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Infinitely many homoclinic orbits for Hamiltonian systems with group symmetries *

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Abstract

This paper deals via variational methods with the existence of infinitely many homoclinic orbits for a class of the first-order time-dependent Hamiltonian systems

$$\dot{z} = JH_z(t,z)$$

without any periodicity assumption on H, providing that $H(t,z)$ is G-symmetric with respect to $z \in \mathbb{R}^{2N}$, is superquadratic as $|z| \to \infty$, and satisfies some additional assumptions.

1 Introduction

This paper is an extension of the work [7]. We consider the existence of infinitely many homoclinic orbits for the first-order time-dependent Hamiltonian systems

$$\dot{z} = JH_z(t,z), \tag{HS}$$

where $z = (p,q) \in \mathbb{R}^{2N}$, $H \in C^1(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R})$, $H(t,0) \equiv 0$, and J is the standard symplectic structure on $\mathbb{R}^{2N}$,

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$$

with I_N being the $N \times N$ identity matrix. By a homoclinic orbit we mean a solution $z \in C^1(\mathbb{R},\mathbb{R}^{2N})$ of (HS) which satisfies $z(t) \not\equiv 0$ and the asymptotic condition $z(t) \to 0$ as $|t| \to \infty$.

Establishing the existence of homoclinic orbits for systems like (HS) is a classical problem. Up to 1990, apart from a few isolated results, the only method for dealing with such a problem was the small-perturbation technique of Melnikov. In very recent years this kind of problem has been deeply investigated through variational methods pioneered by Rabinowitz, Coti-Zelati, Ekeland, Séré, Hofer,

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Wysocki and others, see [2,4-6,8,11-12,14-18]. These papers considered Hamiltonians for the first-order systems (HS) of the form

$$H(t,z)=\frac{1}{2}Az\cdot z+R(t,z),$$

where A is a $2N\times 2N$ symmetric and constant matrix such that each eigenvalue of JA has a nonzero real part, and $R(t,z)$ is periodic in t and globally superquadratic in z. They showed that (HS) has at least one homoclinic orbit. The existence of infinitely many homoclinic orbits of (HS) was also established in [16,17] if, in addition, $R(t,z)$ is convex in z.

Recall that, for the particular case of second order systems of the type

$$-\ddot{q}=-L(t)q+W_q(t,q),$$

where $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function, the works [15] (among other results) and [6,12] obtained some existence results for homoclinic orbits without periodicity assumptions on the Hamiltonian

$$H(t,p,q)=\frac{1}{2}|p|^2-\frac{1}{2}L(t)q\cdot q+W(t,q)\quad (p,q)\in\mathbb{R}^{2N},$$

providing instead that the smallest eigenvalue of $L(t)$ grows without bound as $|t|\to\infty$, and $W(t,q)$ satisfies some growth assumptions.

Motivated by the works of [6,12,15] Ding and Li studied in [9] the Hamiltonian

$$H(t,z)=-\frac{1}{2}M(t)z\cdot z+R(t,z), \tag{1.1}$$

where

$$M(t)=\begin{pmatrix} 0 & L(t)\\ L(t) & 0\end{pmatrix},$$

with L being an $N\times N$ symmetric matrix-valued function. They proved that (HS) has at least one homoclinic orbit under the assumptions:

(L_1) The smallest eigenvalue of $L(t)$ approaches ∞ as $|t|\to\infty$, i.e., $l(t)\equiv\inf_{\xi\in\mathbb{R}^N,|\xi|=1}L(t)\xi\cdot\xi\to\infty$ as $|t|\to\infty$;

(L_2) $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ and there exists $T_0>0$ such that $2L(t)\underline{\pm}\frac{d}{dt}L(t)$ are nonnegative definite for all $|t|\geq T_0$;

(R_1) $R\in C^1(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R})$ and there exists $\mu>2$ such that

$$0<\mu R(t,z)\leq R_z(t,z)\cdot z\quad \forall t\in\mathbb{R}\text{ and }z\neq 0;$$

(R_2) $0<\underline{b}=\inf_{t\in\mathbb{R},|z|=1}R(t,z)$;

(R_3) $|R_z(t,z)|=o(|z|)$ as $z\to 0$ uniformly in t;

(R_4) there exist $0\leq a_1(t)\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, $\gamma>1$ and $a_2>0$ such that

$$|R_z(t,z)|^\gamma\leq a_1(t)+a_2R_z(t,z)\cdot z\quad \forall(t,z).$$

In [7], Ding showed that (HS) possesses infinitely many homoclinic orbits if, in addition, $H(t,z)$ is even in z. The purpose of this paper is to show the same conclusion under a general symmetry condition. Our arguments remain simple even under this general symmetry condition.

To state our result, we recall some standard notations concerning group actions of compact subgroups G of the orthogonal group $\mathcal{O}(2N)$. We let V denote the vector space $\mathbb{R}^{2N}$ considered as a G-space. Hence G acts diagonally on $V^k = (\mathbb{R}^{2N})^k$, i.e., $g(v^1, \cdots, v^k) = (gv^1, \cdots, gv^k)$ for $g \in G$ and $v^i \in V$ ($k \in \mathbb{N}, i = 1, 2, \cdots, k$). If G acts on two subspaces X and Y, then a G-map $f : X \to Y$ is a continuous map which commutes with the action, i.e., $f(gx) = gf(x)$ for any $g \in G$ and $x \in X$. In the special case where the action on Y is trivial ($gy = y$ for all $g \in G$ and $y \in Y$) a G-map is also called invariant. A subset A of V^k is said to be invariant if $gx \in A$ for every $g \in G$ and $x \in A$. We say that G acts admissibly on V if every G-map $\overline{\mathcal{O}} \to V^{k-1}$, $\mathcal{O} \subseteq V^k$ an open bounded invariant neighborhood of 0 in V^k, has a zero on $\partial\mathcal{O}$.

Now we can state the symmetry condition.

(S) There exists a compact subgroup G of $\mathcal{O}(2N)$ acting admissibly on V such that $g^tJg = J$ for every $g \in G$ and $H(t, z)$ is invariant with respect to the action, i.e., $H(t, gz) = H(t, z)$ for all $g \in G$ and $(t, z) \in \mathbb{R} \times \mathbb{R}^{2N}$.

Our result reads as follows.

Theorem 1. *Let H be of the form* (1.1) *with L satisfying* $(L_1) - (L_2)$ *and R satisfying* $(R_1) - (R_4)$. *Suppose, in addition, H satisfies* (S). *Then* (HS) *possesses infinitely many homoclinic orbits $\{z_k\}$ such that*

$$\int_{\mathbb{R}} [-\frac{1}{2} J\dot{z}_k \cdot z_k - H(t, z_k)]dt \to \infty \text{ as } k \to \infty.$$

The Borsuk-Ulam theorem states that $V = \mathbb{R}^{2N}$ with the antipodal action of $G = \{I_{2N}, -I_{2N}\}$ is admissible. So our result generalizes the result in [7].

A simple example of a matrix-valued function satisfying $(L_1)-(L_2)$ is $L(t) = |t|^\theta I_N$ with $\theta > 1$, which arises in the study of generalized harmonic oscillator problems. Consider the functions of the form $R(t, z) = b(t)W(z)$, where $b(t) \in C(\mathbb{R}, \mathbb{R})$, there exist positive constants $\underline{b} \leq \overline{b}$ such that $\underline{b} \leq b(t) \leq \overline{b}$ for all $t \in \mathbb{R}$, and for some integer $m > 0$, $W(z) = \sum_{i=1}^m c_i |z|^{\mu i}$ with $c_i > 0$ $(1 \leq i \leq m)$ and $1 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$. If $\mu > 2$, then $R(t, z)$ satisfies $(R_1) - (R_4)$.

The preliminary results are in Sec.2, and in Sec.3 is the proof of Theorem 1.

2 Preliminaries

An abstract critical point theorem will be used for proving Theorem 1. This abstract theorem is introduced and proved in [3]. So we shall describe it briefly. For details see [3].

Let E be a Hilbert space with an orthogonal action of a compact Lie group G. We are concerned with critical points of an invariant functional $I \in C^1(E, \mathbb{R})$. We need the following assumptions:

(A_1) There exists an admissible representation V of G such that $E = \oplus_{j\in\mathbb{Z}} E^j$ is a G-Hilbert space with $E^j \cong V$ as a representation of G for every $j \in \mathbb{Z}$ (note that $\mathbb{Z}$ can be replaced by $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$, depending on situations).

(A_2) There exists $a \in \mathbb{R}$ such that for each $k \geq 1$

$$\inf_{R>0} \sup_{u\in E_k, \|u\|\geq R} I(u) = \lim_{R\to+\infty} \sup_{u\in E_k, \|u\|\geq R} I(u) < a,$$

where $E_k = \oplus_{j\leq k} E^j$.

(A_3) $b_k = \sup_{r>0} \inf_{u\in E_{k-1}^{\perp}, \|u\|=r} I(u) \to \infty$ as $k \to \infty$.

(A_4) $d_k = \sup_{u\in E_k} I(u) < \infty$.

(A_5) Every sequence $u_n \in F_n = E_{-n-1}^{\perp} = \oplus_{j\geq -n} E^j$ such that $I(u_n) \geq a$ is bounded and $(I|_{F_n})'(u_n) \to 0$ as $n \to \infty$, contains a subsequence which converges in E to a critical point of I, which is the so-called $(\mathrm{PS})^*$ condition. Now we can state the abstract theorem.

Theorem 2.1. *Let E be a G-Hilbert space and $I \in C^1(E, \mathbb{R})$ be a G-invariant functional satisfying $(A_1) - (A_5)$. Then I has an unbounded sequence of critical values. In fact, for each $k \geq 1$ with $b_k > a$ there exists a critical value $c_k \in [b_k, d_k]$.*

Remark 2.2: In [7], an abstract critical point proposition for even functionals is posed to prove its main result. The proposition requires I to satisfy both $(\mathrm{PS})^*$ and $(\mathrm{PS})^{**}$ conditions.

Remark 2.3: The above conditions (A_2) and (A_3) show that the behavior of I is quite interesting. Intuitively, I behaves like fountain (see [3]).

Next we consider the symmetric matrix-valued functions $M \in C(\mathbb{R}, \mathbb{R}^{2N\times 2N})$ of the form

$$M(t) = \begin{pmatrix} 0 & L(t) \\ L(t) & 0 \end{pmatrix}.$$

Suppose that L satisfies (L_1) and (L_2). Let A be the selfadjoint operator $-J\frac{d}{dt} + M$ with the domain $D(A) \subseteq L^2 \equiv L^2(\mathbb{R}, \mathbb{R}^{2N})$, defined as a sum of quadratic forms. Let $\{E(\lambda) | -\infty < \lambda < +\infty\}$ be the resolution of A, and $U = I - E(0) - E(-0)$. Then U commutes with A, $|A|$ and $|A|^{1/2}$, and $A = |A|U$ is the polar decomposition of A (see [10]). $D(A) = D(|A|) = D(I + |A|)$ is a Hilbert space equipped with the norm

$$\|z\|_1 = \|(I + |A|)z\|_{L^2} \text{ for all } z \in D(A),$$

where $\|\cdot\|_{L^2}$ is the norm of L^2. It is not hard to check that $D(A)$ is continuously embedded in $W^{1,2} \equiv W^{1,2}(\mathbb{R}, \mathbb{R}^{2N})$ (see [9]). Moreover we have

Lemma 2.4: *Suppose L satisfies (L_1) and (L_2). Then $D(A)$ is compactly embedded in L^2.*

For the proof of the above lemma, see [9, Lemma 2.1].

Remark 2.5: From Lemma 2.4, it is clear that $(I+|A|)^{-1} : L^2 \to L^2$ is a compact linear operator. Therefore a standard argument shows that $\sigma(A)$, the spectrum of A, consists of eigenvalues numbered by (counted in their multiplicities):

$$\cdots \le \lambda_{-2} \le \lambda_{-1} \le 0 < \lambda_1 \le \lambda_2 \le \cdots$$

with $\lambda_{\pm k} \to \pm\infty$ as $k \to \infty$, and a corresponding system of eigenfunctions $\{e_k\}_{k\in\mathbb{Z}^*}$ of A forms an orthonormal basis in L^2 (for the situation here, we use $\mathbb{Z}^*$, instead of $\mathbb{Z}$).

Now we set $E = D(|A|^{1/2}) = D((I+|A|)^{1/2})$. E is a Hilbert space under the inner product

$$(z_1, z_2)_0 = (|A|^{1/2}z_1, |A|^{1/2}z_2)_{L^2} + (z_1, z_2)_{L^2}$$

and norm

$$\|z\|_0 = (z,z)_0^{1/2} = \|(I+|A|)^{1/2}z\|_{L^2},$$

where $(\cdot,\cdot)_{L^2}$ denotes the L^2 inner product.

Let $E^0 = \ker A$ (note $\dim E^0 < \infty$, by Lemma 2.4), $E^+ = \mathrm{Cl}_E$ (span $\{e_1, e_2, \cdots\}$) and $E^- = (E^0 \oplus E^+)^{\perp_E}$, where $\mathrm{Cl}_E S$ denotes the closure of S in E and $S^{\perp_E}$ denotes the orthogonal complementary subspace of S in E. Then

$$E = E^- \oplus E^0 \oplus E^+. \tag{1.1}$$

Since, by Lemma 2.4, 0 is at most an isolated eigenvalue of A, for later convenience we introduce on E the inner product

$$(z_1, z_2) = (|A|^{1/2}z_1, |A|^{1/2}z_2)_{L^2} + (z_1^0, z_2^0)_{L^2}$$

for all $z_i = z_i^- + z_i^0 + z_i^+ \in E^- \oplus E^0 \oplus E^+ (i = 1, 2)$, and the norm

$$\|z\| = (z,z)^{1/2} \tag{1.2}$$

for all $z \in E$. Clearly, $\|\cdot\|$ is equivalent to $\|\cdot\|_0$. Moreover, E is continuously embedded in $H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$, the Sobolev space of fractional order (see [9]).

Lemma 2.6: *Suppose L satisfies (L_1) and (L_2). Then E is compactly embedded in L^p for all $p \in [2,\infty)$.*

For the proof of the above lemma, see [9, Lemma 2.2].

Finally we introduce

$$a(z,x) = (|A|^{1/2}Uz, |A|^{1/2}x)_{L^2} \tag{1.3}$$

for all $z, x \in E$. The form $a(\cdot,\cdot)$ is the quadratic form associated with A. Clearly, for $z \in D(A)$ and $x \in E$ we have

$$a(z,x) = (Az,x)_{L^2} = \int_{\mathbb{R}} (-J\dot{z} + M(t)z)\cdot x. \tag{1.4}$$

Clearly, E^-, E^0 and E^+ are orthogonal to each other with respect to $a(\cdot,\cdot)$, and furthermore

$$\begin{aligned} a(z,x) &= ((P^+ - P^-)z, x) \quad \text{for } z, x \in E\,, \\ a(z,z) &= \|z^+\|^2 - \|z^-\|^2 \quad \text{for } z \in E, \end{aligned} \tag{1.5}$$

where $P^\pm : E \to E^\pm$ are the orthogonal projectors and $z = z^- + z^0 + z^+ \in E^- \oplus E^0 \oplus E^+$.

3 Proof of Theorem 1

Throughout this section, let the assumptions of Theorem 1 be satisfied. Let $E = D(|A|^{1/2})$ with norm (2.2). By (R_1) and (R_2) we have

$$R(t,z) \ge \underline{b}|z|^\mu \quad \forall t \in \mathbb{R} \quad \text{and} \quad |z| \ge 1. \tag{1.1}$$

Also by (R_4) and (3.1) we have

$$|R_z(t,z)| \le C(1 + |z|^{\gamma'-1}) \quad \forall (t,z), \tag{1.2}$$

where $\gamma' = \frac{\gamma}{\gamma-1}$, which, together with ($\mathrm{R}_3$), yields that for any $\varepsilon > 0$ there is $C_\epsilon > 0$ such that

$$|R_z(t,z)| \le \varepsilon|z| + C_\varepsilon |z|^{\gamma'-1} \ \ \forall (t,z), \tag{1.3}$$

and

$$|R(t,z)| \le \varepsilon|z|^2 + C_\varepsilon |z|^{\gamma'} \ \ \forall (t,z). \tag{1.4}$$

Subsequently, C and C_i stand for generic positive constants, not depending on t and z.
Note that (3.1) and (3.4) imply $\gamma' \ge \mu > 2$.

Let

$$\varphi(z) = \int_{\mathbb{R}} R(t,z) \quad \forall z \in E.$$

Equations (3.1)-(3.4) imply that φ is well-defined, $\varphi \in C^1(E,\mathbb{R})$, and

$$\varphi'(z)x = \int_{\mathbb{R}} R_z(t,z)x \quad \forall x, z \in E \tag{1.5}$$

by Lemma 2.6. In addition, φ' is a compact map. To see this, let $z_n \to z$ weakly in E. By Lemma 2.6 we can assume that $z_n \to z$ strongly in L^p for $p \in [2,\infty)$. By (3.5)

$$\|\varphi'(z_n) - \varphi'(z)\| = \sup_{\|x\|=1} \Big| \int_{\mathbb{R}} (R_z(t,z_n) - R_z(t,z))x \Big|.$$

By (3.3) and the Hölder inequality, for any $R > 0$

$$\Big| \int_{|t|\ge R} (R_z(t,z_n) - R_z(t,z))x \Big|$$

$$\begin{aligned} &\le C\int_{|t|\ge R}(|z_n|+|z|+|z_n|^{\gamma'-1}+|z|^{\gamma'-1})|x| \qquad (1.6)\\ &\le C[\|x\|_{L^2}(\int_{|t|\ge R}|z_n|^2+|z|^2)^{1/2}+\|x\|_{L^{\gamma'}}(\int_{|t|\ge R}|z_n|^{\gamma'}+|z|^{\gamma'})^{(\gamma'-1)/\gamma'}]. \end{aligned}$$

For $\varepsilon>0$, by (3.6) we can take R_0 so large that

$$|\int_{|t|\ge R_0}(R_z(t,z_n)-R_z(t,z))x|<\varepsilon/2 \qquad (1.7)$$

for all $\|x\|=1$ and $n\in\mathbb{N}$. On the other hand, it is well-known (see [13]) that since $z_n\to z$ strongly in L^2,

$$\|R_z(\cdot,z_n)-R_z(\cdot,z)\|_{L^2(B_{R_0})}\to 0$$

as $n\to\infty$, where $B_{R_0}=(-R_0,R_0)$. Therefore, there is $n_0\in\mathbb{N}$ such that

$$|\int_{|t|\le R_0}(R_z(t,z_n)-R_z(t,z))x|<\varepsilon/2 \qquad (1.8)$$

for all $\|x\|=1$ and $n\ge n_0$. Combining (3.7) and (3.8) yields

$$\|\varphi'(z_n)-\varphi'(z)\|<\varepsilon \quad \forall n\ge n_0.$$

Hence φ' is compact.

Let $a(\cdot,\cdot)$ be the quadratic form given by (2.3), and define

$$I(z)=\frac{1}{2}a(z,z)-\varphi(z) \quad \forall z\in E.$$

By (2.5)

$$I(z)=\frac{1}{2}(\|z^+\|^2-\|z^-\|^2)-\varphi(z) \quad \forall z\in E$$

for all $z=z^-+z^0+z^+\in E^-\oplus E^0\oplus E^+$. Then $I\in C^1(E,\mathbb{R})$. Note that by (2.4) a standard argument can show that the nontrivial critical points of I on E are homoclinic orbits of (HS).

Let $\hat{E}_1=E^-\oplus E^0$ and $\hat{E}_2=E^+$ with $\{e_{-n}\}_{n=1}^\infty$ and $\{e_n\}_{n=1}^\infty$ respectively, where $\{e_n\}_{n\in\mathbb{Z}^*}$ is the system of eigenfunctions of A (see Remark 2.5). Then

$$E=\hat{E}_1\oplus\hat{E}_2=\oplus_{j\in\mathbb{Z}^*}E^j\,,$$

where $E^1=\text{span}\{e_1,e_2,\cdots,e_{2N}\}$, $E^2=\text{span}\{e_{2N+1},\cdots,e_{4N}\},\cdots$; $E^{-1}=\text{span}\{e_{-1},e_{-2},\cdots,e_{-2N}\}$, $E^{-2}=\text{span}\{e_{-2N-1},\cdots,e_{-4N}\},\cdots$. Set also $E_n=\oplus_{j\le n}E^j$ and $F_n=E^{\perp}_{-n-1}=\oplus_{j\ge -n}E^j$ for $j,n\in\mathbb{Z}^*$. It remains to check the assumptions of Theorem 2.1. The action of G on E is simply given by $(gz)(t)=gz(t)$. Since g commutes with J and $H(t,z)$ is invariant with respect to the action, it is clear that (A_1) is satisfied. Assumption (A_2) follows from

Lemma 3.1. *For each $k \geq 1$ there exists $R_k > 0$ such that $I(z) < 0$ for all $z \in E_k$ with $\|z\| \geq R_k$.*

Proof. By (3.4), (R_1) and the fact that $|z|^\mu \leq |z|^2$ for $|z| \leq 1$, we have for any ε with $0 < \varepsilon \leq \underline{b}$,

$$R(t,z) \geq \varepsilon(|z|^\mu - |z|^2) \quad \forall (t,z). \tag{1.9}$$

Let $d > 0$ be such that $\|z\|_{L^2}^2 \leq d\|z\|^2$ for all $z \in E$ (by Lemma 2.6) and take $\varepsilon = \min\{\frac{1}{4d}, \underline{b}\}$. Then by (3.9) for $z = z^- + z^0 + z^+ \in E_k$ we have

$$\begin{aligned} I(z) &= \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \int_{\mathbb{R}} R(t,z) \\ &\leq \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 + \varepsilon\|z\|_{L^2}^2 - \varepsilon\|z\|_{L^\mu}^\mu \\ &\leq \|z^+\|^2 - \frac{1}{4}\|z^-\|^2 + \frac{1}{4}\|z^0\|^2 - \varepsilon\|z\|_{L^\mu}^\mu. \end{aligned} \tag{1.10}$$

Since $\dim[E^0 \oplus (\oplus_{0<j\leq k} E^j)] < \infty$, we have

$$\begin{aligned} \|z^0 + z^+\|_{L^2}^2 &= (z_0 + z^+, z)_{L^2} \\ &\leq \|z^0 + z^+\|_{L^{\mu'}} \|z\|_{L^\mu} \\ &\leq C(k)\|z^0 + z^+\|_{L^2}\|z\|_{L^\mu}, \end{aligned}$$

and so $\|z^0 + z^+\| \leq C'(k)\|z\|_{L^\mu}$ or

$$C''(k)\|z^0 + z^+\|^\mu \leq \|z\|_{L^\mu}^\mu, \tag{1.11}$$

where $C(k), C'(k)$ and $C''(k) > 0$ depend on k but not on $z \in E_k$. Equations (3.10) and (3.11) imply

$$I(z) \leq \|z^0 + z^+\|^2 - \frac{1}{4}\|z^-\|^2 - \varepsilon C''(k)\|z^0 + z^+\|^\mu \tag{1.12}$$

for all $z \in E_k$. Equation (3.12) implies that there is $R_k > 0$ such that

$$I(z) < 0 \quad \forall z \in E_k \text{ with } \|z\| \geq R_k.$$

$\diamondsuit$

Note that the above estimate (3.12) also gives $\sup_{z \in E_k} I(z) < \infty$, that is, ($A_4$) holds. Next, ($A_3$) is a consequence of

Lemma 3.2. *There are $r_k > 0$, $a_k > 0$ $(k \geq 1)$ with $a_k \to \infty$ as $k \to \infty$ such that*

$$I(z) \geq a_k \quad \forall z \in E_{k-1}^\perp \text{ with } \|z\| = r_k.$$

Proof. Define

$$\eta_k = \sup_{z \in E_k^{\perp} \setminus \{0\}} \frac{\|z\|_{L^{\gamma'}}}{\|z\|}.$$

Clearly $\eta_k \geq \eta_{k+1} > 0$. We claim that

$$\eta_k \to 0 \quad \text{as} \quad k \to \infty. \tag{1.13}$$

Suppose $\eta_k \to \eta > 0$. Then there is a sequence $z_k \in E_k^{\perp}$ with $\|z_k\| = 1$ and $\|z_k\|_{L^{\gamma^1}} \geq \frac{\eta}{2}$. since $(z_k, e_n) \to 0$ as $k \to \infty$ for each e_n ($n \in \mathbb{Z}^*$), $z_k \to 0$ weakly in E and by Lemma 2.6, $\|z_k\|_{L^{\gamma'}} \to 0$, a contradiction. The claim (3.13) is proved.

By (3.4) with $\varepsilon = \frac{1}{4d}$ (d as in the proof of Lemma 3.1) and $C = C_\varepsilon$ we have, for $z \in E_{k-1}^{\perp}$

$$\begin{aligned} I(z) &= \frac{1}{2}\|z\|^2 - \int_{\mathbb{R}} R(t,z) \\ &\geq \frac{1}{4}\|z\|^2 - C\|z\|_{L^{\gamma'}}^{\gamma'} \\ &\geq \frac{1}{4}\|z\|^2 - C\eta_{k-1}^{\gamma'}\|z\|^{\gamma'}. \end{aligned}$$

Taking $r_k = (2\gamma' C\eta_{k-1}^{\gamma'})^{\frac{-1}{\gamma'-2}}$ and $a_k = (\frac{1}{4} - \frac{1}{2\gamma'})r_k^2$ one obtains

$$I(z) \geq a_k \quad \forall z \in E_{k-1}^{\perp} \text{ with } \|z\| = r_k.$$

Since $\gamma' > 2$, equation (3.13) shows that $a_k \to \infty$ as $k \to \infty$. ♢

Lemma 3.3 *I satisfies (A_5).*

Proof. Let $I_n = I|_{F_n}$. Suppose $z_n \in F_n$ such that $0 \leq I(z_n) \leq C$ and $\varepsilon_n = \|I_n'(z_n)\| \to 0$. By definition and ($\mathrm{R}_1$)

$$\begin{aligned} I(z_n) - \frac{1}{2}I_n'(z_n)z_n &= \int_{\mathbb{R}} (\frac{1}{2}R_z(t,z_n)z_n - R(t,z_n)) \\ &\geq (\frac{1}{2} - \frac{1}{\mu}) \int_{\mathbb{R}} R_z(t,z_n)z_n \\ &\geq (\frac{\mu}{2} - 1) \int_{\mathbb{R}} R(t,z_n). \end{aligned} \tag{1.14}$$

Equation (3.14) and hypothesis (R_4) give $\|R_z(t,z_n)\|_{L^\gamma}^{\gamma} \leq C(1 + \|z_n\|)$, and hence by Lemma 2.6,

$$\begin{aligned} \|z_n^+\|^2 &= I'(z_n)z_n^+ + \int_{\mathbb{R}} R_z(t,z_n)z_n^+ \\ &\leq C\|z_n^+\|(1 + \|R_z(t,z_n)\|_{L^\gamma}). \end{aligned}$$

Thus

$$\|z_n^+\| \le C(1+\|z_n\|^{1/\gamma}). \tag{1.15}$$

Similarly we have

$$\|z_n^-\| \le C(1+\|z_n\|^{1/\gamma}). \tag{1.16}$$

If $E^0=\{0\}$, (3.15) and (3.16) imply $\|z_n\| \le \text{Const } \forall n$. Suppose $E^0 \ne \{0\}$. For $z \in E$, let

$$z^1(t)=\begin{cases} z(t) & \text{if} \quad |z(t)|<1, \\ 0 & \text{if} \quad |z(t)|\ge 1, \end{cases} \quad z^2(t)=\begin{cases} 0 & \text{if} \quad |z(t)|<1, \\ z(t) & \text{if} \quad |z(t)|\ge 1. \end{cases}$$

Since by Lemma 2.6

$$\int_{\mathbb{R}} |z_n^1|^\mu \le \int_{\mathbb{R}} |z_n^1|^2 \le \int_{\mathbb{R}} |z_n|^2 \le C\|z_n\|^2,$$

we have

$$\|z_n^1\|_{L^\mu} \le C\|z_n\|^{2/\mu}. \tag{1.17}$$

By (3.1) and (3.14),

$$\|z_n^2\|_{L^\mu} \le C(1+\|z_n\|^{1/\mu}). \tag{1.18}$$

By L^2 orthogonality and Hölder's inequality with $\mu'=\frac{\mu}{\mu-1}$,

$$\begin{aligned} \|z_n^0\|_{L^2}^2 &= (z_n^0, z_n)_{L^2} \\ &\le \|z_n^0\|_{L^{\mu'}}(\|z_n^1\|_{L^\mu}+\|z_n^2\|_{L^\mu}). \end{aligned}$$

Hence since $\dim E^0<\infty$ and (3.17)-(3.18) hold, one sees

$$\|z_n^0\|_{L^\mu} \le C(\|z_n\|^{2/\mu}+\|z_n\|^{1/\mu}). \tag{1.19}$$

The combination of (3.15)-(3.16) and (3.19) shows that again $\|z_n\| \le$ Const. Finally since φ' is compact, a standard argument shows that $\{z_n\}$ has a convergent subsequence. $\diamondsuit$

Proof of Theorem 1. What we have done so far shows that I satisfies all the assumptions of Theorem 2.1. Hence I has a positive critical value sequence $\{c_k\}$ with $c_k \to \infty$. Let z_k be the critical point of I such that $I(z_k)=c_k$. Then z_k is a homoclinic orbit of (HS) and

$$\int_{\mathbb{R}} (-\frac{1}{2}J\dot{z}_k\cdot z_k - H(t,z_k))dt = I(z_k)=c_k \to \infty$$

as $k\to\infty$. $\diamondsuit$

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References

[1] Ambrosetti, A. & Rabinowitz, P. H., *Dual variational methods in critical point theory and applications*, J. Funct. Anal., 14 (1973), 349-381.

[2] Ambrosetti, A. & Coti-Zelati, V., *Multiple homoclinic orbits for a class of conservative systems*, C. R. Acad. Sci. Paris, 314 (1992), 601-604.

[3] Bartsch, T. & Willem, M., *Periodic solutions of non-autonomous Hamiltonian systems with symmetries*, J. Reine. Angew. Math. 451 (1994), 149-159.

[4] Coti-Zelati, V., Ekland, I. & Séré, E., *A variational approach to homoclinic orbits in Hamiltonian systems*, Math. Ann. 288 (1990), 133-160.

[5] Coti-Zelati, V. & Rabinowitz, P. H. *Homoclinic orbits for a second order Hamiltonian systems possessing superquadratic potentials*, J. Am. Math. Soc., 4 (1991), 693-727.

[6] Ding, Y., *Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems*, Nonlinear Analysis T.M.A., 25 (1995), 1095-1113.

[7] Ding, Y., *Infinitely many Homoclinic orbits for a class of Hamiltonian systems with symmetry*, Chin. Ann. of Math. 19B, 22 (1998), 167-178.

[8] Ding, Y. & Giradi, M., *Periodic and homoclinic solutions to a class of Hamiltonian systems with potentials changing sign*, Dyn. Sys. and Appl., 2 (1993), 131-145.

[9] Ding, Y. & Li, S., *Homoclinic orbits for the first-order Hamiltonian systems*, J. Math. Anal. and Appl., 189 (1995), 585-601.

[10] Edmunds, D. E. & Evans, W. D., *Spectral theory and differential operators*, Clarenden Press, Oxford, 1987.

[11] Hofer, H. & Wysocki, K., *First-order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems*, Math. Ann., 288 (1990), 483-503.

[12] Omana, W. & Willem, M., *Homoclinic orbits for a class of Hamiltonian systems*, Diff. and Int. Eq., 5 (1992), 1115-1120.

[13] Rabinowitz, P. H., *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. in Math., 65, A.M.S. Providence, 1986.

[14] Rabinowitz, P. H., *Homoclinic orbits for a class of Hamiltonian systems*, Proc. Royal. Soc. Edinburgh, 114A (1990), 33-38.

[15] Rabinowitz, P. H., & Tanaka, K., *Some results on connecting orbits for a class of Hamiltonian systems*, Math. Z., 206 (1991), 473-499.

[16] Séré, E., *Existence of infinitely many homoclinic orbits in Hamiltonian systems*, Math. Z., 209 (1992), 27-42.

[17] Séré, E., *Looking for the Bernoulli shift*, Ann. Inst. H. Poincaré Anal. Non. Lineaire 10, no. 5, (1993), 561-590.

[18] Tanaka, K., *Homoclinic orbits in a first-order superquadratic Hamiltonian system: convergence of subharmonic orbits*, J. Diff. Eq., 94 (1991), 315-339.

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