Periodic traveling waves for a nonlocal integro-differential model *

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Abstract
We establish the existence, uniqueness and stability of periodic traveling wave solutions to an integro-differential model for phase transitions.

1 Introduction
In this paper, we are concerned with the following integro-differential model for phase transitions

\[ u_t - Du_{xx} - d(J * u - u) - f(u, t) = 0, \]

where \( x \in \mathbb{R} \) and \( D, d \) are nonnegative constants with \( D + d \neq 0 \); \( J * u(x, t) = \int_{\mathbb{R}} J(x - y)u(y, t)dy \) is the convolution of \( J \) and \( u(x, t) \); \( J \in C^1(\mathbb{R}) \bigcap L^1(\mathbb{R}) \); \( f(u, \cdot) \) is \( T \)-periodic, i.e., \( f(u, t + T) = f(u, t) \) for all \( u, t \in \mathbb{R} \); and \( f(\cdot, t) \) is bistable. Other conditions on \( J \) and \( f \) are specified below. A typical example of \( f \) is the cubic potential function \( f = \rho(1 - u^2)(2u - \gamma(t)) \), where \( \rho > 0 \) is a constant, \( \gamma(t) \) is \( T \)-periodic and \( 0 < \gamma(t) < 2 \).

When \( d = 0 \), equation (1.1) is the classical Allen-Cahn equation [12] for which the results are known. Therefore, we will assume \( d > 0 \) throughout. Equation (1.1) can be considered as a nonlocal version of the Allen-Cahn equation which incorporates spatial long range interaction. When \( d = 0 \) and \( f(u, \cdot) = f(u) \) is independent of \( t \), the traveling wave solution of the form \( u(x, t) = U(x - ct) \) is studied in [12] and [13](see also their references). The nonlocal autonomous case is studied in [5]. X. Chen [9] applied a “squeezing” technique, due to a strong comparison principle, to study the existence, uniqueness and stability of traveling wave solutions for a variety of autonomous nonlocal evolution equations, which includes the Allen-Cahn reaction-diffusion equation, neural networks, the continuum Ising model, and a thalamic model. When \( f(u, t) \) is \( T \)-periodic, periodic traveling wave solutions of the bistable reaction diffusion are studied in [2]. In this paper, we will establish similar results to those in [2] but for the more general equation (1.1).

We assume in this paper that

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H1) $f \in C^2([\mathbb{R} \times \mathbb{R}])$ is periodic in $t$ with period $T$, i.e., there is a $T > 0$ such that $f(u, t) = f(u, t + T)$ for all $u, t \in \mathbb{R}$.

H2) The period map $P(\alpha) := w(\alpha, T)$, where $w(\alpha, t)$ is the solution to

$$w_t = f(w, t), \quad \text{for all } \ t \in \mathbb{R}, \quad w(\alpha, 0) = \alpha,$$  

has exactly three fixed points $\alpha^-, \alpha^0, \alpha^+$, satisfying $\alpha^- < \alpha^0 < \alpha^+$. In addition, they are non-degenerate and $\alpha^\pm$ are stable and $\alpha^0$ is unstable, that is,

$$\frac{d}{d\alpha} P(\alpha^\pm) < 1 < \frac{d}{d\alpha} P(\alpha^0).$$

H3) $J(x) \in C^1(\mathbb{R})$ is nonnegative, $\int_{\mathbb{R}} J(x) \, dx = 1$, and $\int_{\mathbb{R}} |J'(x)| \, dx < \infty$.

In the case $D = 0$, we need the following additional condition,

H4)

$$\sup\{f_u(u, t) : u \in [W^-(t), W^+(t)], t \in [0, T]\} < d,$$  

where $W^\pm(t) = w(\alpha^\pm, t)$.

We are concerned with the periodic traveling wave solutions of (1.1) connecting the two periodic stable solutions $W^\pm(t)$, that is, the solutions of the form $u(x, t) = U(x - ct, t)$, with $U(x, t + T) = U(x, t)$, for all $x, t \in \mathbb{R}$, and $\lim_{x \to \infty} U(\pm x, t) = W^\pm(t)$ uniformly, where $c$ is some real constant (called the wave speed). We claim that the long time behavior of the solutions of (1.1) coupled with the initial condition

$$u(x, 0) = g(x),$$

is governed by the periodic traveling wave solutions $u(x, t) = U(x - ct, t)$ of (1.1). If we work in the traveling wave frame and let $\xi = x - ct$, we are led to study the following problem

$$U_t - cU_{\xi} - DU_{\xi\xi} - d(J * U - U) - f(U, t) = 0,$$  

$$U(\pm\infty, t) = \lim_{\xi \to \pm\infty} U(\xi, t) = w(\alpha^\pm, t), \quad \text{uniformly in } t \in \mathbb{R},$$  

$$U(\cdot, T) = U(\cdot, 0), \quad U(0, 0) = \alpha^0.$$  

The following theorems are our main results concerning the existence, uniqueness and stability of the periodic traveling solutions.

**Theorem 1.1.** Assume (H1), (H2) and (H3) hold. In the case $D = 0$, we also assume (H4). Then there exist a unique smooth function $U(\xi, t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a unique constant $c \in \mathbb{R}$ such that (1.6) - (1.8) hold. Moreover $U(\cdot, t)$ is strictly increasing.
**Theorem 1.2.** The periodic traveling wave solution \( u(x, t) = U(x - ct, t) \), where \( U(\xi, t) \) and \( c \) are as in Theorem 1.1, is uniformly and asymptotically stable.

**Remark 1.1.** (1) If \( u(x, t) = U(x - ct, t) \) is a periodic traveling wave solution of (1.1), so is \( U(x - \xi - ct, t) \), for any \( \xi \in \mathbb{R} \). Therefore, the stability mentioned above is that of the family of spatial translation. Periodic traveling wave solutions are unique modulo a spatial shift.

(2) In the autonomous case, there are discontinuous traveling wave solutions if (H4) fails and the traveling wave solutions are not asymptotically stable. For the periodic case the existence, uniqueness and stability remain open in general, that is, without (H4).

The paper is organized as follows. In Section 2 we study the uniqueness and monotonicity of the wave. In Section 3, we use a homotopy method to prove the existence of the solution to (1.6)-(1.8). And finally in Section 4 we study the uniform and asymptotic stability of the periodic traveling wave solution.

## 2 Uniqueness of Periodic Traveling Waves

In this section, we will establish the uniqueness of smooth periodic traveling wave solutions of (1.6)-(1.8) and prove that the wave is strictly monotone in the spatial direction.

For a metric space \( X \), denote

\[
C_{unif}(X) = \{ u : X \to \mathbb{R}, u \text{ is bounded and uniformly continuous on } X \},
\]

and denote \( ||u|| = \sup_{x \in X} |u(x)| \). First we need the following comparison principle.

**Lemma 2.1.** (Comparison Principle). Suppose that \( R_1 \) is a union of open intervals, \( R_2 = \mathbb{R} \setminus R_1 \), and that, for some \( \tau < t_0 \), \( u(x, t) \in C_{unif}(\mathbb{R} \times [\tau, t_0]) \) has the required derivatives. Assume that \( u(x, t) \geq 0 \) for all \( x \in R_2 \) and \( t \in (\tau, t_0) \), and \( u(x, t) \) satisfies

\[
u_t - Du_{xx} - d(J * u - u) - bu_x - cu \geq 0
\]

on \( R_1 \times (\tau, t_0) \), where \( D \) and \( d \) are nonnegative constants with \( D + d \neq 0 \), \( b = b(x, t) \), \( c = c(x, t) \) are bounded continuous functions on \( R_1 \times (\tau, t_0) \). If \( u(x, \tau) \geq 0 \) for all \( x \in \mathbb{R} \), then \( u(x, t) \geq 0 \) for all \( x \in \mathbb{R} \) and \( t \in (\tau, t_0) \). Moreover, if \( u(x, t) \) is not identically 0 on \( R_1 \times (\tau, t_0) \), then \( u(x, t) > 0 \), for all \( x \in R_1 \) and \( t \in (\tau, t_0) \).

**Proof.** We may assume \( \tau = 0 \). We take \( d > 0 \) since the result is standard for \( d = 0 \). By the assumption that \( u(x, t) \in C_{unif}(\mathbb{R} \times [0, t_0]) \), \( \inf_{x \in \mathbb{R}} u(x, t) \) is continuous on \([0, t_0] \). If the conclusion of the lemma is not true, there exist constants \( \epsilon > 0 \) and \( T_0 > 0 \) such that \( u(x, t) > -\epsilon e^{2Kt} \), for all \( x \in \mathbb{R} \) and \( 0 < t < T_0 \), and
\[ \inf_{x \in \mathbb{R}} \{ u(x, T_0) \} = -\varepsilon e^{2Kt_0}, \text{where } K = 2(D + 4d + b_0 + 4c_0), \quad b_0 = \sup \{ |b(x, t)| : x \in R_1, t \in [\tau, t_0] \}, \quad \text{and } c_0 = \sup \{ |c(x, t)| : x \in R_1, t \in [\tau, t_0] \}. \]

Let \( z(x) \) be a smooth function such that \( 1 \leq z(x) \leq 3, \quad z(0) = 1, \quad z(\pm \infty) = \lim_{x \to \pm \infty} z(\pm x) = 3, \) and \( |z'(x)| \leq 1, \quad |z''(x)| \leq 1. \) Define \( w_\sigma(x, t) = -\varepsilon (z + \sigma z(x)) e^{2Kt} \) for \( \sigma \in [0, 1]. \)

Notice that \( w_1(x, t) < u(x, t) \) and \( w_0(x, t) = -\varepsilon e^{2Kt} \) for \( (x, t) \in R_1 \times (0, t_0). \)

There is a minimum \( \sigma^* \in \left[ \frac{1}{5}, 1 \right) \) such that \( w_{\sigma^*}(x, t) \leq u(x, t), \) for \( x \in \mathbb{R} \) and \( t \in [0, T_0] \) and there exists \((x_1, t_1) \in R_1 \times (0, T_0) \) such that \( u(x_1, t_1) = w_{\sigma^*}(x_1, t_1). \)

Therefore, at \((x_1, t_1),\)

\[
0 \geq (u - w_{\sigma^*})_t - D(u - w_{\sigma^*})_{xx} - d(J * (u - w_{\sigma^*})) - b(u - w_{\sigma^*}) - c(u - w_{\sigma^*}) \\
\geq \varepsilon e^{2Kt_1} \left[ \frac{7}{8} K - D - 4d - b_0 - 4c_0 \right] > 0,
\]

by the choice of \( K, \) which is a contradiction. Therefore \( u(x, t) \geq 0 \) for all \( x \in \mathbb{R} \) and \( t \in (0, t_0]. \) Suppose \( u(x, t) \) is not identically zero on \( R_1 \times (0, t_0] \) and there is a point \((x_2, t_2) \in R_1 \times (0, t_0) \) such that \( u(x_2, t_2) \) achieves the minimum \( 0. \) By a similar argument to the above we deduce that \( (J * u - u)(x_2, t_2) = 0. \) Therefore \( u \equiv 0, \) which is a contradiction. That completes the proof. \( \square \)

Now we are ready to state the uniqueness theorem.

**Theorem 2.2.** Suppose (H1), (H2) and (H3) hold. Then problem (1.6)-(1.8) admits at most one smooth solution.

**Proof.** The proof is similar to that in [2]. Let \((U, c)\) and \((\overline{U}, \overline{c})\) be any two solutions of (1.6)-(1.8) with \( c \geq \overline{c}. \) We prove \( U = \overline{U} \) and \( c = \overline{c}, \) we divide the proof into six steps.

1. By periodicity of \( U(\xi, t) \) and the comparison principle, we have

\[ w(-M^-, kt + t) \leq U(\xi, kt + t) = U(\xi, t) \leq w(M^+, kt + t), \]

where \( M^\pm = \sup_{\xi \in \mathbb{R}} \pm U(\xi, 0). \) Letting \( k \to \infty \) gives \( W^-(t) \leq U(\xi, t) \leq W^+(t). \)

By Lemma 2.1, we have

\[ W^-(t) < U(\xi, t) < W^+(t). \quad (2.3) \]

2. Define \( \nu^\pm = -\frac{1}{4} \int_0^T f_u(W^\pm(t), t)dt. \) Without loss of generality, we may assume \( \nu^+ \geq \nu^- \). The other case can be proved similarly. Let \( \nu = (\nu^+ - \nu^-)/2 \) and \( a^\pm(t) = \exp(\frac{\nu^+}{2} + \int_0^t f_u(W^\pm(\tau), \tau)d\tau). \) Notice that \( P'(a^\pm) = \exp(\int_0^T f_u(W^\pm(\tau), \tau)d\tau) < 1. \) We have \( \nu^+ > 0 \) and \( a^\pm(T) < 1. \) Moreover, there exist two constants \( C_1 \) and \( C_2 \) such that

\[ C_2 a^-\leq a^+(t)e^{\nu t} \leq C_1 a^+(t). \quad (2.4) \]

For \( \nu > 0 \) and \( t \in [0, T], \) let \( I^\pm_\eta(t) := [W^\pm(t) - \eta, W^\pm(t) + \eta] \) and define

\[ \delta_0 = \frac{\sup \{ \eta : |f_u(u, t) - f_u(W^\pm(t), t)| \leq \nu^\pm / 4, \text{ for } t \in [0, T], u \in I^\pm_\eta(t) \}}{2\|a^\pm(\cdot)\|_{C^0([0,T])} + 2\|a^- (\cdot)\|_{C^0([0,T])}}. \]
and let \( \zeta(\xi) \) be a smooth function such that \( 0 \leq \zeta(\xi) \leq 1 \), \( \zeta(\xi) = 0 \) for \( \xi \leq -2 \), and \( \zeta(\xi) = 1 \) for \( \xi \geq 2 \). Let \( a(\xi, t) = e^{\nu t}a^+(t)\zeta(\xi) + a^-(t)(1 - \zeta(\xi)) \). Define

\[
\xi_0 = \inf\{\xi \geq 2 : |d(J * \zeta - \zeta)(\pm \xi)(a^+(t)e^{\nu t} - a^-(t))| \leq \frac{\nu}{4} \min\{a^+(t)e^{\nu t}, a^-(t)\},
\]

and \( |U(\pm \xi, t) - W(\pm t)| < \delta_0 / 2, \forall \xi \geq \xi_0, t \in [0, T] \} \).

This \( \xi_0 \) is well defined since \( \lim_{\xi \to -\infty} U(\pm \xi, t) = W(\pm t) \) uniformly in \( t \) and \( \lim_{x \to \infty} (J * \zeta - \zeta)(\pm x) = 0 \).

For each \( \delta \in (0, \delta_0 / 2] \), define \( U_\delta(\xi, t) = U(\xi, t) + \delta a(\xi, t) \). Then, on \( (\xi_0, +\infty) \),

\[
L'U_\delta(\xi, t) := U_{\delta t} - \delta U_\xi - D U_{\delta \xi} - d(J * U_\delta - U_\delta) - f(U_\delta, t) = f(U, t) - f(U + \delta a^+(t), t) + [\nu^2 / 2 + f_w(W(\xi, t), t) + \nu]a^+(t)e^{\nu t} - \delta d[a^+(t)e^{\nu t} - a^-(t)](J * \zeta - \zeta) \\
= \delta a^+e^\nu[t \nu^2 / 2 + f_w(W, t, t) - \int_0^1 f_w(U + \delta a^+(t), t)d\theta] - \delta d[a^+(t)e^{\nu t} - a^-(t)](J * \zeta - \zeta) \\
\geq \nu^2 / 4 + \nu \delta a^+(t)e^{\nu t} - \delta d[a^+(t)e^{\nu t} - a^-(t)](J * \zeta - \zeta) \\
\geq 0. \tag{2.5}
\]

where we have used the fact that \( a(\xi, t) = a^+(t)e^{\nu t} \) on \( (\xi_0, +\infty) \) and the definitions of \( \xi_0 \) and \( \delta_0 \). Similarly, we have \( L'U_\delta(\xi, t) \geq 0 \), on \( (-\infty, -\xi_0) \). That is, \( U_\delta(\xi, t) \) is a super solution on \( ((-\infty, -\xi_0) \cup (\xi_0, +\infty)) \times \mathbb{R}^+ \).

3. Since \( \lim_{\xi \to -\infty} \bar{U}(\pm \xi, t) = W(\pm t) \) uniformly in \( t \), by (2.3), there exists a large constant \( \hat{\xi}_0 \) such that

\[
\bar{U}(\xi - z + (c - \tau)t, t) \leq \begin{cases} U(\xi, t), & \text{if } \xi \in [-\xi_0, \hat{\xi}_0]; \\
U(\xi, t) + \delta_0, & \text{if } \xi \not\in [-\xi_0, \hat{\xi}_0]; 
\end{cases}
\]

for all \( t \in [0, T] \) and \( z \geq \hat{\xi}_0 \). Define \( \hat{\delta} := \inf\{\delta > 0 : \bar{U}(\xi - z, 0) \leq U(\xi, 0) + \delta, \forall \xi \geq \hat{\xi}_0, \delta \in \mathbb{R}\} \). Obviously, \( \hat{\delta} \leq \delta \). We claim that \( \hat{\delta} = 0 \). In fact, for \( z > \hat{\xi}_0 \), \( L'\bar{U}(\xi - z + (c - \tau)t, t) = 0 \). And on \( [-\xi_0, \hat{\xi}_0] \times (0, T] \),

\[
U_\hat{\delta}(\xi, t) = U(\xi, t) + \hat{\delta} a(\xi, t) \geq U(\xi, t) \geq \bar{U}(\xi - z + (c - \tau)t, t), \tag{2.6}
\]

and

\[
U_\hat{\delta}(\xi, 0) = U(\xi, 0) + \hat{\delta} a(\xi, 0) = U(\xi, 0) + \hat{\delta} \geq \bar{U}(\xi - z, 0) \tag{2.7}
\]

for all \( \xi \in \mathbb{R} \). By Lemma 2.1, we have

\[
\bar{U}(\xi - z + (c - \tau)t, t) \leq U_\hat{\delta}(\xi, t)
\]

for all \( z \geq \hat{\xi}_0, (\xi, t) \in \mathbb{R} \times (0, T] \). Since \( z \geq \hat{\xi}_0 \) is arbitrary, we have

\[
\bar{U}(\xi - z, t) \leq U_\hat{\delta}(\xi, T)
\]
for all $z \geq \hat{z}_0$. By the periodicity of $U(\xi, \cdot)$, we have
\[
U(\xi - z, T) \leq U(\xi, 0) + \hat{\delta}a(\xi, T)
\]
for all $z \geq \hat{z}_0$, $\xi \in \mathbb{R}$. Therefore,
\[
U(\xi - z, 0) \leq U(\xi, 0) + \hat{\delta} \max\{a^+(T)e^{\nu T}, a^-(T)\}
\]
for all $z \geq \hat{z}_0$, and $\xi \in \mathbb{R}$. This contradicts the definition of $\hat{\delta}$ since $a^\pm(T) < 1$.
Therefore,
\[
U(\xi - z, 0) \leq U(\xi, 0)
\]
for all $\xi \in \mathbb{R}$ and $z \geq \hat{z}_0$.

4. By the comparison principle (Lemma 2.1), $U(\xi - z + (c - \tau)t, t) \leq U(\xi, t)$, for all $\xi \in \mathbb{R}$, $t \geq 0$, and $z \geq \hat{z}_0$. Therefore by periodicity, $U((c - \tau)kT - z, 0) \leq U(0, kT) = a^0$. Letting $k \to \infty$, we deduce that $c = \tau$ since $U((c - \tau)kT - z, 0) \to a^+$ if $c > \tau$.

5. Define $z_0 = \inf\{\hat{z}_0 : U(\xi - z, 0) \leq U(\xi, 0), \text{ for all } \xi \in \mathbb{R}, z \geq \hat{z}_0\}$. Similar to the proof in step 3, we can show that $U(\xi - z_0, 0) = U(\xi, 0)$ for $\xi \in \mathbb{R}$.

6. We prove that $z_0 = 0$. If not, $U(\xi - z, 0) < U(\xi, 0)$ for all $\xi \in \mathbb{R}, z > z_0$.
By the comparison principle and periodicity,
\[
U(\xi - z + z_0, 0) = U(\xi - z_0 - z + z_0, 0) = U(\xi - z) < U(\xi, 0),
\]
since $U(\xi - z_0, 0) = U(\xi, 0)$. Therefore $U(\xi, 0)$ is strictly increasing. Since $U(z_0, 0) = U(0, 0) = a_0 = U(0, 0)$, we deduce that $z_0 = 0$. That completes the proof. \(\square\)

**Corollary 2.3.** Under the conditions of Theorem 2.2, any smooth solution to (1.6)-(1.8) is strictly increasing.

### 3 Existence of Periodic Traveling Waves

In this section, we are going to establish the existence of the periodic traveling wave solution to (1.6)-(1.8) by a homotopy argument.

Assume $(U_0, c_0)$ is the unique solution of the following problem, corresponding to the parameter $\theta = \theta_0 \leq 1$,

\[
\begin{align*}
U_t - cU_\xi &- [1 - \theta(1 - D)]U_{\xi\xi} - \theta d(U \ast U - U) - f(U, t) = 0, \quad (3.1) \\
U(\pm\infty, t) &\to 0, \quad \text{uniformly in } t \in \mathbb{R}, \quad (3.2) \\
U(\cdot, T) &\to U(\cdot, 0), \quad U(0, 0) = a^0, \quad (3.3)
\end{align*}
\]
satisfying $U_{0\xi} > 0$, $U_{0\xi}(\xi, t) \to 0$ uniformly in $t$ as $\xi \to \pm\infty$.

Let
\[
X_0 = \{v : v \in C_{unif}(\mathbb{R} \times \mathbb{R}), v(\cdot, t+T) = v(\cdot, t) \text{ and } \lim_{x \to \infty} v(\pm x, t) = 0, \forall t \in \mathbb{R}\}.
\]
and $L = L(U_0, c_0, \theta_0)$ be the linearization of the operator in (3.1)-(3.3) defined by

$$D(L) = X_2 := \{ v \in X_0 : v_{\xi \xi}, v_\xi, v_t \in X_0 \}.$$ 

$$Lv = v_t - [1 - \theta_0(1 - D)]v_{\xi \xi} - \theta_0 d(J * v - v) - c_0 v_\xi - f_u(U_0, t)v \quad (3.4)$$

for $v \in D(L)$. We first establish some lemmas.

Lemma 3.1. $L$ has 0 as a simple eigenvalue.

Proof. Clearly, $p = U_0\zeta$ is an eigenfunction corresponding to the eigenvalue 0. We only need to prove the simplicity. Suppose $\phi(\xi, t) \in X_0$ is another eigenfunction with eigenvalue 0. We prove that $\phi = zp$, for some constant $z \in \mathbb{R}$.

Let $\nu^\pm$ be defined as in Section 2. Without loss of generality we assume $\nu^+ \geq \nu^-$. Let $\nu = (\nu^+ - \nu^-)/2$ be as in Section 2. Suppose $\zeta(\xi)$ is a smooth function such that $\zeta(\xi) = 0$, for $\xi < 0$; $\zeta(\xi) = 1$, for $\xi > 0$; and 0 $\leq \zeta(\xi) \leq 1$, 0 $\leq \zeta'(\xi) \leq 1$, and $|\zeta''(\xi)| \leq 1$, for all $\xi \in \mathbb{R}$. Define

$$A(\xi, t) = \zeta(\xi)a^+(t)e^{\nu t} + (1 - \zeta(\xi))a^-(t),$$

$$B(t) = \int_0^t \max\{a^+(\tau)e^{\nu \tau}, a^-(\tau)\} d\tau,$$

$$K = \frac{\nu^+ - \nu^-/2 + (D + d) + 2c_0 + 2\|f_u\|_{\min(\zeta, t) \in [-\xi_0, \xi_0] \times [0, T]} U_0(\zeta, t)}{\min(\zeta, t) \in [-\xi_0, \xi_0] \times [0, T]} U_0(\zeta, t),$$

where $\|f_u\| = \sup\{|f_u(u, t)| : u \in [W^-(t), W^+(t)], t \in [0, T]\}$ and $\xi_0$ is a large constant to be chosen later. Let $\Psi(\xi, t) = KB(t)U_0\zeta(\xi, t) + A(\xi, t)$, then

$$\Psi(\xi, 0) = 1.$$ We claim that

$$L\Psi(\xi, t) = KBtU_0\zeta(\xi, t) + LA(\xi, t) \geq 0. \quad (3.8)$$

We divide the proof by considering three intervals $(-\infty, -\xi_0)$, $[-\xi_0, \xi_0]$, and $(\xi_0, \infty)$. We assume $\xi_0 > 4$.

On $(\xi_0, \infty)$, $A(\xi, t) = a^+(t)e^{\nu t}$, therefore

$$LA(\xi, t) = [\nu^+ / 2 + f_u(W^+(t), t) + \nu - f_u(U_0(\xi, t), t)a^+(t)e^{\nu t}$$

$$- \theta_0 d(J * \zeta - \zeta)a^+(t)e^{\nu t} - a^-(t)].$$

Notice that $(J * \zeta - \zeta)(\xi) \to 0$, and $U_0(\xi, t) \to W^+(t)$ as $\xi \to \infty$. We deduce, by (2.4), that we can choose $\xi_0$ large enough such that

$$LA(\xi, t) \geq 0,$$ on $(\xi_0, \infty) \times \mathbb{R}^+.$

Similarly we have

$$LA(\xi, t) = [\nu^- / 2 + f_u(W^-(t), t) - f_u(U_0(\xi, t), t)a^-(t)$$

$$- \theta_0 d(J * \zeta - \zeta)a^+(t)e^{\nu t} - a^-(t)] \quad \text{on} \ (-\infty, -\xi_0).$$
Therefore there exists \( \xi_0 >> 1 \) such that
\[
LA(\xi, t) \geq 0, \quad \text{on } (-\infty, -\xi_0) \times \mathbb{R}^+.
\]

We fix \( \xi_0 \) large enough such that \( LA(\xi, t) \geq 0, \) on \(((-\infty, -\xi_0) \cup (\xi_0, \infty)) \times \mathbb{R}^+\). On \([-\xi_0, \xi_0]\),
\[
|LA(\xi, t)| = |A_1 - [1 - \theta_0(1 - D)]A_{\xi \xi} + \theta_0d(U_0(\xi, t) - A) - c_0A_{\xi} - f_u(U_0(\xi, t), t)A(\xi, t)|
\leq \max\{a^+(t)e^{\nu t}, a^-(t)\} \{\nu^+ - \nu^-/2 + [1 - \theta_0(1 - D)] + \theta_0d + 2c_0 + 2\|f_u\|\}
\]

Therefore \( L\Psi(\xi, t) \geq 0, \) on \([-\xi_0, \xi_0\]) by (3.8) and the choice of \( K \) in (3.7).

By the comparison principle, we have
\[
\phi(\xi, t) \leq \Psi(\xi, t)\|\phi(\xi, 0)\|_{\infty}.
\]
Letting \( t = kT \) and letting \( k \to \infty \), we have
\[
|\phi(\xi, 0)| \leq KB(\infty)\|\phi(\xi, 0)\|_{\infty}U_0\xi(\xi, 0),
\]
where \( B(\infty) = \lim_{t \to \infty} B(t) \). The limit exists since \( a^\pm(t) \to 0 \) exponentially and (2.4) holds.

Let \( z_\ast := \sup\{z : \phi(\xi, 0) \geq zU_0\xi(\xi, 0), \text{ for all } \xi \in \mathbb{R}\} \). We claim that \( \phi(\xi, 0) = z_\ast U_0\xi(\xi, 0), \) for all \( \xi \in \mathbb{R} \). If not, there exists a point \( \xi_0 \) such that \( \phi(\xi_0, 0) > z_\ast U_0\xi(\xi_0, 0) \). Then by the comparison principle, \( \phi(\xi, T) > z_\ast U_0\xi(\xi, T) \). Replacing \( \phi \) by \( \phi - z_\ast U_0\xi \), we can assume \( z_\ast = 0 \). So, \( \phi(\xi, 0) > 0 \), for all \( \xi \in \mathbb{R} \). Choose \( \xi \) such that \( KB(\infty)\sup_{|\xi| \geq \xi} U_0\xi(\xi, 0) < 1/4 \) and choose \( \epsilon \) such that \( \phi(\xi, 0) > \epsilon U_0\xi(\xi, 0), \) on \([-\xi, \xi]\). Then
\[
\phi(\xi, 0) - \epsilon U_0\xi(\xi, 0) \geq -\epsilon \sup_{|\xi| \geq \xi} U_0\xi(\xi, 0),
\]
and therefore,
\[
\phi(\xi, t) - \epsilon U_0\xi(\xi, t) \geq -\epsilon \Psi(\xi, t) \sup_{|\xi| \geq \xi} U_0\xi(\xi, 0).
\]
Letting \( t = kT \) and letting \( k \to \infty \), we have
\[
\phi(\xi, 0) - \epsilon U_0\xi(\xi, 0) \geq -\frac{1}{4} \epsilon U_0\xi(\xi, 0),
\]
which contradicts the definition of \( z_\ast \), and completes the proof. \( \Box \)

Since \( J * u - u \) is a bounded operator on \( X_0 \), we know that \( 0 \) is an isolated eigenvalue of \( L \) for \( \theta_0 < 1 \). Now consider the adjoint operator \( L^* = L^*(U_0, c_0, \theta_0) \) of \( L \). Since the comparison principle holds for \( L \), we know that \( 0 \) is an isolated eigenvalue for \( L^* \) with a positive eigenfunction (see Section 11.4 and theorem 9.11 in [17]). We denote by \( \phi^*(x, t) \) the positive eigenfunction of \( L^* \) corresponding to the eigenvalue \( 0 \).
Lemma 3.2. With \( \theta_0, U_0, \) and \( c_0 \) as above with \( \theta_0 < 1 \), there exists \( \eta > 0 \) such that for each \( \theta \in [\theta_0, \theta_0 + \eta] \), (3.1)-(3.3) has a solution \( (U(\theta, \xi, t), c(\theta)) \).

Proof. Consider the operator \( G : (X_2 \times \mathbb{R}) \times \mathbb{R} \rightarrow X_0 \times \mathbb{R} \) defined by

\[
G(w, \theta) = ((U_0 + v)_t - [1 - \theta(1 - D)](U_0 + v)_{\xi\xi} - \theta d(J \ast (U_0 + v) - (U_0 + v)) - (c_0 + c)(U_0 + v)_{\xi} - f(U_0 + v, t), v(0, 0))
\]

for \( w = (v, c) \in X_2 \times \mathbb{R} \). Then \( G \) is of class \( C^1 \), \( G(0, \theta_0) = (0, 0) \) and

\[
\frac{\partial G}{\partial w}(0, \theta_0) = \begin{bmatrix} L & \delta U_{0\xi} \\ \delta & 0 \end{bmatrix},
\]

where \( \delta \) is the \( \delta \)-function. We show that \( \frac{\partial G}{\partial w}(0, \theta_0) \) is invertible. Consider the equation on \( X_0 \times \mathbb{R} \):

\[
\frac{\partial G}{\partial w}(0, \theta_0) \begin{bmatrix} v \\ c \end{bmatrix} = \begin{bmatrix} h \\ b \end{bmatrix}, \quad \text{for } \begin{bmatrix} h \\ b \end{bmatrix} \in X_0 \times \mathbb{R},
\]

i.e.,

\[
Lu + cU_{0\xi} = h, \quad v(0, 0) = b.
\]

By the Fredholm Alternative, (3.9) is solvable if and only if \( h - cU_{0\xi} \perp \phi^* \), i.e.,

\[
\int_0^T \int_{\mathbb{R}} [h\phi^* - cU_{0\xi}\phi^*] \, dx \, dt = 0.
\]

Since \( U_{0\xi} > 0 \) and \( \phi^* > 0 \), \( c \) is uniquely determined by (3.11). After we determine \( c \), the solution \( v \) of (3.9) is determined up to a term \( kU_{0\xi} \), where \( k \) is a constant. Then (3.10) determines \( k \) uniquely. Therefore \( \frac{\partial G}{\partial w}(0, \theta_0) \) is invertible. The lemma now follows from the Implicit Function Theorem. \( \square \)

Lemma 3.3. Suppose that for \( \theta \in [0, \theta] \), where \( \theta \leq 1 \), there exists a solution \( (U(\theta, \xi, t), c(\theta)) \) of (3.1)-(3.3). Then \( \|U(\theta, \cdot, \cdot)\|_{L^\infty(\mathbb{R} \times [0,T])}, \|U_\xi(\theta, \cdot, \cdot)\|_{L^\infty(\mathbb{R} \times [0,T])} \) and \( \|U_t(\theta, \cdot, \cdot)\|_{L^\infty(\mathbb{R} \times [0,T])} \) are uniformly bounded for \( \theta \in [0, \theta] \).

Proof. For the case \( \theta < 1 \), the conclusion of the lemma follows from classical parabolic estimates. Therefore we take \( \theta = 1 \), and prove the lemma for \( \theta \) near 1. We only prove the uniform boundedness of \( U_\xi(\theta, \xi, t) \); all others are similar. Let \( v(\theta, \xi, t) := U_\xi(\theta, \xi, t) \) and \( M = \sup_{\xi, t \in \mathbb{R}} |J^* \ast U(\theta, \xi, t)| \). Then \( v(\theta, \xi, t) \) satisfies

\[
v_t - [1 - \theta(1 - D)]v_{\xi\xi} + \theta dv - c(\theta)v_{\xi} - f_a(U(\theta, \xi, t), t)v = \theta dJ^* \ast U.
\]

Define \( l(\theta) := \theta d - \sup \{ f_a(u, t) : u \in [W^-(t), W^+(t)], t \in [0, T] \} \). For \( \theta \in [0, 1] \) such that \( l(\theta) > 0 \), we have, by the comparison principle for parabolic equations,

\[
v(\theta, \xi, t) \leq e^{-l(\theta)t} \sup_{\xi \in \mathbb{R}} |v(\theta, \xi, 0)| + (1 - e^{-l(\theta)t})M / l(\theta).
\]

By periodicity, we deduce that \( v(\theta, \xi, t) \) is uniformly bounded for \( \theta \in [0, 1] \) with \( l(\theta) > 0 \). \( \square \)
Lemma 3.4. Suppose that there is a sequence \( \theta_j \) such that \( \lim_{\theta_j \to \overline{\theta}} U(\theta, \xi, t) = U(\overline{\theta}, \xi, t) \) uniformly with respect to \((\xi, t) \in \mathbb{R} \times [0, T] \) for some function \( U(\overline{\theta}, \xi, t) \). Then \( \{\overline{c}(\theta_j)\} \) is bounded.

Proof. First we prove the following statement. Suppose \((\overline{V}, \overline{C})\) satisfies, for some \( \overline{\xi} > 0 \),

\[
\begin{align*}
\overline{V} - (1 - \theta)(1 - D)\overline{V} - \theta d(J * \overline{V} - \overline{V}) - \overline{C} \overline{V} - f(\overline{V}, t) & \leq 0, \\
\text{in } (-\infty, \overline{\xi}) \times (0, T], & \\
\overline{V}(\infty, t) < W^{-}(t), \text{ for } t \in [0, T], & \\
\overline{V}(\xi, 0) & \leq \overline{V}(\xi, t), \text{ on } (-\infty, \overline{\xi}),
\end{align*}
\]

and \( \overline{V}(\xi, 0) \) is monotonically increasing. Then \( c(\theta) \leq \overline{C} \).

In fact, if \( c(\theta) > \overline{C} \), then \( U(\theta, \xi, t) \) satisfies

\[
L^C U(\theta, \xi, t) := U_t - (1 - \theta)U_{\xi\xi} - \theta (J * U - U) - \overline{C} U_\xi - f(U, t) = (c(\theta) - \overline{C}) U_\xi > 0.
\]

Let \( m_0 = \inf\{m : U(\theta, \xi, 0) > \overline{V}(\xi - m_0, 0), \text{ for } \xi \in \mathbb{R} \} \). Then by assumption, \( m_0 \) is well defined and \( m_0 \geq 0 \). Moreover, there exists a point \( \xi_0 \in (-\infty, \overline{\xi}) \) such that \( U(\theta, \xi_0, 0) = \overline{V}(\xi_0 - m_0, 0) \). Applying the strong comparison principle on \((-\infty, \overline{\xi}) \times [0, T] \), we get \( U(\theta, \xi, t) > \overline{V}(\xi - m_0, t), \) for all \( \xi \in \mathbb{R}, t \in [0, T] \). This is a contradiction since \( U(\theta, \xi_0, 0) = \overline{V}(\xi_0 - m_0, 0) = \overline{V}(\xi_0 - m_0, 0) \leq \overline{V}(\xi_0 - m_0, T) \), and the claim is proved.

We denote \( \theta_j \) by \( \theta \). Let \( \zeta(s) = [1 + \tanh(s/2)]/2, W_1(t) = w(\alpha^+ - \epsilon, t) \) and \( W_2(t) = w(\alpha^- - \epsilon, t) \), where \( \epsilon \) is a small constant to be chosen. Let \( \overline{V}(\xi, t) = W_1(t)(\zeta(\xi + \xi_0) + W_2(t)(1 - \zeta(\xi + \xi_0)), \) where \( \xi_0 \) is a constant such that \( \zeta(\xi_0) = \alpha^+ - \alpha^- + \epsilon \). Since \( \lim_{\theta_j \to \overline{\theta}} U(\theta, \xi, t) = \overline{V}(\zeta(\overline{\xi})), \) and \( \overline{V}(\overline{\xi}) > 0 \), \( \overline{V}(\overline{\xi}) = \alpha^+ - \epsilon, \) and \( \overline{V}(\overline{\xi}) = \alpha^- - \epsilon \). Since \( \lim_{\theta_j \to \overline{\theta}} U(\theta, \xi, t) = U(\overline{\theta}, \xi, t) \) uniformly and \( U(\theta, +\infty, t) = W^+(t) \), we can choose \( \overline{\xi} \) sufficiently large such that \( U(\theta, \xi, t) > \overline{V}(\xi, t) \), for \( (\xi, t) \in [\overline{\xi}, \infty) \times [0, T] \). For \( \xi < \overline{\xi} \),

\[
L^C(V) = \overline{V}_t - (1 - \theta)(1 - D)\overline{V}_{\xi\xi} - \theta d(J * \overline{V} - \overline{V}) - \overline{C} \overline{V}_\xi - f(\overline{V}, t) = (1 - \zeta)(W_1 - W_2)[(1 - \theta(1 - D))(1 - 2z)] - \theta d(W_1 - W_2)J * \zeta - \zeta f(W_1, t) + (1 - \zeta)J * \zeta f(W_2, t) - f(W_1\zeta + W_2(1 - \zeta), t)
\]

where we use the Taylor's expansion

\[
\zeta f(W_1, t) + (1 - \zeta)J * \zeta f(W_2, t) - f(W_1\zeta + W_2(1 - \zeta), t) = \zeta(1 - \zeta)(W_1 - W_2)^2 f_uW_1(\sigma, t)
\]
for some $\sigma \in [W_2, W_1]$. If we choose $\mathcal{C} = 1 + D + \frac{1}{\sqrt{2}} \sup \{(W^+(t) - W^-(t) + 2|f_{uu}(u, t)|) : u \in [W^-(t) - 1, W^+(t) + 1], t \in [0, T]\} + \sup_{\xi \leq \zeta} d/(1 - \zeta(\xi))$, then $\mathcal{C} < 0$ for $\xi < \zeta$.

Therefore $c(\theta) \leq \mathcal{C}$ by our earlier observation. We can get a lower bound estimate similarly.

We are ready to obtain a solution to (3.1)-(3.3).

**Theorem 3.5.** Under the conditions of Theorem 1.1, there exists a solution $(U(\theta, \xi, t), c(\theta))$ to (3.1)-(3.3) for all $\theta \in [0, 1]$.

**Proof.** By the result in [2], there exists a solution $(U_0, c_0)$ to (3.1)-(3.3) corresponding to $\theta = 0$, such that $U_0 > 0$ and $\lim_{\xi \to \infty} U_0 = 0$ uniformly with respect to $t$. By Lemma 3.2, there exists an interval $[0, \mathcal{T}]$ such that for all $\theta \in [0, \mathcal{T}]$ system (3.1) - (3.3) has a solution $(U(\theta, \xi, t), c(\theta))$ with the required properties. Suppose $[0, \eta]$ is the maximal interval such that (3.1)-(3.3) admits a solution for each $\theta \in [0, \eta)$. Then we claim that $\eta = 1$ and (3.1)-(3.3) admits a solution for each $\theta \in [0, 1]$. By Lemma 3.3 and Helly's theorem, we can choose a subsequence $\eta_j$ such that $\lim_{j \to \infty} \eta_j = \eta$, and $\lim_{j \to \infty} U(\eta_j, \xi, t)$ exists uniformly for all $\xi \in \mathbb{R}$ and each rational $t$. By Lemma 3.3 again, $\|U(t, \xi, t)\|$ is uniformly bounded for all $\xi \in [0, \eta)$. Therefore there exists a uniformly continuous function $U(\eta, \xi, t)$ such that $\lim_{j \to \infty} U(\eta_j, \xi, t) = U(\eta, \xi, t)$ uniformly for all $(\xi, t) \in \mathbb{R} \times [0, T]$. Moreover, by choosing a subsequence if necessary, the derivatives of $U(\eta, \xi, t)$ converge to the corresponding derivatives of $U(\eta, \xi, t)$ uniformly on any compact set of $\mathbb{R} \times [0, T]$. Therefore by Lemma 3.4, we can choose a subsequence of $\{\eta_j\}$ (we label it the same) such that $c(\eta_j) \to c(\eta)$. Therefore $(U(\eta, \xi, t), c(\eta))$ is a solution to (3.1)-(3.3) corresponding to parameter $\eta$, with the same properties as $(U_0, c_0)$. Therefore, either $\eta = 1$, or we can extend the existence interval to $[0, \eta + \epsilon]$ for some $\epsilon > 0$, which would contradict the maximality of $\eta$. Therefore, for all $\theta \in [0, 1]$, (3.1)-(3.3) has a solution. $\square$

4 Stability of the Periodic Traveling Waves

In this section, we study the stability and asymptotic stability of the periodic traveling wave solutions $U(x - ct, t)$ obtained in Section 3.

We denote by $u(x, t; g)$ the solution to the initial value problem

\begin{align*}
  u_t - Du_{xx} - d(J * u - u) - f(u, t) &= 0, \quad \text{in } \mathbb{R} \times (0, \infty), \quad (4.1) \\
  u(x, 0) &= g(x), \quad \text{on } \mathbb{R}, \quad (4.2)
\end{align*}

where $g(\cdot) \in L^\infty(\mathbb{R})$. For the existence and uniqueness of (4.1) and (4.2), we have

**Lemma 4.1.** For any $g(\cdot) \in L^\infty(\mathbb{R})$, there exists a unique solution $u(x, t; g) \in C^1([0, \infty), L^\infty(\mathbb{R}))$ of (4.1) and (4.2). Moreover, $u(\cdot, t; g)$ is continuous from $[0, \infty) \times C_{\text{unif}}(\mathbb{R})$ to $C_{\text{unif}}(\mathbb{R})$. 

Proof. The case $D \neq 0$ follows from standard parabolic theory. We only need to consider the case where $D = 0$. Write (4.1) and (4.2) as

$$u(x, t) = g(x) + \int_0^t (d(J * u - u) + f(u, t)) \, dt. \quad (4.3)$$

Then the local existence and uniqueness follow from the contraction mapping theorem in the usual way. Let $M = \sup_{x \in \mathbb{R}} |g(x)|$. Then $w(\pm M, t)$ are super- and sub-solutions of (4.1) respectively. By the comparison principle,

$$w(-M, t) \leq u(x, t; g) \leq w(M, t)$$

for $t > 0$. Global existence follows since $w(\pm M, t)$ are bounded. The continuous dependence can easily be proved using (4.3). \qed

We claim that the asymptotic behavior of the solution to (4.1) and (4.2) is governed by the periodic traveling wave solution $U(x - ct, t)$. We have the following result:

**Theorem 4.2.** (1) (Uniform Stability) For any $\epsilon > 0$, there is a $\delta > 0$ such that for any $g \in C_{\text{unif}}(\mathbb{R})$ with $\|g(\cdot) - U(\cdot, 0)\| < \delta$, one has

$$\|u(\cdot, t; g) - U(\cdot - ct, t)\| < \epsilon \quad (4.4)$$

for all $t > 0$.

(2). (Asymptotic Stability) For any $g \in C_{\text{unif}}(\mathbb{R})$ satisfying

$$\liminf_{x \to \infty} g(x) > W^0(0), \quad \limsup_{x \to -\infty} g(x) < W^0(0), \quad (4.5)$$

where $W^0(t) = w(\alpha^0, t)$ and $w(\alpha^0, t)$ is the solution of (1.2). Then there is $\xi_0 \in \mathbb{R}$ such that

$$\|u(\cdot, t; g) - U(\cdot - ct + \xi_0, t)\| \to 0 \quad (4.6)$$

exponentially as $t \to \infty$.

In order to prove the theorem we need the following lemmas. The first lemma use the monotonicity of $U(\cdot, t)$ to construct super- and sub- solutions.

**Lemma 4.3.** There exist $\beta_1 > 0, \delta_1 > 0$ and $\sigma_1 > 0$ such that, for any $\delta \in (0, \delta_1)$, $\tau \in \mathbb{R}^+$ and $\xi_0 \in \mathbb{R}$, $v^\pm(x, t)$ are super- and sub- solutions of (4.1), respectively, on $[\tau, \infty]$, where

$$v^\pm(x, t) = U(x - c(t - \tau) + \xi_0 \pm \sigma_1 \delta (1 - e^{-\beta_1(t-\tau)}), t) \pm \delta e^{-\beta_1(t-\tau)} \quad (4.7)$$

for $x \in \mathbb{R}$ and $t \in [\tau, \infty)$.

**Proof.** The proof of the lemma is similar to that of Lemma 2.2 in [9]. We omit it. \qed
The next lemma is an analog of the strong comparison principle of parabolic equations. This is the key lemma to apply the “squeezing” technique employed in [9] to prove the stability.

**Lemma 4.4.** There is a positive function \( \eta(\cdot, t) \) satisfying \( 0 \leq \eta(\cdot, t) \leq 1 \) for \( t \in [0, T] \) such that \( \eta(\cdot, t) \) is non-increasing and for any super-solution \( u_1(x, t) \) and sub-solution \( u_2(x, t) \) of (4.1) on \( \mathbb{R}^+ \) satisfying \( u_1(x, \tau) \geq u_2(x, \tau) \) for all \( x \in \mathbb{R} \) and for some \( \tau \in \mathbb{R} \), and \( |u_1(x, t)| \leq K_0 = \sup_{t \in \mathbb{R}} \{ |W^-(t)| + 1, |W^+(t)| + 1 \} \) for all \( x \in \mathbb{R} \) and \( t \geq \tau \), the following holds

\[
    u_1(x, t) - u_2(x, t) \geq \eta(M, t - \tau) \int_{z}^{z+1} [u_1(y, \tau) - u_2(y, \tau)] dy
\]

for all \( x \in \mathbb{R} \) with \( |x - z| \leq M \) and \( t \geq \tau \).

For the proof of this lemma, we refer the reader to a similar result in [9].

To prove the stability, we first need to show that, for given initial data as in (4.5), the solution with this initial data first forms a vague front of periodic traveling waves as the system evolves. In order to prove that, we need to construct various super- and sub-solutions.

**Lemma 4.5.** Let \( \zeta(s) = \frac{1}{2}(1 + \tanh \frac{s}{2}) \). For any given \( T_0 > 0 \) and \( m_{\pm} \in \mathbb{R} \) with \( m_- < m_+ \), there exist positive constants \( K, C, \epsilon_0 \), and a positive function \( \rho(\cdot) \) satisfying \( \lim_{\epsilon \to 0} \rho(\epsilon) = 0 \), such that, for all \( 0 < \epsilon \leq \epsilon_0 \) and \( h \in \mathbb{R} \),

\[
    (1) \quad v_1^+(x, t) = w(m_{\pm}, t) \zeta(\epsilon(x - h) + Ct) + w(m_{\pm}, t)(1 - \zeta(\epsilon(x - h) + Ct)) \pm \rho(\epsilon)e^{Kt}
\]

are super- and sub-solutions of (4.1) on \([0, T_0]\), respectively, where \( w(m_{\pm}, t) \) are solutions of (1.2) with \( w(m_{\pm}, 0) = m_{\pm} \), and

\[
    (2) \quad v_2^+(x, t) = w(m_{\pm}, t) \zeta(\epsilon(x - h) - Ct) + w(m_{\pm}, t)(1 - \zeta(\epsilon(x - h) - Ct)) \pm \rho(\epsilon)e^{Kt}
\]

are super- and sub-solutions of (4.1) on \([0, T_0]\), respectively.

**Proof.** We only prove that \( v_1^+(x, t) \) is a super-solution. The other claims can be proved similarly. Denote \( v(x, t) = w(m_{\pm}, t) \zeta(\epsilon(x - h) + Ct) + w(m_{\pm}, t)(1 -
\[ \zeta(\epsilon(x - h) + Ct). \] Then

\[
v_{1t}^+ - Dv_{1xx}^+ - d(J * v_{1}^+ - v_{1}^+) - f(v_{1}^+, t)
= f(w(m_+, t), t)\zeta(\epsilon(x - h) + Ct) + f(w(m_-, t), t)(1 - \zeta(\epsilon(x - h) + Ct))
- f(v_{1}^+, t) + C(w(m_+, t) - w(m_-, t))\zeta(\epsilon(x - h) + Ct)
(1 - \zeta(\epsilon(x - h) + Ct)) + \rho(\epsilon)Ke^{Kt}
+ (w(m_+, t) - w(m_-, t))[Dc^2\zeta(\epsilon(x - h) + Ct)(1 - \zeta(\epsilon(x - h) + Ct))
(1 - 2\zeta(\epsilon(x - h) + Ct)) - d(J * \zeta(\epsilon(x - h) + Ct) - \zeta(\epsilon(x - h) + Ct))]
= f(w(m_+, t), t)\zeta(\epsilon(x - h) + Ct) + f(w(m_-, t), t)(1 - \zeta(\epsilon(x - h) + Ct))
- f(v(x, t), t) + [f(v(x, t), t) - f(v_{1}^+(x, t), t)] + \rho(\epsilon)Ke^{Kt}
+ C(w(m_+, t) - w(m_-, t))\zeta(\epsilon(x - h) + Ct)(1 - \zeta(\epsilon(x - h) + Ct))
- Dc^2(w(m_+, t) - w(m_-, t))\zeta(\epsilon(x - h) + Ct)(1 - \zeta(\epsilon(x - h) + Ct))
(1 - 2\zeta(\epsilon(x - h) + Ct)) - d(w(m_+, t) - w(m_-, t))(J * \zeta(\epsilon(x - h) + Ct)
- \zeta(\epsilon(x - h) + Ct))
= I + II + III + IV + V + VI \tag{4.11}
\]

By Taylor’s expansion,

\[
I = f_{u_0}(u^*(x, t), t)(u^{**}(x, t) - w(m_-, t))
(w(m_+, t) - w(m_-, t))\zeta(\epsilon(x - h) + Ct)(1 - \zeta(\epsilon(x - h) + Ct)),
\]

where \( u^*(x, t), u^{**}(x, t) \) are between \( w(m_-, t) \) and \( w(m_+, t) \). Therefore there exists a constant \( M_1 \), independent of \( \epsilon \) such that

\[
|I| \leq M_1(w(m_+, t) - w(m_-, t))\zeta(\epsilon(x - h) + Ct)(1 - \zeta(\epsilon(x - h) + Ct)). \tag{4.12}
\]

Let

\[
\rho(\epsilon) = \sup\{|VI| : x, t, h, C \in \mathbb{R}\}. \tag{4.13}
\]

Since \( w(m_+, t) \) is bounded and \( \zeta(\cdot) \) is uniformly continuous, it is easy to see that \( \lim_{\epsilon \to 0} \rho(\epsilon) = 0 \). For \( II \), let \( M_2 = \sup\{|f_{u_0}(u, t)| : u \in [w(m_-, t) - 1, w(m_+, t) + 1], t \in \mathbb{R}\} \), and \( K = 1 + M_2 \). Choose \( \epsilon_0 \) such that \( \rho(\epsilon_0)e^{K\epsilon_0} \leq 1 \). Then, for \( 0 < \epsilon \leq \epsilon_0 \),

\[
|II| \leq \int_0^1 |f_u(v + \rho(\epsilon)\theta e^{Kt}, t)| d\theta \rho(\epsilon)e^{Kt} \leq M_2\rho(\epsilon)e^{Kt}. \tag{4.14}
\]

Now choose \( C = M_1 + D \). Then, by (4.11) - (4.14),

\[
v_{1t}^+ - Dv_{1xx}^+ - d(J * v_{1}^+ - v_{1}^+) - f(v_{1}^+, t) \geq 0.
\]

\[\square\]
Lemma 4.6. Suppose that $g \in C^{\text{unif}}(\mathbb{R})$ satisfies
\begin{equation}
\liminf_{x \to -\infty} g(x) > W^0(0) \quad \text{and} \quad \limsup_{x \to \infty} g(x) < W^0(0) .
\end{equation}
Then, for any $\delta > 0$, there are constants $H > 0$ and $T_0 > 0$ such that
\begin{equation}
U(x - H, T_0) - \delta \leq u(x, T_0; g) \leq U(x + H, T_0) + \delta .
\end{equation}

Proof. Without loss of generality, we assume, for some $0 < \delta_0 < 1$, that
\begin{equation}
W^-(0) - \delta_0 \leq g(x) \leq W^+(0) + \delta_0 .
\end{equation}
By assumption (H2), for $\delta << 1$, there is a $T_0 > 0$ such that
\begin{equation}
W^+(T_0) - \delta/4 < w(m_+, T_0) < W^+(T_0) + \delta/4
\end{equation}
for $m_+ = W^0(0) + \delta_0$ or $m_+ = W^+(0) + 2\delta_0$, and that
\begin{equation}
W^-(T_0) - \delta/4 < w(m_-, T_0) < W^-(T_0) + \delta/4
\end{equation}
for $m_- = W^0(0) - \delta_0$ or $m_- = W^-(0) - 2\delta_0$, where $w(m_+, t)$ are as in Lemma 4.6.

Fix $\epsilon < \epsilon_0$ small enough such that
\begin{equation}
\rho(\epsilon)e^{KT_0} < \delta/4 .
\end{equation}
By (4.15), there is an $h$ large enough such that
\begin{equation}
v^-(x, 0) \leq g(x) \leq v^+(x, 0) .
\end{equation}
Hence, by the comparison principle,
\begin{equation}
v^-(x, t) \leq u(x, t; g) \leq v^+(x, t)
\end{equation}
for all $x \in \mathbb{R}$ and $t \in [0, T_0]$. Since $\lim_{x \to -\infty} U(x - CT_0, T_0) = W^\pm(T_0)$, by (4.16)-(4.21), there exists $H$ large enough such that
\begin{equation}
U(x - H, T_0) - \delta \leq v^-(x, T_0) \leq u(x, T_0; g) \leq v^+(x, T_0) \leq U(x + H, T_0) + \delta
\end{equation}
for all $x \in \mathbb{R}$. This completes the proof.
The following lemma is the “squeezing technique” employed in [9].

**Lemma 4.7.** There exists $\epsilon^* > 0$ such that if $u(x, t)$ is a solution of (4.1), and if for some $\tau \in \mathbb{R}^+$, $\xi \in \mathbb{R}$, $\delta \in (0, \frac{\beta_1}{2})$, and $h > 0$, one has

$$U(x - ct + \xi, \tau) - \delta \leq u(x, \tau) \leq U(x - ct + \xi + h, \tau) + \delta$$

(4.23)

for all $x \in \mathbb{R}$, then for every $t \geq \tau + 1$, there exist $\hat{\xi}(t), \hat{\delta}(t) \geq 0$ and $\hat{h}(t) \geq 0$ satisfying

$$\hat{\xi}(t) \in [\xi - \sigma_1 \delta, \xi + h + \sigma_1 \delta],$$

(4.24)

$$0 \leq \hat{\delta}(t) \leq e^{-\beta_1 (t-\tau-1)} [\delta + \epsilon^* \min\{h, 1\}],$$

(4.25)

$$0 \leq \hat{h}(t) \leq [h - \sigma_1 \epsilon^* \min\{h, 1\}] + 2\sigma_1 \delta,$$

(4.26)

such that (4.23) holds with $\tau$, $\xi$, $\delta$ and $h$ being replaced by $t \geq \tau + 1$, $\hat{\xi}(t)$, $\hat{\delta}(t)$ and $\hat{h}(t)$ respectively, where $\beta_1$, $\delta_1$, and $\sigma_1$ are as in Lemma 4.3.

**Proof.** The proof is similar to that of Lemma 3.3 in [9]. In the proof of that lemma, the only properties used are given by Lemma 4.3 and Lemma 4.4. For details, see [9] and [23].

**Proof of Theorem 4.2.** (1). Let $\epsilon > 0$ be given. Since $U(\cdot, \cdot)$ is uniformly continuous on $\mathbb{R} \times [0, T]$, there is a constant $k_0 > 0$ such that

$$|U(x + k, t) - U(x, t)| < \epsilon/2,$$

(4.27)

for all $x \in \mathbb{R}$, $t \in [0, T]$ and all $k$ with $|k| \leq k_0$.

Let $\beta_1$, $\delta_1$ and $\sigma_1$ be given as in Lemma 4.3. Choose $\delta > 0$ such that $\delta < \min\{\delta_1, \epsilon/2, k_0/\sigma_1\}$. Then for any $g \in C_{unif}(\mathbb{R})$ satisfying $\|g(\cdot) - U(\cdot, 0)\| < \delta$, by Lemma 4.3, we have

$$U(x - ct - \sigma_1 \delta(1 - e^{-\beta_1 t}), t) - \delta e^{-\beta_1 t} \leq u(x, t; g) \leq U(x - ct + \sigma_1 \delta(1 - e^{-\beta_1 t}), t) + \delta e^{-\beta_1 t}$$

for $x \in \mathbb{R}$ and $t \in [0, \infty)$. By (4.27) and the choice of $\delta$, we have

$$\|u(\cdot, t; g) - U(\cdot - ct, t)\| < \epsilon,$$

for all $t > 0$.

(2). Let $\epsilon^*$ be given as in Lemma 4.7 and $\beta_1$, $\delta_1$ and $\sigma_1$ be given as in Lemma 4.3. Let $\overline{\theta} = \min\{\delta_1/2, \epsilon^*/4\}$, and $\overline{\sigma} = \sigma_1 \epsilon^* - 2\sigma_1 \overline{\theta}$. Let $t_0$ be chosen such that $e^{-\beta_1 (t_0 - 1)(\overline{\theta} + \epsilon^*)} \leq (1 - \overline{\sigma})\overline{\theta}$. By Lemma 4.6, there are $\xi_0 \in \mathbb{R}$, $h > 0$ and $T_0 > 0$ such that

$$U(x - cT_0 + \xi_0, T_0) - \overline{\theta} \leq u(x, T_0; g) \leq U(x - cT_0 + \xi_0 + h, T_0) + \overline{\theta}$$

(4.28)
for all \( x \in \mathbb{R} \). First, we may assume \( 0 < h \leq 1 \). In fact, if \( h > 1 \), we can choose integer \( N > 0 \) such that \( 0 < h - N \gamma \leq 1 \). Applying Lemma 4.7 repeatedly, we conclude that

\[
U(x-c(kt_0+T_0)+\xi_k,kt_0+T_0)-\delta_k \leq u(x,kt_0+T_0;g)
\]

\[
\leq U(x-c(kt_0+T_0)+\xi_k+h_k,kt_0+T_0)+\delta_k \tag{4.29}
\]

for all \( x \in \mathbb{R} \), where \( \xi_k \in [\xi_{k-1}-\sigma_{k-1}\delta_{k-1},\xi_{k-1}+\sigma_{k-1}\delta_{k-1}+h_{k-1}] \), \( \delta_k \leq (1-\gamma)^k\delta \), \( h_k \leq h_{k-1}-\gamma \), and \( \delta_0 = \delta \). Therefore (4.28) holds with \( \xi_0, \delta, h, \) and \( T_0 \) being replaced by \( \xi_N, \delta_N, h_N, \) and \( T_N = Nt_0+T_0 \), respectively.

Now we assume \( h \leq 1 \) and (4.28) holds. Define \( T_k = kt_0, \delta_k = (1-\gamma)^k\delta \), and \( h_k = h_{k-1}-\gamma \). Then we can show by induction that (4.29) still holds. Define \( \delta(t) = \delta_k, \xi(t) = \xi_k-\sigma_1\delta_k \), and \( h(t) = h_k+2\sigma_1\delta_k \), for \( t \in [T_k+T_0,T_{k+1}+T_0] \) and \( k = 0, 1, \ldots \). Then, by Lemma 4.3,

\[
U(x-c(t)+\xi(t),t)-\delta(t) \leq u(x,t;g) \leq U(x-c(t)+\xi(t)+h(t),t)+\delta(t) \tag{4.30}
\]

for \( x \in \mathbb{R} \) and \( t \geq T_0 \). Note that \( \delta(t) \to 0, h(t) \to 0 \) and \( \xi(t) \to \xi(\infty) \) exponentially as \( t \to \infty \). Therefore,

\[
u(x,t;g) \to U(x-c(t)+\xi(\infty),t)
\]

exponentially as \( t \to \infty \).

References


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