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THE FUNDAMENTAL SOLUTION FOR A CONSISTENT COMPLEX MODEL OF THE SHALLOW SHELL EQUATIONS

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Abstract

The calculation of the Fourier transforms of the fundamental solution in shallow shell theory ostensibly was accomplished by J.L. Sanders [J. Appl. Mech. 37 (1970), 361–366]. However, as is shown in detail in this paper, the complex model used by Sanders is, in fact, inconsistent. This paper provides a consistent version of Sanders's complex model, along with the Fourier transforms of the fundamental solution for this corrected model. The inverse Fourier transforms are then calculated for the particular cases of the shallow spherical and circular cylindrical shells, and the results of the latter are seen to be in agreement with results appearing elsewhere in the literature.

§1. Introduction

The study of shells is quite an important area in the field of structural mechanics. Often it is not possible to find exact solutions for the equations of shell theory, in which case they must be solved numerically. However, in order to apply many of the available numerical methods, especially the boundary element methods – BEMs, it is first necessary to know the fundamental solution of the problem in question. It was exactly this reasoning which led to the calculation of the fundamental solution for the shallow cylindrical shell ([3]).

It was while writing [3] that the authors were informed that, in fact, this fundamental solution had already been calculated by the applied mathematicians J.L. Sanders and J.G. Simmonds in [28] and [30] (in 1970!). However, upon careful study of these works, this author discovered that the complex equations for shallow shell theory developed by Sanders in [28], and used to solve the above problem in [30], are inconsistent. The model treated in [3] is the consistent real (as opposed to complex) model which is given in [28] and from which Sanders derives his questionable complex model; this model is equivalent to the models for the cylindrical shell developed in [7], [16] and [35], after including all simplifications therein.

The advantage of a complex model is that, using certain symmetries in the shell equations (the so-called *static-geometric analogy*), the order of the problem is effectively halved – e.g., the model treated in [3] has order eight, whereas the corresponding complex model would have order four. Thus, it would be very convenient to have a consistent complex model for the general equations of shallow shell theory. Novozhilov ([23]) seems to have provided such; however, the approach used by Sanders in [28] is much more amenable to the calculation of fundamental solutions.

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The purpose of this paper, then, is to provide a corrected, consistent version of Sanders's complex model of the shallow shell theory, and also to use this model to calculate the Fourier transforms of its fundamental solution. The organization of the paper is as follows: In Section 2, we give a careful derivation of Sanders's complex equations, pointing out in detail where the model fails, and modifying it so that, while remaining true to Sanders's basic approach, our new model is consistent. In Section 3, we calculate the Fourier transforms of the fundamental solution for the general model from Section 2. In Section 4, we invert these transforms for the case of the shallow spherical shell, thus providing a correct derivation for the fundamental solution in this particular case. Finally, in Section 5, we do the same for the shallow cylindrical shell, and we show in the Appendix that our results do agree with those appearing elsewhere in the literature.

§2. A consistent model of the equations of shallow shell theory in complex form

In this section we look carefully at the complex model for the general shallow shell developed by J.L. Sanders in [28]. We point out where this model fails (in more detail than was done in [3]) and, in the process, we provide a corrected, consistent version. We note that our approach, along with Sanders's, is similar to that used by Novozhilov, except that we strive to keep Sanders's relationship between complex stresses and changes in curvature, a relationship which Novozhilov's model does not satisfy.

We wish to point out that, although our ultimate aim is to be able to calculate the fundamental solution for various types of shallow shell, the purpose of this section is only to develop a consistent model. The model must **not** depend on the smoothness of the quantities involved – in particular, it must be consistent when the quantities involved have derivatives of arbitrary order.

We start with the real model for the shallow shell equations given in [28]. This model is equivalent to those models for the spherical and the circular cylindrical shell treated in [16] and [35], after including all simplifications therein. It is also a special case of the general (real) shell equations developed in [23].

Sanders gives the fundamental equations of shallow shell theory in dimensionless form for a shell with quadratic middle surface

$$z = (ax^2 + 2bxy + cy^2)/2\mu, \tag{2.1}$$

where $\mu = \frac{L^2\sqrt{12(1-\nu^2)}}{Rh}$. Here, L is a "reference length" and R a "reference radius of curvature". Also, h is the constant shell thickness and ν is Poisson's ratio. Using lower case letters to denote real quantities (reserving capitals for complex quantities), the equations are (again, [28, p. 362]):

CONSTITUTIVE RELATIONS:

$$e_{11} = n_{11} - \nu n_{22},\ e_{22} = n_{22} - \nu n_{11}, e_{12} = (1+\nu)n_{12}; \tag{2.2}$$

$$m_{11} = k_{11} + \nu k_{22},\ m_{22} = k_{22} + \nu k_{11},\ m_{12} = (1-\nu)k_{12}; \tag{2.3}$$

EQUILIBRIUM EQUATIONS:

$$\frac{\partial n_{11}}{\partial x}+\frac{\partial n_{12}}{\partial y}=-p_1, \tag{2.4}$$

$$\frac{\partial n_{12}}{\partial x}+\frac{\partial n_{22}}{\partial y}=-p_2, \tag{2.5}$$

$$\frac{\partial^2 m_{11}}{\partial x^2}+2\frac{\partial^2 m_{12}}{\partial x\partial y}+\frac{\partial^2 m_{22}}{\partial y^2}+an_{11}+2bn_{12}+cn_{22}=-p; \tag{2.6}$$

COMPATIBILITY EQUATIONS:

$$\frac{\partial k_{22}}{\partial x}-\frac{\partial k_{12}}{\partial y}=0, \tag{2.7}$$

$$-\frac{\partial k_{12}}{\partial x}+\frac{\partial k_{11}}{\partial y}=0, \tag{2.8}$$

$$\frac{\partial^2 e_{22}}{\partial x^2}-2\frac{\partial^2 e_{12}}{\partial x\partial y}+\frac{\partial^2 e_{11}}{\partial y^2}-ak_{22}+2bk_{12}-ck_{11}=0; \tag{2.9}$$

STRAIN-DISPLACEMENT RELATIONS:

$$e_{11}=\frac{\partial u}{\partial x}-aw,\ e_{22}=\frac{\partial v}{\partial y}-cw,\ e_{12}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-bw, \tag{2.10}$$

$$k_{11}=-\frac{\partial^2 w}{\partial x^2},\ k_{22}=-\frac{\partial^2 w}{\partial y^2},\ k_{12}=-\frac{\partial^2 w}{\partial x\partial y}. \tag{2.11}$$

Here, the n_{ij} are the stresses per unit length; m_{ij}, the moments per unit length; e_{ij} the strains; k_{ij}, the changes of curvature; p_1, p_2 and p the x-, y- and z-direction forces, respectively; and u, v and w the x-, y- and z-direction displacements, respectively.

Now, we notice a certain symmetry between each equilibrium (force/moment) equation and the corresponding compatibility (strain/curvature) equation – the so-called *static-geometric analogy*. This symmetry suggests that we extend the real quantities to complex quantities by way of the following definitions, as Sanders does:

$$\begin{aligned} N_{11}&=n_{11}+ik_{22}, & N_{22}&=n_{22}+ik_{11}, & N_{12}&=n_{12}-ik_{12},\\ K_{11}&=k_{11}-in_{22}, & K_{22}&=k_{22}-in_{11}, & K_{12}&=k_{12}+in_{12}, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} E_{11}&=N_{11}-\nu N_{22}, & E_{22}&=N_{22}-\nu N_{11}, & E_{12}&=(1+\nu)N_{12},\\ M_{11}&=K_{11}+\nu K_{22}, & M_{22}&=K_{22}+\nu K_{11}, & M_{12}&=(1-\nu)K_{12}. \end{aligned} \tag{2.13}$$

We note here that equations (2.12) imply that $N_{11}=iK_{22}$, $N_{22}=iK_{11}$, $N_{12}=-iK_{12}$ ([28,(13), p. 363]).

The above quantities satisfy the following "complex equilibrium equations":

$$\frac{\partial N_{11}}{\partial x}+\frac{\partial N_{12}}{\partial y}=-p_1, \tag{2.14}$$

$$\frac{\partial N_{12}}{\partial x}+\frac{\partial N_{22}}{\partial y}=-p_2, \tag{2.15}$$

$$\begin{aligned}&\frac{\partial^2 M_{11}}{\partial x^2}+2\frac{\partial^2 M_{12}}{\partial x\partial y}+\frac{\partial^2 M_{22}}{\partial y^2}+aN_{11}+2bN_{12}+cN_{22}=\\ &\quad -p+2i\nu\left(\frac{\partial p_1}{\partial x}+\frac{\partial p_2}{\partial y}\right),\end{aligned} \tag{2.16}$$

as well as the "complex compatibility equations"

$$\frac{\partial K_{22}}{\partial x}-\frac{\partial K_{12}}{\partial y}=ip_1, \tag{2.17}$$

$$-\frac{\partial K_{12}}{\partial x}+\frac{\partial K_{11}}{\partial y}=ip_2, \tag{2.18}$$

$$\frac{\partial^2 E_{22}}{\partial x^2}-2\frac{\partial^2 E_{12}}{\partial x\partial y}+\frac{\partial^2 E_{11}}{\partial y^2}-aK_{22}+2bK_{12}-cK_{11}=-ip, \tag{2.19}$$

where p_1, p_2 and p are still the real forces from above.

Since our equations (2.16) and (2.19) differ from those obtained by Sanders ([28, p. 363, (3) and (9)′]), let us provide a derivation of each. First, let us rewrite equations (2.6) and (2.9):

$$\frac{\partial^2 m_{11}}{\partial x^2}+2\frac{\partial^2 m_{12}}{\partial x\partial y}+\frac{\partial^2 m_{22}}{\partial y^2}=\frac{\partial^2}{\partial x^2}(k_{11}+\nu k_{22})+2\frac{\partial^2}{\partial x\partial y}[(1-\nu)k_{12}]+\frac{\partial^2}{\partial y^2}(k_{22}+\nu k_{11})$$

$\Rightarrow$ (2.6) can be written as

$$\begin{aligned}&\frac{\partial^2 k_{11}}{\partial x^2}+2\frac{\partial^2 k_{12}}{\partial x\partial y}+\frac{\partial^2 k_{22}}{\partial y^2}+\nu\left(\frac{\partial^2 k_{22}}{\partial x^2}-2\frac{\partial^2 k_{12}}{\partial x\partial y}+\frac{\partial^2 k_{11}}{\partial y^2}\right)\\ &\quad +an_{11}+2bn_{12}+cn_{22}=-p;\end{aligned} \tag{2.20}$$

$$\frac{\partial^2 e_{22}}{\partial x^2}-2\frac{\partial^2 e_{12}}{\partial x\partial y}+\frac{\partial^2 e_{11}}{\partial y^2}=\frac{\partial^2}{\partial x^2}(n_{22}-\nu n_{11})-2\frac{\partial^2}{\partial x\partial y}[(1+\nu)n_{12}]+\frac{\partial^2}{\partial y^2}(n_{11}-\nu n_{22})$$

$\Rightarrow$ (2.9) can be written as

$$\begin{aligned}&\frac{\partial^2 n_{22}}{\partial x^2}-2\frac{\partial^2 n_{12}}{\partial x\partial y}+\frac{\partial^2 n_{11}}{\partial y^2}-\nu\left(\frac{\partial^2 n_{11}}{\partial x^2}+2\frac{\partial^2 n_{12}}{\partial x\partial y}+\frac{\partial^2 n_{22}}{\partial y^2}\right)\\ &\quad -ak_{22}+2bk_{12}-ck_{11}=0.\end{aligned} \tag{2.21}$$

Further, we note that

$$\begin{aligned}\frac{\partial^2 n_{11}}{\partial x^2}+2\frac{\partial^2 n_{12}}{\partial x\partial y}+\frac{\partial^2 n_{22}}{\partial y^2}&=\frac{\partial}{\partial x}\left(\frac{\partial n_{11}}{\partial x}+\frac{\partial n_{12}}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial n_{12}}{\partial x}+\frac{\partial n_{22}}{\partial y}\right)\\ &=-\frac{\partial p_1}{\partial x}-\frac{\partial p_2}{\partial y},\\ \frac{\partial^2 k_{22}}{\partial x^2}-2\frac{\partial^2 k_{12}}{\partial x\partial y}+\frac{\partial^2 k_{11}}{\partial y^2}&=\frac{\partial}{\partial x}\left(\frac{\partial k_{22}}{\partial x}-\frac{\partial k_{12}}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial k_{12}}{\partial x}+\frac{\partial k_{11}}{\partial y}\right)\\ &=0.\end{aligned}$$

The latter implies that we can further simplify (2.20):

$$\frac{\partial^2 k_{11}}{\partial x^2}+2\frac{\partial^2 k_{12}}{\partial x\partial y}+\frac{\partial^2 k_{22}}{\partial y^2}+an_{11}+2bn_{12}+cn_{22}=-p. \tag{2.22}$$

Now for the proofs of equations (2.16) and (2.19):

PROOF OF EQUATION (2.16):

$$\begin{aligned}
&\frac{\partial^2 M_{11}}{\partial x^2}+2\frac{\partial^2 M_{12}}{\partial x\partial y}+\frac{\partial^2 M_{22}}{\partial y^2}+aN_{11}+2bN_{12}+cN_{22}\\
&=\frac{\partial^2}{\partial x^2}(K_{11}+\nu K_{22})+2\frac{\partial^2}{\partial x\partial y}[(1-\nu)K_{12}]+\frac{\partial^2}{\partial y^2}(K_{22}+\nu K_{11})+aN_{11}+2bN_{12}+cN_{22}\\
&=\frac{\partial^2}{\partial x^2}[(k_{11}-in_{22})+\nu(k_{22}-in_{11})]+2\frac{\partial^2}{\partial x\partial y}[(1-\nu)(k_{12}+in_{12})]\\
&\quad+\frac{\partial^2}{\partial y^2}[(k_{22}-in_{11})+\nu(k_{11}-in_{22})]+a[n_{11}+ik_{22}]+2b[n_{12}-ik_{12}]\\
&\quad+c[n_{22}+ik_{11}]\\
&=\frac{\partial^2 k_{11}}{\partial x^2}+2\frac{\partial^2 k_{12}}{\partial x\partial y}+\frac{\partial^2 k_{22}}{\partial y^2}+\nu\left(\frac{\partial^2 k_{22}}{\partial x^2}-2\frac{\partial^2 k_{12}}{\partial x\partial y}+\frac{\partial^2 k_{11}}{\partial y^2}\right)\\
&\quad+an_{11}+2bn_{12}+cn_{22}\\
&\quad+i\left[-\frac{\partial^2 n_{22}}{\partial x^2}+2\frac{\partial^2 n_{12}}{\partial x\partial y}-\frac{\partial^2 n_{11}}{\partial y^2}+\nu\left(-\frac{\partial^2 n_{11}}{\partial x^2}-2\frac{\partial^2 n_{12}}{\partial x\partial y}-\frac{\partial^2 n_{22}}{\partial y^2}\right)\right.\\
&\quad\left.+ak_{22}-2bk_{12}+ck_{11}\right]\\
&=\frac{\partial^2 k_{11}}{\partial x^2}+2\frac{\partial^2 k_{12}}{\partial x\partial y}+\frac{\partial^2 k_{22}}{\partial y^2}+\nu\left(\frac{\partial^2 k_{22}}{\partial x^2}-2\frac{\partial^2 k_{12}}{\partial x\partial y}+\frac{\partial^2 k_{11}}{\partial y^2}\right)+an_{11}+2bn_{12}+cn_{22}\\
&\quad-i\left[\frac{\partial^2 n_{22}}{\partial x^2}-2\frac{\partial^2 n_{12}}{\partial x\partial y}+\frac{\partial^2 n_{11}}{\partial y^2}-\nu\left(\frac{\partial^2 n_{11}}{\partial x^2}+2\frac{\partial^2 n_{12}}{\partial x\partial y}+\frac{\partial^2 n_{22}}{\partial y^2}\right)\right.\\
&\quad\left.-ak_{22}+2bk_{12}-ck_{11}\right]\\
&\quad-2i\nu\left(\frac{\partial^2 n_{11}}{\partial x^2}+2\frac{\partial^2 n_{12}}{\partial x\partial y}+\frac{\partial^2 n_{22}}{\partial y^2}\right)\\
&=-p+2i\nu\left(\frac{\partial p_1}{\partial x}+\frac{\partial p_2}{\partial y}\right).
\end{aligned}$$

PROOF OF EQUATION (2.19):

$$\begin{aligned}
&\frac{\partial^2 E_{22}}{\partial x^2}-2\frac{\partial^2 E_{12}}{\partial x\partial y}+\frac{\partial^2 E_{11}}{\partial y^2}-aK_{22}+2bK_{12}-cK_{11}\\
&=\frac{\partial^2}{\partial x^2}(N_{22}-\nu N_{11})-2\frac{\partial^2}{\partial x\partial y}[(1+\nu)N_{12}]+\frac{\partial^2}{\partial y^2}(N_{11}-\nu N_{22})\\
&\quad-aK_{22}+2bK_{12}-cK_{11}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{\partial^2}{\partial x^2}[n_{22}+ik_{11}-\nu(n_{11}+ik_{22})]-2\frac{\partial^2}{\partial x\partial y}[(1+\nu)(n_{12}-ik_{12})]\\
&+\frac{\partial^2}{\partial y^2}[n_{11}+ik_{22}-\nu(n_{22}+ik_{11})]-a(k_{22}-in_{11})+2b(k_{12}+in_{12})-c(k_{11}-in_{22})\\
=&\frac{\partial^2 n_{22}}{\partial x^2}-2\frac{\partial^2 n_{12}}{\partial x\partial y}+\frac{\partial^2 n_{11}}{\partial y^2}+\nu\left(-\frac{\partial^2 n_{11}}{\partial x^2}-2\frac{\partial^2 n_{12}}{\partial x\partial y}-\frac{\partial^2 n_{22}}{\partial y^2}\right)-ak_{22}+2bk_{12}-ck_{11}\\
&+i\left[\frac{\partial^2 k_{11}}{\partial x^2}+2\frac{\partial^2 k_{12}}{\partial x\partial y}+\frac{\partial^2 k_{22}}{\partial y^2}+\nu\left(-\frac{\partial^2 k_{22}}{\partial x^2}+2\frac{\partial^2 k_{12}}{\partial x\partial y}-\frac{\partial^2 k_{11}}{\partial y^2}\right)\right.\\
&\left.+an_{11}+2bn_{12}+cn_{22}\right]\\
=&\,i\left[\frac{\partial^2 k_{11}}{\partial x^2}+2\frac{\partial^2 k_{12}}{\partial x\partial y}+\frac{\partial^2 k_{22}}{\partial y^2}+an_{11}+2bn_{12}+cn_{22}\right]=-ip.
\end{aligned}
$$

$\diamondsuit$

The next question is: How do we define the imaginary parts of the complex displacements U, V and W? This is where Sanders's complex model becomes inconsistent – he chooses to define W so that

$$K_{11}=-\frac{\partial^2 W}{\partial x^2},\quad K_{22}=-\frac{\partial^2 W}{\partial y^2},\quad K_{12}=-\frac{\partial^2 W}{\partial x\partial y}, \tag{2.23}$$

$$N_{11}=-i\frac{\partial^2 W}{\partial y^2},\quad N_{22}=-i\frac{\partial^2 W}{\partial x^2},\quad N_{12}=i\frac{\partial^2 W}{\partial x\partial y}. \tag{2.24}$$

Now, for sufficiently smooth forces p_1, p_2 and p, the remaining quantities will also be sufficiently smooth so that the order of partial differentiation doesn't matter. That being the case, if we use (2.24) to define N_{11}, N_{22} and N_{12}, the equilibrium equations (2.14) and (2.15) become

$$
\begin{aligned}
\frac{\partial N_{11}}{\partial x}+\frac{\partial N_{12}}{\partial y}&=\frac{\partial}{\partial x}\left(-i\frac{\partial^2 W}{\partial y^2}\right)+\frac{\partial}{\partial y}\left(i\frac{\partial^2 W}{\partial x\partial y}\right)=0=-p_1,\\
\frac{\partial N_{12}}{\partial x}+\frac{\partial N_{22}}{\partial y}&=\frac{\partial}{\partial x}\left(i\frac{\partial^2 W}{\partial x\partial y}\right)+\frac{\partial}{\partial y}\left(-i\frac{\partial^2 W}{\partial x^2}\right)=0=-p_2,
\end{aligned} \tag{2.25}
$$

each of which is a contradiction unless the corresponding force is zero. Similarly, the compatibility equations (2.17) and (2.18) become

$$
\begin{aligned}
\frac{\partial K_{22}}{\partial x}-\frac{\partial K_{12}}{\partial y}&=\frac{\partial}{\partial x}\left(-\frac{\partial^2 W}{\partial y^2}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial^2 W}{\partial x\partial y}\right)=0=ip_1,\\
-\frac{\partial K_{12}}{\partial x}+\frac{\partial K_{11}}{\partial y}&=-\frac{\partial}{\partial x}\left(-\frac{\partial^2 W}{\partial x\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial^2 W}{\partial x^2}\right)=0=ip_2,
\end{aligned} \tag{2.26}
$$

again leading to a contradiction for nonzero p_1 or p_2. Thus, the complex model developed and used by Sanders in [28] (and by Sanders and Simmonds in [30]) is, indeed, inconsistent. (We will exhibit later in the paper a third inconsistency, involving the normal force, p.)

In order to avoid these inconsistencies, we introduce the real quantities F_{11}, F_{22} and F_{12}, and we define complex W so that

$$K_{11} = -\frac{\partial^2 W}{\partial x^2} - iF_{22}, \quad K_{22} = -\frac{\partial^2 W}{\partial y^2} - iF_{11}, \quad K_{12} = -\frac{\partial^2 W}{\partial x \partial y} + iF_{12}, \tag{2.27}$$

$$\begin{aligned} N_{11} &= iK_{22} = F_{11} - i\frac{\partial^2 W}{\partial y^2}, \quad N_{22} = iK_{11} = F_{22} - i\frac{\partial^2 W}{\partial x^2}, \\ N_{12} &= -iK_{12} = F_{12} + i\frac{\partial^2 W}{\partial x \partial y}. \end{aligned} \tag{2.28}$$

Introducing these additional quantities allows us three extra degrees of freedom with which we may avoid the above inconsistencies. Further, in requiring the F_{ij} to be real, we do not lose the important relationships

$$\text{Re } K_{11} = -\frac{\partial^2 w}{\partial x^2}, \quad \text{Re } K_{22} = -\frac{\partial^2 w}{\partial y^2}, \quad \text{Re } K_{12} = -\frac{\partial^2 w}{\partial x \partial y}.$$

The introduction of these three quantities is certainly not a new idea – e.g., it is similar to the introduction of the expressions T_1^*, T_2^* and S^* by Novozhilov in his consistent complex model ([23, p. 73]).

Inserting (2.27) and (2.28) into the equilibrium equations (2.14)–(2.16) leads to

$$\frac{\partial F_{11}}{\partial x} + \frac{\partial F_{12}}{\partial y} = -p_1, \tag{2.29}$$

$$\frac{\partial F_{12}}{\partial x} + \frac{\partial F_{22}}{\partial y} = -p_2, \tag{2.30}$$

$$\begin{aligned} &\Delta^2 W + i\left(a\frac{\partial^2 W}{\partial y^2} - 2b\frac{\partial^2 W}{\partial x \partial y} + c\frac{\partial^2 W}{\partial x^2}\right) + i\left(\frac{\partial^2 F_{22}}{\partial x^2} - 2\frac{\partial^2 F_{12}}{\partial x \partial y} + \frac{\partial^2 F_{11}}{\partial y^2}\right) \\ &\quad - (aF_{11} + 2bF_{12} + cF_{22}) = p - i\nu\left(\frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial y}\right), \end{aligned} \tag{2.31}$$

where $\Delta^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2$ is the biharmonic operator in two dimensions.

We may now define complex U and V so that

$$E_{11} = \frac{\partial U}{\partial x} - aW, \quad E_{22} = \frac{\partial V}{\partial y} - cW, \quad E_{12} = \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) - bW. \tag{2.32}$$

Then equations (2.32) imply

$$\frac{\partial U}{\partial x} = aW + F_{11} - i\frac{\partial^2 W}{\partial y^2} - \nu F_{22} + i\nu\frac{\partial^2 W}{\partial x^2}, \tag{2.33}$$

$$\frac{\partial V}{\partial y} = cW + F_{22} - i\frac{\partial^2 W}{\partial x^2} - \nu F_{11} + i\nu\frac{\partial^2 W}{\partial y^2}, \tag{2.34}$$

$$\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = 2bW + 2i(1+\nu)\frac{\partial^2 W}{\partial x \partial y} + 2(1+\nu)F_{12}. \tag{2.35}$$

At this point, equations (2.29)–(2.31), (2.33)–(2.35) give us six equations in the six unknowns U, V, W, F_{11}, F_{22} and F_{12}. However, we choose to replace (2.31), as follows. First, taking $\frac{\partial^2}{\partial y^2}$ of (2.33) plus $\frac{\partial^2}{\partial x^2}$ of (2.34) minus $\frac{\partial^2}{\partial x \partial y}$ of (2.35) results in

$$\begin{aligned}\Delta^2 W + i\left(a\frac{\partial^2 W}{\partial y^2} - 2b\frac{\partial^2 W}{\partial x \partial y} + c\frac{\partial^2 W}{\partial x^2}\right) \\ + i\left(\frac{\partial^2 F_{22}}{\partial x^2} - 2\frac{\partial^2 F_{12}}{\partial x \partial y} + \frac{\partial^2 F_{11}}{\partial y^2}\right) = -i\nu\left(\frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial y}\right).\end{aligned} \tag{2.36}$$

Then we replace (2.31) with the equation which results from subtracting (2.36) from (2.31), i.e., with

$$aF_{11} + 2bF_{12} + cF_{22} = -p. \tag{2.37}$$

We note here that (2.37) also follows from the compatibility equation (2.19). We also note here that, without the quantities F_{11}, F_{22} and F_{12}, the insertion of (2.32) into this compatibility equation would lead to a contradiction similar to those found in (2.25) and (2.26). Hence, Sanders's model is inconsistent even when $p_1 = p_2 = 0$.

Finally, it is easy to show that Sanders's final system of four PDEs ([28, p. 365, (42) and (43)]) is inconsistent as well.

§3. The Fourier transform of the fundamental solution for an arbitrary shallow shell

We are now in a position to find the Fourier transform of the fundamental solution for the system developed above. To this end, we set the forces p_1, p_2 and p equal to constant multiples of the Dirac delta function $\delta(x, y) = \delta(x)\delta(y)$ (i.e., we allow them to be concentrated forces, acting at the origin). Our system (2.29), (2.30), (2.33)–(2.35), (2.37) then becomes

$$\frac{\partial F_{11}}{\partial x} + \frac{\partial F_{12}}{\partial y} = -\lambda_1 \delta(x, y), \tag{3.1}$$

$$\frac{\partial F_{12}}{\partial x} + \frac{\partial F_{22}}{\partial y} = -\lambda_2 \delta(x, y), \tag{3.2}$$

$$aF_{11} + 2bF_{12} + cF_{22} = -\lambda \delta(x, y), \tag{3.3}$$

$$\frac{\partial U}{\partial x} = aW - i\frac{\partial^2 W}{\partial y^2} + i\nu\frac{\partial^2 W}{\partial x^2} + F_{11} - \nu F_{22}, \tag{3.4}$$

$$\frac{\partial V}{\partial y} = cW - i\frac{\partial^2 W}{\partial x^2} + i\nu\frac{\partial^2 W}{\partial y^2} + F_{22} - \nu F_{11}, \tag{3.5}$$

$$\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = 2bW + 2i(1+\nu)\frac{\partial^2 W}{\partial x \partial y} + 2(1+\nu)F_{12}, \tag{3.6}$$

where λ_1, λ_2 and λ are arbitrary constants.

We note here that we now treat the problem in a distributional setting – we mention this only because it is not clear whether Sanders ([28]) considers the problem in such a setting.

For a tempered distribution f on $\mathbb{R}^2$, define its Fourier transform by

$$\hat{f}(\alpha, \beta) = \mathcal{F}(f)(\alpha, \beta) = \iint_{\mathbb{R}^2} e^{-i(\alpha x + \beta y)} f(x, y)\,dx\,dy.$$

Then the inverse Fourier transform of $\hat{f}$ is

$$f(x,y) = \mathcal{F}^{-1}(\hat{f})(x,y) = \frac{1}{4\pi^2} \iint\limits_{\mathbb{R}^2} e^{i(\alpha x+\beta y)} \hat{f}(\alpha,\beta) d\alpha d\beta.$$

The transform of system (3.1)–(3.6) then becomes

$$\alpha\widehat{F}_{11} + \beta\widehat{F}_{12} = i\lambda_1, \tag{3.7}$$
$$\alpha\widehat{F}_{12} + \beta\widehat{F}_{22} = i\lambda_2, \tag{3.8}$$
$$a\widehat{F}_{11} + 2b\widehat{F}_{12} + c\widehat{F}_{22} = -\lambda, \tag{3.9}$$
$$i\alpha\widehat{U} = a\widehat{W} + i\beta^2\widehat{W} - i\nu\alpha^2\widehat{W} + \widehat{F}_{11} - \nu\widehat{F}_{22}, \tag{3.10}$$
$$i\beta\widehat{V} = c\widehat{W} + i\alpha^2\widehat{W} - i\nu\beta^2\widehat{W} + \widehat{F}_{22} - \nu\widehat{F}_{11}, \tag{3.11}$$
$$i\beta\widehat{U} + i\alpha\widehat{V} = 2b\widehat{W} - 2i(1+\nu)\alpha\beta\widehat{W} + 2(1+\nu)\widehat{F}_{12}. \tag{3.12}$$

We eliminate $\widehat{F}_{11}, \widehat{F}_{22}$ and $\widehat{F}_{12}$ and then solve the remaining three equations for $\widehat{U}, \widehat{V}$ and $\widehat{W}$. Rather than presenting the results in general form, we present them, as Sanders does, for the three cases: I, normal force ($\lambda = 1, \lambda_1 = \lambda_2 = 0$); II, x-direction tangential force ($\lambda_1 = 1, \lambda = \lambda_2 = 0$); and III, y-direction tangential force ($\lambda_2 = 1, \lambda = \lambda_1 = 0$). In each case, $\Lambda_1 = (\alpha^2+\beta^2)^2 - i\Lambda_2$, where $\Lambda_2 = a\beta^2 - 2b\alpha\beta + c\alpha^2$.

I. NORMAL FORCE:

$$\widehat{W}_1 = \frac{1}{\Lambda_1} - \frac{i}{\Lambda_2}, \tag{3.13}$$
$$\widehat{U}_1 = -\frac{\nu\alpha}{\Lambda_1} - \frac{1}{\Lambda_1\Lambda_2}[a\alpha^3 + (2a-c)\alpha\beta^2 + 2b\beta^3], \tag{3.14}$$
$$\widehat{V}_1 = -\frac{\nu\beta}{\Lambda_1} - \frac{1}{\Lambda_1\Lambda_2}[c\beta^3 + (2c-a)\alpha^2\beta + 2b\alpha^3]; \tag{3.15}$$

II. x-DIRECTION TANGENTIAL FORCE:

$$\widehat{W}_2 = -\widehat{U}_1, \tag{3.16}$$
$$\begin{aligned}\widehat{U}_2 = {} & \frac{1}{\Lambda_1}[(1-\nu^2)\alpha^2 + 2(1+\nu)\beta^2 - 2ia\nu)] \\ & - \frac{i}{\Lambda_1\Lambda_2}[(a^2+c^2)\alpha^2 - 4bc\alpha\beta + 2(a^2+2b^2)\beta^2],\end{aligned} \tag{3.17}$$
$$\widehat{V}_2 = -\frac{2b\nu i}{\Lambda_1} - \frac{(\nu+1)^2\alpha\beta}{\Lambda_1} - \frac{i}{\Lambda_1\Lambda_2}[2ab\alpha^2 - (a-c)^2\alpha\beta + 2bc\beta^2]; \tag{3.18}$$

III. y-DIRECTION TANGENTIAL FORCE:

$$\widehat{W}_3 = -\widehat{V}_1, \tag{3.19}$$
$$\widehat{U}_3 = \widehat{V}_2, \tag{3.20}$$
$$\begin{aligned}\widehat{V}_3 = {} & \frac{1}{\Lambda_1}[(1-\nu^2)\beta^2 + 2(1+\nu)\alpha^2 - 2ic\nu] \\ & - \frac{i}{\Lambda_1\Lambda_2}[(a^2+c^2)\beta^2 - 4ab\alpha\beta + 2(2b^2+c^2)\alpha^2].\end{aligned} \tag{3.21}$$

We note the many symmetries which are apparent – not only do we have that $\widehat{W}_2 = -\widehat{U}_1$, $\widehat{W}_3 = -\widehat{V}_1$ and $\widehat{U}_3 = \widehat{V}_2$ (which are also satisfied by Sanders's incorrect Fourier transforms), but we also have the following:

If we denote $\widehat{U}_1 = f(\alpha, \beta, a, b, c)$, then $\widehat{V}_1 = f(\beta, \alpha, c, b, a)$. The same relationship is satisfied by $\widehat{U}_2$ and $\widehat{V}_3$.

In those cases where the expression Λ_1 can be factored, the method of partial fraction expansions can be used to write the above in a form for which the inverse transform may be found using methods such as those which were used in [3]. We illustrate this statement in the next section, where we solve the problem for the case of the spherical shell.

§4. The fundamental solution for the shallow spherical shell

The dimensionless equation for the middle surface of a shallow spherical shell (see [28, p. 366]) is

$$z = -\frac{1}{2}x^2 - \frac{1}{2}y^2. \tag{4.1}$$

Therefore, we treat (1.1) for the case $a = c = -\mu$, $b = 0$. In this case,

$$\Lambda_2 = -\mu(\alpha^2 + \beta^2), \Lambda_1 = (\alpha^2 + \beta^2)^2 + i\mu(\alpha^2 + \beta^2)$$

and (4.2)

$$\frac{1}{\Lambda_1} = \frac{i}{\mu}\left(\frac{1}{\alpha^2 + \beta^2 + i\mu} - \frac{1}{\alpha^2 + \beta^2}\right).$$

The transforms from Section 2 then become

I. NORMAL FORCE:

$$\widehat{W}_1 = \frac{i}{\mu}\frac{1}{\alpha^2 + \beta^2 + i\mu}, \tag{4.3}$$

$$\widehat{U}_1 = -(\nu + 1)\frac{\alpha}{\Lambda_1}, \tag{4.4}$$

$$\widehat{V}_1 = -(\nu + 1)\frac{\beta}{\Lambda_1}; \tag{4.5}$$

II. x-DIRECTION TANGENTIAL FORCE:

$$\widehat{W}_2 = -\widehat{U}_1, \tag{4.6}$$

$$\widehat{U}_2 = 2i\mu(\nu + 1)\frac{1}{\Lambda_1} + (1 - v^2)\frac{\alpha^2}{\Lambda_1} + 2(\nu + 1)\frac{\beta^2}{\Lambda_1}, \tag{4.7}$$

$$\widehat{V}_2 = -(\nu + 1)^2\frac{\alpha\beta}{\Lambda_1}; \tag{4.8}$$

III. y-DIRECTION TANGENTIAL FORCE:

$$\widehat{W}_3 = -\widehat{V}_1, \tag{4.9}$$

$$\widehat{U}_3 = \widehat{V}_2, \tag{4.10}$$

$$\widehat{V}_3(\alpha, \beta) = \widehat{U}_2(\beta, \alpha). \tag{4.11}$$

Now, to find the inverse transforms, we will need (see [1], [3])

$$\mathcal{F}^{-1}\left(\frac{1}{\alpha^2+\beta^2}\right) = -\frac{1}{4\pi}\ln(x^2+y^2), \tag{4.12}$$

$$\mathcal{F}^{-1}\left(\frac{1}{\alpha^2+\beta^2+i\mu}\right) = \frac{i}{4}H_0^{(1)}(\omega^3\sqrt{\mu}\; r), \text{ where } \omega = e^{\frac{i\pi}{4}}, r = \sqrt{x^2+y^2}, \tag{4.13}$$

$$\mathcal{F}^{-1}(\alpha\hat{f}(\alpha,\beta)) = \frac{1}{i}\frac{\partial}{\partial x}\mathcal{F}^{-1}(\hat{f}(\alpha,\beta)), \tag{4.14}$$

$$\frac{d}{dz}[H_0^{(1)}(z)] = -H_1^{(1)}(z), \tag{4.15}$$

$$\frac{d}{dz}[H_1^{(1)}(z)] = H_0^{(1)}(z) - \frac{1}{z}H_1^{(1)}(z), \tag{4.16}$$

where $H_n^{(1)}(z)$ is the Hankel function of the first kind, of order n. Applying (4.12)–(4.16) to (4.3)–(4.11), and after much simplification, we have

I. NORMAL FORCE:

$$W_1 = \frac{1}{4\mu}H_0^{(1)}(\omega^3\sqrt{\mu}\; r), \tag{4.17}$$

$$U_1 = -\frac{\omega(\nu+1)}{4\sqrt{\mu}}\frac{x}{r}H_1^{(1)}(\omega^3\sqrt{\mu}\; r) - \frac{\nu+1}{2\pi\mu}\frac{x}{r^2}, \tag{4.18}$$

$$V_1(x,y) = U_1(y,x); \tag{4.19}$$

II. x-DIRECTION TANGENTIAL FORCE:

$$W_2 = -U_1, \tag{4.20}$$

$$\begin{aligned} U_2 = &-\frac{i(\nu+1)^2}{4}\frac{x^2}{r^2}H_0^{(1)}(\omega^3\sqrt{\mu}\; r) + \frac{\omega^3(\nu+1)^2}{4\sqrt{\mu}}\frac{y^2-x^2}{r^3}H_1^{(1)}(\omega^3\sqrt{\mu}\; r) \\ &-\frac{\nu+1}{\pi}\ln r + \frac{i(\nu+1)^2}{2\pi\mu}\frac{y^2-x^2}{r^4}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} V_2 = &-\frac{i(\nu+1)^2}{4}\frac{xy}{r^2}H_0^{(1)}(\omega^3\sqrt{\mu}\; r) - \frac{\omega^3(\nu+1)^2}{2\sqrt{\mu}}\frac{xy}{r^3}H_1^{(1)}(\omega^3\sqrt{\mu}\; r) \\ &-\frac{i(\nu+1)^2}{\pi\mu}\frac{xy}{r^4}; \end{aligned} \tag{4.22}$$

III. y-DIRECTION TANGENTIAL FORCE

$$W_3 = -V_1, \quad U_3 = V_2 \quad V_3(x,y) = U_2(y,x), \qquad (4.23)\text{–}(4.25)$$

where, again, $r = \sqrt{x^2+y^2}$ and $\omega = e^{\frac{i\pi}{4}}$.

It is interesting to compare these results with those obtained by Sanders ([28, p. 366, (69)–(74)]). Using the facts that $K_0(z) = \frac{\pi i}{2}H_0^{(1)}(iz)$ and $K_1(z) = -\frac{\pi}{2}H_1^{(1)}(iz)$ for $-\pi < ayz \leq \frac{\pi}{2}$, where $K_n(z)$ is the modified Bessel function of the second kind, of order n (see [1]), we see, surprisingly, that Sanders's real parts are identical to ours. His imaginary parts differ from ours, of course, given our introduction of the functions F_{ij} in (2.27) and (2.28). However, we can compare them by looking at the stress measures n_{ij}. For example, we have (from (2.28))

$$n_{11} = F_{11} + \mathrm{Im}\ W_{yy}, \tag{4.26}$$

while Sanders has ([28, p. 362, (5) and (13)])

$$n_{11} = \mathrm{Im}\ W_{yy}. \tag{4.27}$$

Likewise for n_{12} and n_{22}. We see, after solving for the function F_{ij}, that Sanders's results again agree with ours! We are astounded that Sanders had the intuition to arrive at the correct results, using an inconsistent model. However, it is because of his use of an inconsistent model that we must consider the results in this paper as a justification for his results, and not vice versa.

§5. The fundamental solution for the shallow cylindrical shell

The dimensionless equation for the middle surface of a shallow cylindrical shell (see [30, p. 368]) is

$$z = -\frac{1}{2}y^2. \tag{5.1}$$

Therefore, we treat (1.1) for the case $c = -\mu, a = b = 0$. In this case,

$$\Lambda_2 = -\mu\alpha^2, \Lambda_1 = (\alpha^2 + \beta^2)^2 + i\mu\alpha^2$$

and

$$\begin{aligned} \frac{1}{\Lambda_1} &= \frac{\omega}{2\sqrt{\mu}}\left[\frac{1}{\alpha(\alpha^2+\beta^2+\omega^2\sqrt{\mu}\ \alpha)} - \frac{1}{\alpha(\alpha^2+\beta^2-\omega^3\sqrt{\mu}\ \alpha)}\right] \\ &= \frac{\omega}{2\sqrt{\mu}}\left[\frac{1}{\alpha D_+} - \frac{1}{\alpha D_-}\right]. \end{aligned} \tag{5.2}$$

The transforms from Section 3 become:

I. NORMAL FORCE:

$$\widehat{W}_1 = \frac{1}{\Lambda_1} + \frac{i}{\mu}\frac{1}{\alpha^2}, \tag{5.3}$$

$$\widehat{U}_1 = -\frac{\nu\alpha}{\Lambda_1} + \frac{\beta^2}{\alpha\Lambda_1}, \tag{5.4}$$

$$\widehat{V}_1 = -(\nu+2)\frac{\beta}{\Lambda_1} - \frac{\beta^3}{\alpha^2\Lambda_1}; \tag{5.5}$$

II. x-DIRECTION TANGENTIAL FORCE:

$$\widehat{W}_2 = -\widehat{U}_1, \tag{5.6}$$

$$\widehat{U}_2 = \frac{i\mu}{\Lambda_1} + (1-\nu^2)\frac{\alpha^2}{\Lambda_1} + 2(1+\nu)\frac{\beta^2}{\Lambda_1}, \tag{5.7}$$

$$\widehat{V}_2 = -(\nu+1)^2\frac{\alpha\beta}{\Lambda_1} - \frac{i\mu\beta}{\alpha\Lambda_1}; \tag{5.8}$$

III. y-DIRECTION TANGENTIAL FORCE:

$$\widehat{W}_3 = -\widehat{V}_1, \tag{5.9}$$

$$\widehat{U}_3 = \widehat{V}_2, \tag{5.10}$$

$$\widehat{V}_3 = \frac{2i\mu(\nu+1)}{\Lambda_1} + 2(\nu+1)\frac{\alpha^2}{\Lambda_1} + (1-\nu^2)\frac{\beta^2}{\Lambda_1} + i\mu\frac{\beta^2}{\alpha^2\Lambda_1}. \tag{5.11}$$

To find the inverse transforms we will need (again, see [1], [3])

$$H(x) = \text{Heaviside function}, \; H'(x) = \delta(x), \tag{5.12}$$
$$\operatorname{sgn} x = H(x) - H(-x),$$

$$\mathcal{F}^{-1}\left(\frac{1}{\alpha^{n+1}\beta^{m+1}}\right) = -\frac{i^{n+m}}{4n!m!}(x^n \operatorname{sgn} x)(y^m \operatorname{sgn} y), \quad n,m = 0,1,2,\ldots \tag{5.13}$$

$$\mathcal{F}^{-1}\left(\frac{1}{\alpha^{n+1}}\right) = \frac{i^{n+1}}{2n!}\delta(y)x^n \operatorname{sgn} x, \quad n = 0,1,2,\ldots \tag{5.14}$$

$$\mathcal{F}^{-1}\left(\frac{1}{\beta^{n+1}}\right) = \frac{i^{n+1}}{2n!}\delta(x)y^n \operatorname{sgn} y, \quad n = 0,1,2,\ldots \tag{5.15}$$

$$\mathcal{F}^{-1}(\hat{f}\hat{g}) = f * g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-x_1, y-y_1)g(x_1,y_1)dx_1dy_1$$
$$\text{(convolution of } f \text{ and } g\text{)}, \tag{5.16}$$

and

$$\mathcal{F}^{-1}\left(\frac{1}{D_+}\right) = \frac{i}{4}e^{\frac{\omega\sqrt{\mu}\;x}{2}}H_0^{(1)}\left(\frac{\omega^2\sqrt{\mu}}{2}r\right) = F(x,y), \tag{5.17}$$

$$\mathcal{F}^{-1}\left(\frac{1}{D_-}\right) = F(-x,y). \tag{5.18}$$

Now, formally, we have

$$\begin{aligned}\mathcal{F}^{-1}\left(\frac{1}{\alpha D_+}\right) &= \left[\frac{i}{2}\delta(y) \operatorname{sgn} x\right] * F(x,y)\\ &= \frac{i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(y-y_1) \operatorname{sgn}(x-x_1)\frac{i}{4}e^{\frac{\omega\sqrt{\mu}\;x_1}{2}}H_0^{(1)}\left(\frac{\omega^3\sqrt{\mu}}{2}\rho\right)dx_1dy_1 \qquad (5.19)\\ &= -\frac{1}{8}\int_{-\infty}^{\infty}\operatorname{sgn}(x-x_1)e^{\frac{\omega\sqrt{\mu}\;x_1}{2}}H_0^{(1)}\left(\frac{\omega^3\sqrt{\mu}}{2}\rho\right)dx_1, \rho = x_1^2+y^2.\end{aligned}$$

However, the above diverges "at $x_1 = \infty$" since, for large $|x_1|$, $H_0^{(1)}\left(\frac{\omega^3\sqrt{\mu}}{2}\rho\right)$ behaves like (see [1])

$$\frac{1}{(x_1^2+y^2)^{1/4}}e^{-\frac{\omega\sqrt{\mu}|x_1|}{2}}$$

and, thus, the integrand behaves like

$$\frac{1}{(x_1^2+y^2)^{1/4}}e^{\frac{\omega\sqrt{\mu}(x_1-|x_1|)}{2}}.$$

Similarly,

$$\mathcal{F}^{-1}\left(\frac{1}{\beta D_\pm}\right) = -\frac{1}{8}e^{\pm\frac{\omega\sqrt{\mu}\ x}{2}} \int_{-\infty}^{2} \operatorname{sgn}(y-y_1)H_0^{(1)}\left(\frac{\omega^2\sqrt{\mu}}{2}\rho\right)dy_1, \rho = x^2+y_1^2, \tag{5.20}$$

but these integrals *converge*. We can then write

$$\frac{1}{\alpha D_\pm} = \frac{1}{\beta^2}\left(\frac{1}{\alpha} - \frac{\alpha}{D_\pm} \mp \frac{\sqrt{\mu}\ \omega^3}{D_\pm}\right). \tag{5.21}$$

Finally, we also will need

$$\mathcal{F}^{-1}\left(\frac{1}{\beta^2 D_\pm}\right) = -\frac{i}{8}e^{\pm\frac{\omega\sqrt{\mu}\ x}{2}} \int_{-\infty}^{\infty} (y-y_1)\ \operatorname{sgn}(y-y_1)H_0^{(1)}\left(\frac{\omega^3\sqrt{\mu}}{2}r_1\right)dy_1, \tag{5.22}$$

$$\mathcal{F}^{-1}\left(\frac{1}{\beta^3 D_\pm}\right) = \frac{1}{16}e^{\pm\frac{\omega\sqrt{\mu}\ x}{2}} \int_{-\infty}^{\infty} (y-y_1)^2\ \operatorname{sgn}(y-y_1)H_0^{(1)}\left(\frac{\omega^3\sqrt{\mu}}{2}r_1\right)dy_1, \tag{5.23}$$

where, in these and below, we have $r_1 = \sqrt{x^2+y_1^2}$. We now proceed to find the inverse transforms of (5.3)–(5.11). Following [3], let us define

$$F_{03}(x,y) = F(-x,y), \quad F_{07}(x,y) = F(x,y), \tag{5.24}$$

where F was defined in (5.17). (For the "official" definition of F_{0j} and F_{1j}, see the Appendix.) We then have

$$\frac{\partial}{\partial x}F_{03}(x,y) = i\left[\frac{\omega^3\sqrt{\mu}}{2}F_{03}(x,y) - F_{13}(x,y)\right], \tag{5.25}$$

$$\frac{\partial}{\partial x}F_{07}(x,y) = i\left[\frac{-\omega^3\sqrt{\mu}}{2}F_{03}(x,y) - F_{17}(xy)\right], \tag{5.26}$$

$$\text{where}\quad F_{13}(x,y) = \frac{\omega^3\sqrt{\mu}}{8}\frac{x}{r}e^{\frac{-\omega\sqrt{\mu}\ x}{2}}H_1^{(1)}\left(\frac{\omega^3\sqrt{\mu}}{2}r\right), \tag{5.27}$$

$$F_{17}(x,y) = \frac{\omega^3\sqrt{\mu}}{8}\frac{x}{r}e^{\frac{\omega\sqrt{\mu}\ x}{2}}H_1^{(1)}\left(\frac{\omega^3\sqrt{\mu}}{2}r\right). \tag{5.28}$$

Using this notation, and after much computation, our solutions are

I. NORMAL FORCE:

$$\begin{aligned} W_1 = &-\frac{1}{8}\int_{-\infty}^{\infty}(y-y_1)\ \operatorname{sgn}(y-y_1)[F_{03}(x,y_1)+F_{07}(x,y_1)]dy_1 \\ &+\frac{\omega}{4\sqrt{\mu}}\int_{-\infty}^{\infty}(y-y_1)\ \operatorname{sgn}(y-y_1)[F_{13}(x,y_1)-F_{17}(x,y_1)]dy_1 \\ &-\frac{i}{2\mu}x\ \operatorname{sgn}\ x\ \delta(y). \end{aligned} \tag{5.29}$$

(Please note: in the rest of this paper, $F_{i3} = F_{i3}(x,y)$ unless it is part of an integrand, in which case $F_{i3} = F_{i3}(x,y_1)$; similarly for F_{i7}. Also, $\int$ means $\int_{-\infty}^{\infty}$.)

$$\begin{aligned} U_1 = {} & \frac{\omega(1+\nu)}{2\sqrt{\mu}}(F_{03}-F_{07}) - \frac{\omega^3\sqrt{\mu}}{8}\int (y-y_1)\ \mathrm{sgn}(y-y_1)(F_{03}-F_{07})dy_1 \\ & - \frac{1}{4}\int (y-y_1)\ \mathrm{sgn}(y-y_1)(F_{13}+F_{17})dy_1 \\ & - \frac{i}{4} y\ \mathrm{sgn}\ x\ \mathrm{sgn}\ y; \end{aligned} \tag{5.30}$$

$$\begin{aligned} V_1 = {} & \frac{i(1-\nu)}{8}\int \ \mathrm{sgn}(y-y_1)(F_{03}+F_{07})dy_1 \\ & + \frac{\mu}{16}\int (y-y_1)^2\ \mathrm{sgn}(y-y_1)(F_{03}+F_{07})dy_1 \\ & + \frac{\omega^3(1+\nu)}{4\sqrt{\mu}}\int \ \mathrm{sgn}(y-y_1)(F_{13}-F_{17})dy_1 \\ & - \frac{\omega\sqrt{\mu}}{8}\int (y-y_1)^2\ \mathrm{sgn}(y-y_1)(F_{13}-F_{17})dy_1 \\ & - \frac{i}{4} x\ \mathrm{sgn}\ x\ \mathrm{sgn}\ y; \end{aligned} \tag{5.31}$$

II. x-DIRECTION TANGENTIAL FORCE:

$$W_2 = U_1, \tag{5.32}$$

$$\begin{aligned} U_2 = {} & \frac{(3-\nu)(1+\nu)}{4}(F_{03}+F_{07}) - \frac{i\mu}{8}\int (y-y_1)\ \mathrm{sgn}(y-y_1)(F_{03}+F_{07})dy_1 \\ & - \frac{\omega(1+\nu)^2}{2\sqrt{\mu}}(F_{13}-F_{17}) + \frac{\omega^3\sqrt{\mu}}{4}\int (y-y_1)\ \mathrm{sgn}(y-y_1)(F_{13}-F_{17})dy_1, \end{aligned} \tag{5.33}$$

$$\begin{aligned} V_2 = {} & -\frac{\omega^3(1+\nu)^2}{2\sqrt{\mu}}\frac{\partial}{\partial y}(F_{03}-F_{07}) + \frac{\omega\sqrt{\mu}}{4}\int \ \mathrm{sgn}(y-y_1)(F_{03}-F_{07})dy_1, \\ & - \frac{\omega^3\mu^{3/2}}{16}\int (y-y_1)^2\ \mathrm{sgn}(y-y_1)(F_{03}-F_{07})dy_1 \\ & - \frac{\mu}{8}\int (y-y_1)^2\ \mathrm{sgn}(y-y_1)(F_{13}+F_{17})dy_1 \\ & - 8i\mu y^2\ \mathrm{sgn}\ x\ \mathrm{sgn}\ y; \end{aligned} \tag{5.34}$$

III. y-DIRECTION TANGENTIAL FORCE:

$$W_3 = V_1, \tag{5.35}$$

$$U_3 = V_2, \tag{5.36}$$

$$\begin{aligned} V_3 = {} & \frac{(3-\nu)(1+\nu)}{4}(F_{03}+F_{07}) + \frac{i\mu(1-2\nu)}{8}\int (y-y_1)\ \mathrm{sgn}(y-y_1)(F_{03}+F_{07})dy_1 \\ & + \frac{\mu^2}{48}\int (y-y_1)^3\ \mathrm{sgn}(y-y_1)(F_{03}+F_{07})dy_1 \\ & + \frac{\omega(1+\nu)^2}{2\sqrt{\mu}}(F_{13}-F_{17}) \end{aligned} \tag{5.37}$$

$$+\frac{\omega^3\sqrt{\mu}(1+2\nu)}{4}\int(y-y_1)\ \mathrm{sgn}(y-y_1)(F_{13}-F_{17})dy_1$$
$$-\frac{\omega\mu^{3/2}}{24}\int(y-y_1)^3\ \mathrm{sgn}(y-y_1)(F_{13}-F_{17})dy_1$$
$$+\frac{i}{4}xy\ \mathrm{sgn}\ x\ \mathrm{sgn}\ y.$$

The real parts of these results are identical to the (real) results in [3], as is shown in the Appendix. Since the latter results also have been verified by direct substitution, it is seen that our results are, indeed, correct.

It is difficult to compare these results to those obtained by Sanders and Simmonds in [30], as they use "classical" methods and add additional terms which do not have Fourier transforms, in what seems an ad hoc manner. At any rate, as they use the inconsistent model developed and used in [28], we are suspect of their results, as ingenious as their methods may be.

§6. Closing remarks

We have shown that the complex model for the shallow shell equation developed and used in [28] and [30] is inconsistent, and we have corrected that model. We have provided consistent complex solutions to the only two cases for which the denominator Λ_1, in the Fourier transforms of the solution, can be factored (into polynomials in α and β).

Further, the PDE

$$\Delta^2 w - i(w_{xx}+Kw_{yy}) = f, \tag{6.1}$$

where $|K|\leq 1$ and f is the applied surface load, is seen often in the literature of shallow shell theory (e.g., see [32] and [33]). Equation (5.1) is easily seen to be the w-equation in Sanders's system of PDEs ([28, p. 365], (42)), with $p_1\equiv p_2\equiv 0$ and after a change of variables. As that system of PDEs is inconsistent, we believe that special care must be taken when using results from those papers.

Appendix. The equivalence of our solutions with those in [3].

Chen et al., in [3], use the dimensional form of the variables, and the equivalent dimensional form of the real model given in this paper in (2.1)–(2.11), to derive the fundamental solution for the circular cylindrical shell.

We show that our solutions are equivalent to those in [3] for Case II: x-DIRECTION TANGENTIAL FORCE (i.e., for $\lambda_1=1$, $\lambda=\lambda_2=0$), the remaining two cases proceeding similarly.

First, we give the relationship between the dimensional form of the variables, used in [3], and denoted by $\tilde{x},\tilde{y}$ etc., and the dimensionless variables in this paper. From [28, p. 362], we have

$$\begin{array}{lll}\tilde{x}=Lx & \tilde{u}=\frac{\sigma L}{E}u & \tilde{w}=\frac{\mu\sigma R}{E}w\\ \tilde{y}=Ly & \tilde{v}=\frac{\sigma L}{E}v & \hat{p}_1=\frac{\sigma h}{L}p_1\end{array}\tag{A.1}$$

where L and R are the reference lengths which were used in the definition of u, E is Young's modulus and σ is a "reference stress". Also, [3] uses the quantity $\tilde{\mu}$:

$$\tilde{u}^4=\frac{12(1-\nu^2)}{\widetilde{R}h}$$

where $\widetilde{R}$ is the radius of the circular cylindrical shell, h is the shell thickness and ν is Poisson's ratio.

We choose to let $L = 1$ and $R = \widetilde{R}$, in which case we have

$$u = \tilde{u}^2 \quad \tilde{;}x = x \quad \tilde{;}y = y \quad \tilde{;}p_1 = \sigma h p_1, \tag{A.2}$$

and we need to show that the solutions u_2, v_2, w_2 and $\tilde{u}_2, \tilde{v}_2$ and $\tilde{w}_2$ satisfy

$$\hat{u}_2 = \frac{\sigma}{E} u_2 \quad \tilde{;}v_2 = \frac{\sigma}{E} v_2 \quad \hat{;}w_2 = \frac{u\sigma R}{E} w_2 \tag{A.3}$$

for some choice of the parameter σ.

Now, Chen et al. ([3, p. 20, (A.19) and p. 21, (A.23)]) define

$$\begin{aligned} F_{0j}(x,y) &= \frac{i}{4} e^{\frac{i\omega^j \sqrt{\mu}\ x}{2}} H_0^{(1)}\left(\frac{\tau_j \omega^j \sqrt{\mu}}{2}\sqrt{x^2+y^2}\right), \\ F_{1j}(x,y) &= \frac{\tau_j \omega^j \sqrt{\mu}}{8} \frac{x}{\sqrt{x^2+y^2}} e^{\frac{i\omega^j \sqrt{\mu}\ x}{2}} H_1^{(1)}\left(\frac{\tau_j \omega^j \sqrt{\mu}}{2}\sqrt{x^2+y^2}\right), \\ & j = 1,3,5,7; \omega = e^{\frac{\pi i}{4}}, \end{aligned} \tag{A.4}$$

where $\tau_1 = \tau_3 = 1, \tau_5 = \tau_7 = -1$, and where we have used $\tilde{x} = x$, $\tilde{y} = y$ and $\tilde{u}^2 = u$. Then it is easy to show that

$$F_{01} \pm F_{05} = \overline{F_{03} \pm F_{07}}, \quad F_{11} \pm F_{15} = -\overline{(F_{13} \pm F_{13})}, \tag{A.5}$$

from which we also have

$$\begin{aligned} \sum F_{0j} &= 2\ \mathrm{Re}(F_{03} + F_{07}) \\ \sum \omega^j F_{0j} &= \sqrt{2}\ i[\mathrm{Re}(F_{03} - F_{07}) - \mathrm{Im}(F_{03} - F_{07}) \\ \sum \omega^{2j} F_{0j} &= 2\ \mathrm{Im}(F_{03} + F_{07}) \\ \sum \omega^{3j} F_{0j} &= \sqrt{2}\ i[\mathrm{Re}(F_{03} - F_{07}) + \mathrm{Im}(F_{03} - F_{07})] \end{aligned}$$

and (A.6)

$$\begin{aligned} \sum F_{1j} &= 2i\ \mathrm{Im}(F_{13} + F_{17}) \\ \sum \omega^j F_{1j} &= -\sqrt{2}[\mathrm{Re}(F_{13} - F_{17}) + \mathrm{Im}(F_{13} - F_{17})] \\ \sum \omega^{2j} F_{1j} &= -2i\ \mathrm{Re}(F_{13} + F_{17}) \\ \sum \omega^{3j} F_{1j} &= \sqrt{2}[\mathrm{Re}(F_{13} - F_{17}) - \mathrm{Im}(F_{13} - F_{17})], \end{aligned}$$

where by $\sum a_j$ we mean $a_1 + a_3 + a_5 + a_7$.

We use equations (A.6) to simplify the expressions for $\tilde{u}_2$, $\tilde{v}_2$ and $\tilde{w}_2$ ([3, (4.31)–(4.33)]), resulting in

$$\begin{aligned}
\tilde{w}_2 = \tilde{\lambda}_1 \Big\{ &- \frac{3\sqrt{2}(1+\nu)}{h^2 R \mu^{3/2}} [\mathrm{Re}(F_{03} - F_{07}) - \mathrm{Im}(F_{03} - F_{07})] \\
&- \frac{3}{2\sqrt{2} h^2 R \sqrt{\mu}} \int (y - y_1) \operatorname{sgn}(y - y_1) [\mathrm{Re}(F_{03} + F_{07}) + \mathrm{Im}(F_{03} - F_{07})] dy_1 \qquad \text{(A.7)} \\
&+ \frac{3}{h^2 R \mu} \int (y - y_1) \operatorname{sgn}(y - y_1) \ \mathrm{Re}(F_{13} + F_{17}) dy_1 \Big\},
\end{aligned}$$

$$\begin{aligned}
\tilde{u}_2 = \tilde{\lambda}_1 \Big\{ &\frac{3(3-\nu)(1+\nu)}{h^2 R^2 \mu^2} \ \mathrm{Re}(F_{03} + F_{07}) + \frac{3}{2h^2 R^2 \mu} \int (y - y_1) \operatorname{sgn}(y - y_1) \ \mathrm{Im}(F_{03} + F_{07}) dy_1 \\
&- \frac{3\sqrt{2}(1+\nu)^2}{h^2 R^2 \mu^{5/2}} [\mathrm{Re}(F_{13} - F_{17}) - \mathrm{Im}(F_{13} - F_{17})] \qquad \text{(A.8)} \\
&- \frac{3}{\sqrt{2}\ h^2 R^2 \mu^{3/2}} \int (y - y_1) \operatorname{sgn}(y - y_1) [\mathrm{Re}(F_{13} + F_{17}) + \mathrm{Im}(F_{13} + F_{17})] dy_1 \Big\},
\end{aligned}$$

$$\begin{aligned}
\tilde{v}_2 = \tilde{\lambda}_1 \Big\{ &\frac{3\sqrt{2}(1+\nu)^2}{h^2 R^2 \mu^{5/2}} \frac{\partial}{\partial y} [\mathrm{Re}(F_{03} - F_{07}) + \mathrm{Im}(F_{03} - F_{07})] \\
&- \frac{3}{\sqrt{2}\ h^2 R^2 \mu^{3/2}} \int \operatorname{sgn}(y - y_1) [\mathrm{Re}(F_{03} - F_{07}) - \mathrm{Im}(F_{03} - F_{07})] dy_1 \qquad \text{(A.9)} \\
&+ \frac{3}{4\sqrt{2}\ h^2 R^2 \sqrt{\mu}} \int (y - y_1)^2 \operatorname{sgn}(y - y_1) [\mathrm{Re}(F_{03} - F_{07}) + \mathrm{Im}(F_{03} - F_{07})] dy_1 \\
&- \frac{3}{2h^2 R^2 \mu} \int (y - y_1)^2 \operatorname{sgn}(y - y_1) \ \mathrm{Re}(F_{13} + F_{17}) dy_1 \Big\},
\end{aligned}$$

where we have set $\tilde{\lambda} = \tilde{\lambda}_2 = 0$ and where, as above, $\int$ means $\int_{-\infty}^{\infty}$. Again, we have replaced $\tilde{\mu}$ by $\sqrt{\mu}$.

Now, the question is: what value of $\tilde{\lambda}_1$ ([3, p. 12, (4.6)]) corresponds to $\lambda_1 = 1$? First, $\lambda_1 = 1$ gives a load on the right side of our equation (2.1) equal to

$$-p_1 = -\delta(x, y). \qquad \text{(A.10)}$$

¿From (A.2), this corresponds to

$$\tilde{p}_1 = \sigma h \delta(x, y). \qquad \text{(A.11)}$$

Finally, from ([3, p. 10, (4.4) and (4.6)]), we have

$$\frac{1-\nu^2}{Eh} \tilde{p}_1 = \tilde{\lambda}_1 \delta(x, y). \qquad \text{(A.12)}$$

Therefore, (A.11) and (A.12) combine to give us

$$\tilde{\lambda}_1 = \frac{1-\nu^2}{Eh} \sigma h = \frac{(1-\nu^2)\sigma}{E}. \qquad \text{(A.13)}$$

We insert this value of $\tilde{\lambda}_1$ into (A.7)–(A.9), and we compute the real parts of (5.32)–(5.34) in order to compare. Taking real parts of (5.32)–(5.34) results in

$$\begin{aligned} w_2 = &-\frac{1+\nu}{2\sqrt{2}\ \sqrt{\mu}}[\mathrm{Re}(F_{03}-F_{07})-\mathrm{Im}(F_{03}-F_{07})] \\ &-\frac{\sqrt{\mu}}{8\sqrt{2}}\int (y-y_1)\ \mathrm{sgn}(y-y_1)[\mathrm{Re}(F_{03}-F_{07})+\mathrm{Im}(F_{03}-F_{07})]dy_1 \qquad (A.14) \\ &+\frac{1}{4}\int (y-y_1)\ \mathrm{sgn}(y-y_1)\ \mathrm{Re}(F_{13}+F_{17})dy_1, \\ u_2 = &\ \frac{(3-\nu)(1+\nu)}{4}\ \mathrm{Re}(F_{03}+F_{07})+\frac{u}{8}\int (y-y_1)\ \mathrm{sgn}(y-y_1)\ \mathrm{Im}(F_{03}+F_{07})dy_1 \\ &-\frac{(1+\nu)^2}{2\sqrt{2}\ \sqrt{\mu}}[\mathrm{Re}(F_{13}-F_{17})-\mathrm{Im}(F_{13}-F_{17})] \qquad (A.15) \\ &-\frac{\sqrt{\mu}}{4\sqrt{2}}\int (y-y_1)\ \mathrm{sgn}(y-y_1)[\mathrm{Re}(F_{13}-F_{17})+\mathrm{Im}(F_{13}-F_{17})]dy_1, \\ v_2 = &\ \frac{(1+\nu)^2}{2\sqrt{2}\ \sqrt{\mu}}\frac{\partial}{\partial y}[\mathrm{Re}(F_{03}-F_{07})+\mathrm{Im}(F_{07}-F_{07})] \\ &+\frac{\sqrt{\mu}}{4\sqrt{2}}\int\ \mathrm{sgn}(y-y_1)[\mathrm{Re}(F_{03}-F_{07})-\mathrm{Im}(F_{03}-F_{07})]dy_1 \\ &+\frac{\mu^{3/2}}{16\sqrt{2}}\int (y-y_1)^2\ \mathrm{sgn}(y-y_1)[\mathrm{Re}(F_{03}-F_{07})+\mathrm{Im}(F_{03}-F_{07})]dy_1 \qquad (A.16) \\ &-\frac{\mu}{8}\int (y-y_1)^2\ \mathrm{sgn}(y-y_1)\ \mathrm{Re}(F_{13}+F_{17})dy_1. \end{aligned}$$

We see that the $\tilde{u}_2$ and $u_2, \tilde{v}_2$ and v_2, and $\tilde{w}_2$ and w_2 involve the same terms. Therefore, we need only compare coefficients, which we do in tabular form:

$\tilde{w}_2$:	w_2:	RATIO $\left(\frac{\tilde{w}_2\ \text{COEFF.}}{w_2\ \text{COEFF.}}\right)$:
$\frac{-3\sqrt{2}(1+\nu)(1-\nu^2)\sigma}{Eh^2R\mu^{3/2}}$	$-\frac{1+\nu}{2\sqrt{2}\ \sqrt{\mu}}$	$\frac{12(1-\nu^2)\sigma}{Eh^2R\mu} = \frac{\mu\sigma R}{E}$
$\frac{-3(1-\nu^2)\sigma}{2\sqrt{2}\ Eh^2R\sqrt{\mu}}$	$-\frac{\sqrt{\mu}}{8\sqrt{2}}$	$\frac{12(1-\nu^2)\sigma}{3h^2R\mu} = \frac{\mu\sigma R}{E}$
$\frac{3(1-\nu^2)\sigma}{Eh^2R\mu}$	$\frac{1}{4}$	$\frac{12(1-\nu^2)\sigma}{Eh^2R\mu} = \frac{\mu\sigma R}{E}$

$\tilde{\mu}_2$:	μ_2:	RATIO:
$\frac{3(3-\nu)(1+\nu)(1-\nu^2)\sigma}{Eh^2R^2\mu^2}$	$\frac{(3-\nu)(1+\nu)}{4}$	$\frac{12(1-\nu^2)\sigma}{Eh^2R^2\mu^2} = \frac{\sigma}{E}$
$\frac{3(1-\nu^2)\sigma}{2Eh^2R^2\mu}$	$\frac{\mu}{8}$	$\frac{12(1-\nu^2)\sigma}{Eh^2R^2\mu^2} = \frac{\sigma}{E}$
$-\frac{3\sqrt{2}(1+\nu)^2(1-\nu^2)\sigma}{Eh^2R^2\mu^{5/2}}$	$-\frac{(1+\nu)^2}{2\sqrt{2}\ \sqrt{\mu}}$	$\frac{12(1-\nu^2)\sigma}{Eh^2R^2\mu^2} = \frac{\sigma}{E}$
$-\frac{3(1-\nu^2)\sigma}{\sqrt{2}Eh^2R^2\mu^{3/2}}$	$-\frac{\sqrt{\mu}}{4\sqrt{2}}$	$\frac{12(1-\nu^2)\sigma}{Eh^2R^2\mu^2} = \frac{\sigma}{E}$

$\tilde{v}_2$:	v_2:	RATIO:
$\frac{3\sqrt{2}(1+\nu^2)(1-\nu^2)\sigma}{Eh^2R^2\mu^{5/2}}$	$\frac{(1+\nu)^2}{2\sqrt{2}\ \sqrt{m}}$	$\frac{12(1-\nu^2)\sigma}{Eh^2R^2\mu^2} = \frac{\sigma}{E}$
$\frac{3(1-\nu^2)\sigma}{\sqrt{2}\ Eh^2R^2\mu^{3/2}}$	$\frac{\sqrt{\mu}}{4\sqrt{2}}$	$\frac{12(1-\nu^2)\sigma}{Eh^2R^2\mu^2} = \frac{\sigma}{E}$
$\frac{3(1-\nu^2)\sigma}{4\sqrt{2}\ Eh^2R^2\sqrt{\mu}}$	$\frac{\mu^{3/2}}{16\sqrt{2}}$	$\frac{12(1-\nu^2)\sigma}{Eh^2R^2\mu^2} = \frac{\sigma}{E}$
$-\frac{3(1-\nu^2)\sigma}{2Eh^2R^2\mu}$	$-\frac{\mu}{8}$	$\frac{12(1-\nu^2)\sigma}{Eh^2R^2\mu^2} = \frac{\sigma}{E}$

and we see that equations (A.3) are satisfied for any choice of the parameter σ.

In closing, let us note that, for the case $\lambda_2 = 1$, we have $\tilde{\lambda}_2 = \frac{(1-\nu^2)\sigma}{E}$, while, corresponding to $\lambda = 1$, we need $\tilde{\lambda} = -\frac{(1-\nu^2)\sigma R}{Eh^2\mu}$.

REFERENCES

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.

[2] B. Budiansky and J.L. Sanders, On the "best" first-order linear shell theory, in *Progress in Applied Mechanics – The Prager Anniversary Volume*, pp. 129–140, Macmillan, New York, 1963.

[3] G. Chen, M.P. Coleman, D. Ma, P.J. Morris and P. You, The fundamental solution for shallow circular cylindrical shells; part I: derivations, to appear in Int. J. Engineering Sci.

[4] G. Chen and J. Zhou, *Boundary Element Methods*, Academic Press, London, 1992.

[5] G.N. Chernyshev, Relations between dislocations and concentrated loadings in the theory of shells, J. Appl. Math. Mech. 23 (1959), 249–257.

[6] G.N. Chernyshev, On the action of concentrated forces and moments on an elastic thin shell of arbitrary shape, J. Appl. Math. Mech. 27 (1963), 172–184.

[7] L.H. Donnell, *Beams, Plates and Shells*, McGraw-Hill, New York, 1976.

[8] J. Dundurs and A. Jahanshahi, Concentrated forces on unidirectional stretched plates, Quart. J. Mech. Appl. Math. 18 (1965), 129–139.

[9] W. Flügge, Concentrated forces on shells, Proc. 11th Int. Cong. of Appl. Mech., Springer-Verlag, Berlin, 1964, 270–276.

[10] W. Flügge and D.A. Conrad, Thermal singularities for cylindrical shells, Proc. 3rd U.S. National Cong. of Appl. Mech., 1958, p. 321.

[11] W. Flügge and R.E. Elling, Singular solutions for shallow shells, Int. J. Solids and Structures, 8 (1972), 227–247.

[12] E. Hansen, An integral equation method for stress concentration problems in cylindrical shells, J. Elasticity 7 (1977), 283–305.

[13] R. Kanwal, *Generalized Functions: Theory and Technique*, Academic Press, New York, 1983.

[14] W.T. Koiter, A consistent first approximation in the general theory of elastic shells, *Proc. Symp. on the Theory of Thin Elastic Shells*, pp. 155–169, Delft, 1959.

[15] W.T. Koiter, A spherical shell under point loads at its poles, in *Progress in Applied Mechanics – The Prager Anniversary Volume*, pp. 155–169, Macmillan, New York, 1963.

[16] H. Kraus, *Thin Elastic Shells*, Wiley, New York, 1967.

[17] S. Lukasiewicz, Concentrated loading on shells, *Applied Mechanics – Proceedings Eleventh Int. Cong. of Appl. Math.*, pp. 272–282, Springer-Verlag, 1964.

[18] S. Lukasiewicz, Concentrated loads on shallow spherical shells, Quart. J. Mech. Appl. Math. 20 (1967), 293–305.

[19] S. Lukasiewicz, *Local Loads in Plates and Shells*, PWN-Polish Scientific Publishers, Warsaw, 1979.

[20] T. Matsui and O. Matsuoka, The fundamental solution in the theory of shallow shells, Int. J. Solids and Structures 14 (1978), 971–986.

[21] P.M. Naghdi, Further results in the derivation of the general equations of elastic shells, Int. J. Eng. Sci. 2 (1964), 269–273.

[22] F. Niordson, *Shell Theory*, North Holland, Amsterdam, 1985.

[23] V.V. Novozhilov, *The Theory of Thin Shells*, Noordhoff, Gröningen, The Netherlands, 1959.

[24] E. Reissner, Stresses and small displacement of shallow spherical shells, I, II, J. Math. and Phys. 25 (1947), 80–85, 279–300.

[25] E. Reissner, On the determination of stresses and displacements for unsymmetrical deformation of shallow spherical shells, J. Math. and Phys. 38, No. 1 (1959), 16–35.

[26] J.L. Sanders, An improved first-approximation theory for thin shells, NASA Rept. 24 (1959).

[27] J.L. Sanders, On the shell equations in complex form, IUTAM Symposium Copenhagen, in *Theory of Thin Shells*, Springer-Verlag, Berlin, 1969.

[28] J.L. Sanders, Singular solutions to the shallow shell equations, J. Appl. Mech. 37 (1970), 361–366.

[29] J.L. Sanders, Cutouts in shallow shells, J. Appl. Mech. 37 (1970), 374–383.

[30] J.L. Sanders and J.G. Simmonds, Concentrated forces on shallow cylindrical shells, J. Appl. Mech. 37 (1970), 367–373.

[31] J.G. Simmonds, Green's functions for closed, elastic spherical shells: exact and accurate approximate solutions, Nederl. Akad. Wetensch. Proc. Ser. B, 71 (1968), 236–249.

[32] J.G. Simmonds and M.R. Bradley, The fundamental solution for a shallow shell with an arbitrary quadratic mid surface, J. Appl. Mech. 43 (1976), 286–290.

[33] J.G. Simmonds and C.G. Tropf, The fundamental (normal point load) solution for a shallow hyperbolic paraboloidal shell, SIAM J. Appl. Math. 27 (1974),

102–120.

[34] I. Stakgold, *Boundary Value Problems of Mathematical Physics*, Vol. II, Macmillan, New York, 1967.

[35] S. Timoshenko and S. Woinowsky-Krieger, Theory of Plates and Shells, 2nd ed., McGraw-Hill, New York, 1959.

[36] C.J. Tsai and J.L. Sanders, Elliptical cutouts in cylindrical shells, J. Appl. Mech. 42 (1975), 326–334.

[37] V.Z. Vlasov, *Priklad. Mat. Mekhan.*, Vol. 11, Moscow, 1947.

[38] W.P. Wagner, Die Singularitäten functionen des gespannten Platte und der Kreiszylinderschale, ZAMP 35 (1984), 723–727.

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