POSITING SOLUTIONS OF A NONLINEAR
THREE-POINT BOUNDARY-VALUE PROBLEM

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ABSTRACT. We study the existence of positive solutions to the boundary-value problem

\[ u'' + a(t)f(u) = 0, \quad t \in (0, 1) \]
\[ u(0) = 0, \quad \alpha u(\eta) = u(1), \]

where \(0 < \eta < 1\) and \(0 < \alpha < 1/\eta\). We show the existence of at least one positive solution if \(f\) is either superlinear or sublinear by applying the fixed point theorem in cones.

1. Introduction

The study of multi-point boundary-value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [7, 8]. Then Gupta [5] studied three-point boundary-value problems for nonlinear ordinary differential equations. Since then, the more general nonlinear multi-point boundary value problems have been studied by several authors by using the Leray-Schauder Continuation Theorem, Nonlinear Alternatives of Leray-Schauder, and coincidence degree theory. We refer the reader to [1-3, 6, 10-12] for some recent results of nonlinear multi-point boundary value problems.

In this paper, we consider the existence of positive solutions to the equation

\[ u'' + a(t)f(u) = 0, \quad t \in (0, 1) \]  

with the boundary condition

\[ u(0) = 0, \quad \alpha u(\eta) = u(1), \]

where \(0 < \eta < 1\). Our purpose here is to give some existence results for positive solutions to (1.1)-(1.2), assuming that \(\alpha \eta < 1\) and \(f\) is either superlinear or sublinear. Our proof is based upon the fixed point theorem in a cone.

From now on, we assume the following:

(A1) \(f \in C([0, \infty), [0, \infty))\);

(A2) \(a \in C([0, 1], [0, \infty))\) and there exists \(x_0 \in [\eta, 1]\) such that \(a(x_0) > 0\)

1991 Mathematics Subject Classification: 34B15.

Key words and phrases: Second-order multi-point BVP, positive solution, cone, fixed point.

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Partially supported by NNSF (19801028) of China and NSF (ZR-96-017) of Gansu Province
Set
\[ f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}. \]

Then \( f_0 = 0 \) and \( f_\infty = \infty \) correspond to the superlinear case, and \( f_0 = \infty \) and \( f_\infty = 0 \) correspond to the sublinear case. By the positive solution of (1.1)-(1.2) we understand a function \( u(t) \) which is positive on \( 0 < t < 1 \) and satisfies the differential equation (1.1) and the boundary conditions (1.2).

The main result of this paper is the following

**Theorem 1.** Assume (A1) and (A2) hold. Then the problem (1.1)-(1.2) has at least one positive solution in the case
(i) \( f_0 = 0 \) and \( f_\infty = \infty \) (superlinear) or
(ii) \( f_0 = \infty \) and \( f_\infty = 0 \) (sublinear).

The proof of above theorem is based upon an application of the following well-known Guo’s fixed point theorem \[4\].

**Theorem 2.** Let \( E \) be a Banach space, and let \( K \subset E \) be a cone. Assume \( \Omega_1, \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1 \), \( \Omega_1 \subset \Omega_2 \), and let
\[ A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K \]
be a completely continuous operator such that
(i) \( \| Au \| \leq \| u \|, \quad u \in K \cap \partial \Omega_1, \) and \( \| Au \| \geq \| u \|, \quad u \in K \cap \partial \Omega_2; \) or
(ii) \( \| Au \| \geq \| u \|, \quad u \in K \cap \partial \Omega_1, \) and \( \| Au \| \leq \| u \|, \quad u \in K \cap \partial \Omega_2. \)

Then \( A \) has a fixed point in \( K \cap (\overline{\Omega}_2 \setminus \Omega_1) \).

2. THE PRELIMINARY LEMMAS

**Lemma 1.** Let \( \alpha \eta \neq 1 \) then for \( y \in C[0,1] \), the problem
\[
\begin{align*}
  u'' + y(t) &= 0, \quad t \in (0,1) \\
  u(0) &= 0, \quad \alpha u(\eta) = u(1)
\end{align*}
\]
has a unique solution
\[
  u(t) = - \int_0^t (t-s) y(s) \, ds - \frac{\alpha t}{1 - \alpha \eta} \int_0^\eta (\eta-s) y(s) \, ds + \frac{t}{1 - \alpha \eta} \int_0^1 (1-s) y(s) \, ds
\]

The proof of this lemma can be found in \[6\].

**Lemma 2.** Let \( 0 < \alpha < \frac{1}{\eta} \). If \( y \in C[0,1] \) and \( y \geq 0 \), then the unique solution \( u \) of the problem (2.1)-(2.2) satisfies
\[
  u \geq 0, \quad t \in [0,1]
\]

*Proof* From the fact that \( u''(x) = -y(x) \leq 0 \), we know that the graph of \( u(t) \) is concave down on \((0,1)\). So, if \( u(1) \geq 0 \), then the concavity of \( u \) and the boundary condition \( u(0) = 0 \) imply that \( u \geq 0 \) for \( t \in [0,1] \).

If \( u(1) < 0 \), then we have that
\[
  u(\eta) < 0
\]
and
\[
  u(1) = \alpha u(\eta) > \frac{1}{\eta} u(\eta).
\]
This contradicts the concavity of \( u \).
Lemma 3. Let $\alpha \eta > 1$. If $y \in C[0,1]$ and $y(t) \geq 0$ for $t \in (0,1)$, then (2.1)-(2.2) has no positive solution.

Proof Assume that (2.1)-(2.2) has a positive solution $u$.

If $u(1) > 0$, then $u(\eta) > 0$ and
\[
\frac{u(1)}{1} = \frac{\alpha u(\eta)}{1} > \frac{u(\eta)}{\eta}
\]
(this contradicts the concavity of $u$).

If $u(1) = 0$ and $u(\tau) > 0$ for some $\tau \in (0,1)$, then
\[
u(\eta) = u(1) = 0, \quad \tau \neq \eta
\]
If $\tau \in (0,\eta)$, then $u(\tau) > u(\eta) = u(1)$, which contradicts the concavity of $u$. If $\tau \in (\eta,1)$, then $u(0) = u(\eta) < u(\tau)$ which contradicts the concavity of $u$ again.

In the rest of the paper, we assume that $\alpha \eta < 1$. Moreover, we will work in the Banach space $C[0,1]$, and only the sup norm is used.

Lemma 4. Let $0 < \alpha < \frac{1}{\eta}$. If $y \in C[0,1]$ and $y \geq 0$, then the unique solution $u$ of the problem (2.1)-(2.2) satisfies
\[
\inf_{t \in [\eta,1]} u(t) \geq \gamma \|u\|
\]
where $\gamma = \min\{\alpha \eta, \frac{\alpha(1-\eta)}{1-\alpha \eta}, \eta\}$.

Proof. We divide the proof into two steps.

Step 1. We deal with the case $0 < \alpha < 1$. In this case, by Lemma 2, we know that
\[
u(\eta) \geq u(1).
\]
Set
\[
u(\bar{t}) = \|u\|.
\]
If $\bar{t} \leq \eta < 1$, then
\[
\min_{t \in [\eta,1]} u(t) = u(1)
\]
and
\[
u(\bar{t}) \leq u(1) + \frac{u(1) - u(\eta)}{1-\eta}(0-1)
\]
\[
= u(1)[1 - \frac{1 - \frac{1}{\alpha}}{1-\eta}]
\]
\[
= u(1) \frac{1 - \alpha \eta}{\alpha(1-\eta)}.
\]
This together with (2.9) implies that
\[
\min_{t \in [\eta,1]} u(t) \geq \frac{\alpha(1-\eta)}{1-\alpha \eta} \|u\|.
\]
If \( \eta < \bar{\tau} < 1 \), then
\[
\min_{t \in [\eta, 1]} u(t) = u(1) \tag{2.11}
\]
From the concavity of \( u \), we know that
\[
\frac{u(\eta)}{\eta} \geq \frac{u(\bar{\tau})}{\bar{\tau}} \tag{2.12}
\]
Combining (2.12) and boundary condition \( \alpha u(\eta) = u(1) \), we conclude that
\[
\frac{u(1)}{\alpha \eta} \geq \frac{u(\bar{\tau})}{\bar{\tau}} \geq u(\bar{\tau}) = \| u \|.
\]
This is
\[
\min_{t \in [\eta, 1]} u(t) \geq \alpha \eta \| u \|. \tag{2.13}
\]

Step 2. We deal with the case \( 1 \leq \alpha < \frac{1}{\eta} \). In this case, we have
\[
u(\eta) \leq u(1). \tag{2.14}\]
Set
\[
u(\bar{\tau}) = \| u \| \tag{2.15}
\]
then we can choose \( \bar{\tau} \) such that
\[
\eta \leq \bar{\tau} \leq 1 \tag{2.16}
\]
(we note that if \( \bar{\tau} \in [0, 1] \setminus [\eta, 1] \), then the point \((\eta, u(\eta))\) is below the straight line determined by \((1, u(1))\) and \((\bar{\tau}, u(\bar{\tau}))\). This contradicts the concavity of \( u \). From (2.14) and the concavity of \( u \), we know that
\[
\min_{t \in [\eta, 1]} u(t) = u(\eta). \tag{2.17}
\]
Using the concavity of \( u \) and Lemma 2, we have that
\[
\frac{u(\eta)}{\eta} \geq \frac{u(\bar{\tau})}{\bar{\tau}}. \tag{2.18}
\]
This implies
\[
\min_{t \in [\eta, 1]} u(t) \geq \eta \| u \|. \tag{2.19}
\]
This completes the proof.

3 Proof of main theorem

Proof of Theorem 1. Superlinear case. Suppose then that \( f_0 = 0 \) and \( f_\infty = \infty \).
We wish to show the existence of a positive solution of (1.1)-(1.2). Now (1.1)-(1.2)
has a solution \( y = y(t) \) if and only if \( y \) solves the operator equation
\[
y(t) = -\int_0^t (t - s) a(s)f(y(s))ds - \frac{\alpha t}{1 - \alpha \eta} \int_0^\eta (\eta - s) a(s)f(y(s))ds
+ \frac{t}{1 - \alpha \eta} \int_0^1 (1 - s) a(s)f(y(s))ds \tag{3.1}
\]
\( \overset{\text{def}}{=} Ay(t) \).
Denote
\[ K = \{ y \mid y \in C[0, 1], y \geq 0, \min_{\eta \leq t \leq 1} y(t) \geq \gamma \| y \| \}. \] (3.2)

It is obvious that \( K \) is a cone in \( C[0, 1] \). Moreover, by Lemma 4, \( AK \subset K \). It is also easy to check that \( A : K \to K \) is completely continuous.

Now since \( f_0 = 0 \), we may choose \( H_1 > 0 \) so that \( f(y) \leq \epsilon y \), for \( 0 < y < H_1 \), where \( \epsilon > 0 \) satisfies
\[ \frac{\epsilon}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)ds \leq 1. \] (3.3)

Thus, if \( y \in K \) and \( \| y \| = H_1 \), then from (3.1) and (3.3), we get
\[
Ay(t) \leq \frac{t}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)f(y(s))ds \\
\leq \frac{t}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)\epsilon y(s)ds \\
\leq \frac{\epsilon}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)\| y \| \\
\leq \frac{\epsilon}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)dsH_1. \] (3.4)

Now if we let
\[ \Omega_1 = \{ y \in C[0, 1] \mid \| y \| < H_1 \}, \] (3.5)
then (3.4) shows that \( \| Ay \| \leq \| y \| \), for \( y \in K \cap \partial \Omega_1 \).

Further, since \( f_\infty = \infty \), there exists \( \tilde{H}_2 > 0 \) such that \( f(u) \geq \rho u \), for \( u \geq \tilde{H}_2 \), where \( \rho > 0 \) is chosen so that
\[ \rho \frac{\eta \gamma}{1 - \eta \alpha} \int_\eta^1 (1 - s)a(s)ds \geq 1. \] (3.6)

Let \( H_2 = \max\{2H_1, \frac{\tilde{H}_2}{\gamma} \} \) and \( \Omega_2 = \{ y \in C[0, 1] \mid \| y \| < H_2 \} \), then \( y \in K \) and \( \| y \| = H_2 \) implies
\[ \min_{\eta \leq t \leq 1} y(t) \geq \gamma \| y \| \geq \tilde{H}_2. \]
and so

\[
Ay(\eta) = - \int_0^\eta (\eta - s)a(s)f(y(s))dt - \frac{\alpha \eta}{1 - \alpha \eta} \int_0^\eta (\eta - s)a(s)f(y(s))ds
\]

\[
+ \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)f(y(s))ds
\]

\[
= - \frac{1}{1 - \alpha \eta} \int_0^\eta (\eta - s)a(s)f(y(s))ds + \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)f(y(s))ds
\]

\[
= - \frac{1}{1 - \alpha \eta} \int_0^\eta a(s)f(y(s))ds + \frac{1}{1 - \alpha \eta} \int_0^\eta sa(s)f(y(s))ds
\]

\[
+ \frac{\eta}{1 - \alpha \eta} \int_0^1 a(s)f(y(s))ds - \frac{\eta}{1 - \alpha \eta} \int_0^1 sa(s)f(y(s))ds
\]

\[
= \eta - \frac{1}{1 - \alpha \eta} \int_0^1 sa(s)f(y(s))ds
\]

\[
\geq \eta - \frac{1}{1 - \alpha \eta} \int_0^1 a(s)f(y(s))ds - \frac{\eta}{1 - \alpha \eta} \int_0^1 sa(s)f(y(s))ds \quad \text{(by } \eta < 1) \]

\[
= \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)f(y(s))ds.
\]

Hence, for \( y \in K \cap \partial \Omega_2, \)

\[
\|Ay\| \geq \rho \frac{\eta \gamma}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)ds\|y\| \geq \|y\|.
\]

Therefore, by the first part of the Fixed Point Theorem, it follows that \( A \) has a fixed point in \( K \cap (\overline{\Omega_2 \setminus \Omega_1}) \), such that \( H_1 \leq \|u\| \leq H_2 \). This completes the superlinear part of the theorem.

**Sublinear case.** Suppose next that \( f_0 = \infty \) and \( f_\infty = 0 \). We first choose \( M_3 > 0 \) such that \( f(y) \geq M_3 y \) for \( 0 < y < H_3 \), where

\[
M_3 \gamma \left( \frac{\eta}{1 - \alpha \eta} \right) \int_0^1 (1 - s)a(s)ds \geq 1. \tag{3.8}
\]

By using the method to get (3.7), we can get that

\[
Ay(\eta) = - \int_0^\eta (\eta - s)a(s)f(y(s))dt - \frac{\alpha \eta}{1 - \alpha \eta} \int_0^\eta (\eta - s)a(s)f(y(s))ds
\]

\[
+ \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)f(y(s))ds
\]

\[
\geq \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)f(y(s))ds
\]

\[
\geq \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)My(s)ds
\]

\[
\geq \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)a(s)M_3 \gamma ds\|y\|
\]

\[
\geq H_3
\]
Thus, we may let \( \Omega_3 = \{ y \in C[0, 1] \mid \|y\| < H_3 \} \) so that
\[
\|Ay\| \geq \|y\|, \quad y \in K \cap \partial \Omega_3.
\]

Now, since \( f_\infty = 0 \), there exists \( \hat{H}_4 > 0 \) so that \( f(y) \leq \lambda y \) for \( y \geq \hat{H}_4 \), where \( \lambda > 0 \) satisfies
\[
\frac{\lambda}{1-\alpha \eta} \int_0^1 (1-s)a(s)ds \leq 1. \tag{3.10}
\]

We consider two cases:

Case (i). Suppose \( f \) is bounded, say \( f(y) \leq N \) for all \( y \in [0, \infty) \). In this case choose
\[
H_4 = \max\{2H_3, \frac{N}{1-\alpha \eta} \int_0^1 (1-s)a(s)ds\}
\]
so that for \( y \in K \) with \( \|y\| = H_4 \) we have
\[
Ay(t) = -\int_0^t (t-s)a(s)f(y(s))ds - \frac{\alpha t}{1-\alpha \eta} \int_0^{\eta} (\eta-s)a(s)f(y(s))ds
\]
\[
+ \frac{t}{1-\alpha \eta} \int_0^1 (1-s)a(s)f(y(s))ds
\]
\[
\leq \frac{t}{1-\alpha \eta} \int_0^1 (1-s)a(s)N ds
\]
\[
\leq H_4,
\]
and therefore \( \|Ay\| \leq \|y\| \).

Case (ii). If \( f \) is unbounded, then we know from (A1) that there is \( H_4 : H_4 > \max\{2H_3, \frac{1}{7}\hat{H}_4\} \) such that
\[
f(y) \leq f(H_4) \quad \text{for} \quad 0 < y \leq H_4.
\]
(We are able to do this since \( f \) is unbounded). Then for \( y \in K \) and \( \|y\| = H_4 \) we have
\[
Ay(t) = -\int_0^t (t-s)a(s)f(y(s))ds - \frac{\alpha t}{1-\alpha \eta} \int_0^{\eta} (\eta-s)a(s)f(y(s))ds
\]
\[
+ \frac{t}{1-\alpha \eta} \int_0^1 (1-s)a(s)f(y(s))ds
\]
\[
\leq \frac{t}{1-\alpha \eta} \int_0^1 (1-s)a(s)f(H_4)ds
\]
\[
\leq \frac{1}{1-\alpha \eta} \int_0^1 (1-s)a(s)H_4 ds
\]
\[
\leq H_4.
\]
Therefore, in either case we may put
\[
\Omega_4 = \{ y \in C[0, 1] \mid \|y\| < H_4 \},
\]
and for \( y \in K \cap \partial \Omega_4 \) we may have \( \|Ay\| \leq \|y\| \). By the second part of the Fixed Point Theorem, it follows that BVP (1.1)-(1.2) has a positive solution. Therefore, we have completed the proof of Theorem 1.
REFERENCES


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