X) \to F(x, r, p, X)$ whenever x is a Lebesgue point of $F(\cdot, r, p, X)$, and almost every x has this property for all r, p, X by $F \in \mathcal{SC}$ (see [4], page 382), which together with (4.5) shows that (4.13) holds pointwise a.e. and (locally) dominated, hence in $L^n_{\mathrm{loc}}(\Omega)$. ☐

We now turn to the parabolic analogue of Theorem 4.1. In this case the initial boundary value problem can be rewritten as before as

$$u_t + F(x, t, u, Du, D^2u) = f(x, t) \quad \text{in} \quad Q = \Omega \times (0, T], \quad u = \psi \quad \text{on} \quad \partial_p Q, \tag{4.14}$$

where

$$F(x, t, 0, 0, 0) \equiv 0. \tag{4.15}$$

The proof of the theorem below is similar to the one in the elliptic case and is therefore omitted, save for the remarks to follow.

Theorem 4.5. *Let $F \in \mathcal{SC}$ satisfy (4.1) and (4.15), let $f \in L^{n+1}(Q)$, $\psi \in C(\partial_p Q)$ and let Ω satisfy a uniform exterior cone condition. Then (4.14) has an L^{n+1}-viscosity solution.*

The parabolic version of the compactness result Proposition 4.2 is

Proposition 4.6. *Let Ω satisfy a uniform exterior cone condition and $\mathcal{C} \subset C(\partial_p Q)$ be compact, $R > 0$ and $B_R = \{f \in L^{n+1}(Q) \colon \|f\|_{L^{n+1}(Q)} \leq R\}$. Then the set of all functions $u \in C(\overline{Q})$ such that there exists $\psi \in \mathcal{C}$ and $f \in B_R$ for which u is an L^{n+1}-viscosity solution of both*

$$u_t + \mathcal{P}^-(D^2u) - \gamma|Du| \leq f \quad \textit{and} \quad -f \leq u_t + \mathcal{P}^+(D^2u) + \gamma|Du| \tag{4.16}$$

in Q and $u = \psi$ on $\partial_p Q$ is precompact in $C(\overline{Q})$.

A version of the maximum principle and an existence result sufficient for the proof of this proposition is given in [7]. The interior Hölder continuity is established in [10], Section 5. The limit theorem needed to complete the proof of Theorem 4.5 is proved in [10], Section 6.

Remark 4.7. The proofs of the existence results above do not require full Propositions 4.2 and 4.6 but rather their versions with B_R replaced by $B_R \cap C(\Omega)$ (or $B_R \cap C(Q)$). In this case, the proofs of the versions of the maximum principles and equi-Hölder continuity results found in Caffarelli [2], Trudinger [30] (elliptic case), and Wang [32] (parabolic case) could be used. This leaves aside (4.7), upon which we have commented. The parabolic analogue is proved in [7]. The proofs of the various maximum principles sketched in [7] might interest the reader in any case.

Remark 4.8. We note again, for emphasis, that Theorems 4.1 and 4.5 are also true if n and $n+1$ are replaced by p in (parameter dependent) appropriate ranges of the form $n-\delta < p$ and $n+1-\delta < p$ respectively. To document this fully in the elliptic case requires results from [4], [28] while [10] contains the parabolic story.

Acknowledgments. Crandall was supported by NSF grant DMS93-02995 and by an appointment as a Miller Research Professor at the University of California, Berkeley. Kocan was supported by the Australian Research Council and an Alexander von Humboldt Fellowhip. Lions was supported by URA CNRS 749 and NSF grant DMS93-02995. Święch was supported by NSF grants DMS97-06760, DMS93-02995.

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