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A minmax problem for parabolic systems with competitive interactions *

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Abstract

In this paper we model the evolution and interaction between two competing populations as a system of parabolic partial differential equations. The interaction between the two populations is quantified by the presence of non-local terms in the system of equations. We model the whole system as a two-person zero-sum game where the gains accrued by one population necessarily translate into the others loss.

For a suitably chosen objective functional(pay-off) we establish and characterize the saddle point of the game. The controls(strategies) are kernels of the interaction terms.

1 Introduction

In 1914 Lancaster [4] proposed the first analytic model for describing combat between two forces. The Lancaster model, a coupled system of ordinary differential equations, fails to account for the spatial movement of opposing forces on the battlefield. To overcome this obvious shortcoming and include spatial dependence, Protopopescu et al. [9] introduced a more general model, namely, they replaced the system of ordinary differential equations with a system of **semilinear parabolic equations with competitive interactions.** These systems belong to the class of reaction-diffusion systems which lately have become one of the mainstream fields in pure, applied and numerical PDE's [3]. This competition model can be used to represent other phenomenon, e.g., densities of competing biological populations and concentration of chemical reactants.

In this paper we consider competitive systems as a two-person zero-sum game which implies that one player's gains necessarily translate to the other players's losses. Each player controls some of the game parameters which the player can manipulate to navigate the evolution of the system towards a desired state. A quantitative measurement of the players performance is modeled in terms of a minimizing(respectively maximizing) functional which depends upon the state of the system and the controls.

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We will show that under suitable conditions the game admits a unique saddle point and then we will characterize the saddle point of the game as a solution of an optimality system. The optimality system will consist of the parabolic state equations coupled with two adjoint equations. The controls will be the attrition kernels of the non-local interaction terms. The attrition kernels represent the effect of "weapons" in the combat model case. In general, the interactions between the two populations are non-local and the kernels measure the range over which one population can affect the other population.

Lenhart et al. [8] considered the steady state case with the operator $-\Delta$ and Lenhart et al. [5] have also considered the parabolic case where the controls are the source terms.

The outline of this part of the paper is the following. In the next section we give the statement of the problem. The payoff(cost) functional is defined and for given controls, the unique solution of the state system is constructed. The existence of the unique saddle point is established in Section 3. In Section 4 the optimality system is defined and the saddle point is represented as the solution of this optimality system.

2 Statement of the Problem

Throughout this part of the paper, C denotes generic constants, unless otherwise indicated.

Let Ω be a bounded domain in $\mathbf{R}^{\mathbf{m}}$ with $\partial\Omega \in C^{1,1}$, let $T > 0$ and $Q = \Omega \times (0,T)$. For $\Gamma > 0$, define the control set,

$$C_\Gamma = \{c \in L^\infty_+(Q \times Q) \mid \|c\| \leq \Gamma\}.$$

where L^∞_+ is the set of positive L^∞ functions.

For any $c, d \in C_\Gamma$, let the pair $(u, v) = (u(c,d), v(c,d))$ denote the solution of the *state system*

$$\begin{aligned} L_1 u(x,t) &= f(x,t) - u(x,t)\int_Q c(x,y,t,\tau)v(y,\tau)\,dyd\tau \text{ on } Q \\ L_2 v(x,t) &= g(x,t) - v(x,t)\int_Q d(x,y,t,\tau)u(y,\tau)\,dyd\,\tau \\ & u = u^0\,,\ v = v^0 \text{ on } \Omega \times \{0\} \\ & u = 0\,,\ v = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \end{aligned} \tag{2.1}$$

where $u, v \in L^2(0,T,H^1_0(\Omega)) = V$, and

$$L_k u = u_t - (a^k_{ij} u_{x_i})_{x_j} + b^k_i u_{x_i} + c^k u,\ k = 1, 2\,.$$

Here we have used the summation convention with respect to repeated indices.

The solutions u, v represent the concentration of the two competing populations. The sources f, g are given and the attrition kernels c, d are the controls. The first player controls d with the purpose of maximizing $\mathcal{J}$ (the payoff); the second player controls c to minimize $\mathcal{J}$ (the cost). Given two target functions, $\tilde{u}$, $\tilde{v} \in L^2$, and $K, L \geq 0$ and $M, N > 0$, the *payoff (cost) functional* $\mathcal{J}$ is defined by

$$\begin{aligned} \mathcal{J}(c,d) &= \frac{1}{2}\int_Q \left\{K[u(c,d)-\tilde{u}]^2 - L[v(c,d)-\tilde{v}]^2\right\} dx\,dt \\ &\quad +\frac{1}{2}\int_Q\int_Q (Nc^2 - Md^2)\,dx\,dt\,dy\,d\tau. \end{aligned} \tag{2.2}$$

The *saddle point* (c^*, d^*) (if it exists) is defined as a pair of *strategies* $(c^*, d^*) \in C_\Gamma \times C_\Gamma$, such that

$$\mathcal{J}(c^*, d^*) = \sup_{d\in C_\Gamma} \mathcal{J}(c^*, d) = \inf_{c\in C_\Gamma} \mathcal{J}(c, d^*)$$

We make the following assumptions:

$$a^k_{ij}, b^k_i, c^k \in L^\infty(Q), k = 1, 2,\ i, j = 1, ..., m \tag{2.3}$$

$$\theta\zeta_i zeta_i \leq a^k_{ij}\zeta_i\zeta_j \leq \theta^{-1}\zeta_i\zeta_i,\ \theta > 0 \text{ for all } \boldsymbol{\zeta}\ \in \mathbf{R^m} \tag{2.4}$$

$$u^0, v^0 \in L^\infty_+(\Omega) \tag{2.5}$$

$$c^k(x,t) > c_0 > 0\,, k = 1, 2\,. \tag{2.6}$$

Finally to set up solutions in V, we define the following bilinear form on $H^1_0(\Omega)$:

$$a^k(t,\phi,\psi) = \int_\Omega a^k_{ij}\phi_{x_i}\psi_{x_j}\,dx + \int_\Omega b^k_i\phi_{x_i}\psi\,dx + \int_\Omega c^k\phi\psi\,dx \text{ for each t } \in (0,T). \tag{2.7}$$

Then for all $\phi, \psi \in L^2(0,T,H^1_0(\Omega))$, solutions (u,v) in $V \times V$ of (2.1) satisfy

$$\begin{aligned} &\int_0^T (\langle u_t,\psi\rangle + a^1(t,u,\psi))\,dt \\ &\quad = \int_0^T\int_\Omega (f - u(x,t)\int_Q c(x,y,t,\tau)v(y,\tau)\,dyd\tau)\psi\,dx\,dt \qquad (2.8) \\ &\int_0^T (\langle v_t,\phi\rangle + a^2(t,v,\phi))\,dt \\ &\quad = \int_0^T\int_\Omega (f - v(x,t)\int_Q d(x,y,t,\tau)u(y,\tau)\,dyd\,\tau)\phi\,dx\,dt \end{aligned}$$

where $\langle,\rangle$ denotes the duality between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$.

Using an iterative scheme we have the following existence result.

Proposition 2.1 (Existence of solutions of the state system). *Given $c, d \in C_\Gamma$, there exists a solution (u,v) of the state system (2.1) in $V \times V$ and there exists a constant $C_1 > 0$ such that $0 \leq u \leq C_1$, $0 \leq v \leq C_1$.*

Proof: Since the bilinear form $a^k(t,\phi,\psi)$ is continuous and coercive for all $\phi,\,\psi \in H_0^1(\Omega)$, we obtain, using standard linear theory [2], positive solutions u_0, $\tilde v$ in V to

$$\begin{aligned} L_1 u_0 &= f && \text{in } Q,\\ u_0 &= u^0 && \text{at } t=0,\\ u_0 &= 0 && \text{on } \Sigma, \end{aligned}$$

and

$$\begin{aligned} L_2 \tilde v &= g && \text{in } Q,\\ \tilde v &= v^0 && \text{at } t=0,\\ \tilde v &= 0 && \text{on } \Sigma. \end{aligned}$$

Solution $u_0, \tilde v$ will be the supersolutions for the iterates to be constructed. Also, by standard existence theory[2], there exists a constant $C_1 > 0$ such that $\|u_0\|_\infty, \|\tilde v\|_\infty \le C_1$.

Now let v_0 be the solution in V of

$$\begin{aligned} L_2 v_0 &= g - \tilde v \int_Q d(x,y,t,\tau) u_0 \, dy d\tau && \text{in } Q,\\ v_0 &= v^0 && \text{at } t=0,\\ v_0 &= 0 && \text{on } \Sigma. \end{aligned}$$

Let (u_k, v_k) be the solutions of the pair of linear Initial - Boundary value problem

$$\begin{aligned} &L_1 u_k + \mu u_k = f - u_{k-1}\int_Q c v_{k-1} + \mu u_{k-1} \\ &L_2 v_k + \mu v_k = g - v_{k-1}\int_Q d u_{k-1} + \mu v_{k-1} && \text{in } Q,\\ &u_k = u^0, v_k = v^0 && \text{at } t=0,\\ &u_k = 0, v_k = 0 && \text{on } \Sigma, \end{aligned} \tag{2.9}$$

where μ is a constant which makes the right hand side of the first equation in (2.9) an increasing function of u and the right hand side of the second equation an increasing function of v for iterates in the range $0 \le u_k, v_k \le C_1$. We have monotone convergence of the iterates,

$$u_k \searrow u, \quad v_k \nearrow v \text{ pointwise}, \quad 0 \le u_k \le C_1, \quad 0 \le v_k \le C_1\,,$$

through comparision results[5]. From *a priori* estimates of u_k, v_k from the system we get uniform bounds on $\|u_k\|_V$, $\|v_k\|_V$. Thus, u_k and v_k convege weakly to u and v in V. Now we show that u, v solve the state system in the sense of (2.8). The uniform bounds on u_k and v_k in V combined with the state equation give uniform bounds for $(u_k)_t$ and $(v_k)_t$ in $L^2(0,T,H^{-1}(\Omega))$. Using compactness results [7, Chapter 4, Prop. 4.2] implies that u_k, v_k converge strongly in $L^2(Q)$. Passing to the limit in the weak formulation of the system (2.9) we obtain $u = u(c,d)$ and $v = v(c,d)$ which solve the system (2.1). $\diamondsuit$

Proposition 2.2 (Uniqueness of solutions of the state system.) *For a fixed pair (c,d) in $[C_\Gamma]^2$ and for c_0 sufficiently large, the state system (2.1) admits a unique solution.*

Proof. Suppose (u,v) and $(\bar u,\bar v)$ solve the state system for given initial conditions u^0 and v^0. Using test functions $(u-\bar u)$, $(v-\bar v)$ and then subtracting the $(\bar u,\bar v)$ system from (u,v) system

$$\begin{aligned}&\int_Q [(u-\bar u)_t(u-\bar u)+a^1_{ij}(u-\bar u)_{x_i}(u-\bar u)_{x_j}+b^1_i(u-\bar u)_{x_i}(u-\bar u)+c^1(u-\bar u)^2\\&+\int_Q [(v-\bar v)_t(v-\bar v)+a^2_{ij}(v-\bar v)_{x_i}(v-\bar v)_{x_j}+b^2_i(v-\bar v)_{x_i}(v-\bar v)+c^2(v-\bar v)^2]=\\&-\int_Q u(u-\bar u)\int_Q cv+\int_Q \bar u(u-\bar u)\int_Q c\bar v-\int_Q v(v-\bar v)\int_Q du+\int_Q \bar v(v-\bar v)\int_Q d\bar u.\end{aligned} \tag{2.10}$$

We will estimate the various terms in the above equality and show that the resulting relationship can only be satisfied if $u=\bar u$ and $v=\bar v$. Note that
(i)

$$\int_Q [(u-\bar u)_t(u-\bar u)+(v-\bar v)_t(v-\bar v)]=\frac{1}{2}\int_{\Omega\times T}[(u-\bar u)^2+(v-\bar v)^2],$$

(ii)

$$\int_Q a^1_{ij}(u-\bar u)_{x_i}(u-\bar u)_{x_j}\ge\theta\int_Q[|\nabla(u-\bar u)|^2,$$

(iii)

$$\int_Q a^2_{ij}(v-\bar v)_{x_i}(v-\bar v)_{x_j}\ge\theta\int_Q[|\nabla(v-\bar v)|^2,$$

where θ is the ellipticity constant in (2.4),
(iv)

$$\left|\int_Q b^1_i(u-\bar u)_{x_i}(u-\bar u)\right|\le C(\epsilon)\int_Q|\nabla(u-\bar u)|^2+C(\frac{1}{\epsilon})\int_Q(u-\bar u)^2$$

and
(v)

$$\left|\int_Q b^2_i(v-\bar v)_{x_i}(v-\bar v)\right|\le C(\epsilon)\int_Q|\nabla(v-\bar v)|^2+C(\frac{1}{\epsilon})\int_Q(v-\bar v)^2.$$

We choose ϵ such that the $C(\epsilon)$'s in (iv) and (v) equal θ in (ii) and (iii). Now we estimate the double integrals:

$$\begin{aligned}&-\int_Q u(u-\bar u)\int_Q cv+\int_Q \bar u(u-\bar u)\int_Q c\bar v\\&=\ -\int_Q u(u-\bar u)\int_Q cv+\int_Q \bar u(u-\bar u)\int_Q cv\\&\quad\ -\int_Q \bar u(u-\bar u)\int_Q cv+\int_Q \bar u(u-\bar u)\int_Q c\bar v\end{aligned}$$

$$
\begin{aligned}
&= -\int_Q (u-\bar{u})^2 \int_Q cv - \int_Q \bar{u}(u-\bar{u}) \int_Q c(v-\bar{v}) \\
&\leq -\int_Q \bar{u}(u-\bar{u}) \int_Q c(v-\bar{v}) \\
&\leq C\{\int_Q (u-\bar{u})^2 + \int_Q (v-\bar{v})^2\}. \qquad (2.11)
\end{aligned}
$$

By choosing c_0 sufficiently large, we absorb the $\int_Q (u-\bar{u})^2$ and $\int_Q (v-\bar{v})^2$ terms on the left hand side in (2.10). Thus we get

$$(c_0 - C)\{\int_Q (u-\bar{u})^2 + \int_Q (v-\bar{v})^2\} \leq 0$$

We conclude that $u = \bar{u}$ and $v = \bar{v}$. ♢

3 Existence of the Saddle Point

Sufficient conditions for the objective functional $\mathcal{J}(c,d)$ to admit a saddle point are [1]:
(1) The mapping $c \mapsto \mathcal{J}(c,d)$ is strictly convex and lower semi-continuous in the weak toplogy of $L^2(Q \times Q)$.
(2) The mapping $d \mapsto \mathcal{J}(c,d)$ is strictly concave and upper semi-continuous in the weak topology of $L^2(Q \times Q)$.
We will prove (1), *and* (2) follows similarly.

For c, $\bar{c}$ given in C_Γ define a new function $J : [\mathbf{0}, \mathbf{1}] \to \mathbf{R}$ as

$$J(\nu) = \mathcal{J}(\nu c + (1-\nu)\bar{c}, d).$$

The strict convexity of the map $c \mapsto \mathcal{J}(c,d)$ is equivalent to showing $J''(\nu) > 0$ for all ν in $[0,1]$.

Since $\mathcal{J}$ is a function of the state variables and the state variables themselves are functions of the controls, we begin by estimating the first and second derivatives of u and v with respected to the control c. The derivatives involved are directional derivatives, in the distributional sense. We begin by deriving a useful apriori estimate.

Consider the Gelfand triple

$$V \subset L^2(Q) \subset V'$$

and for any (c,d) in $[C_\Gamma]^2$ **define** the operator $\mathcal{L} : V^2 \to (V^{'})^2$ by the formula

$$\mathcal{L}\begin{pmatrix} \zeta \\ \chi \end{pmatrix} = \begin{pmatrix} L_1\zeta + u\int_Q c\chi + \zeta \int_Q cv \\ L_2\chi + v\int_Q d\zeta + \chi\int_Q du \end{pmatrix}. \qquad (3.12)$$

Proposition 3.1 *For any $\epsilon > 0$, there exists $c_0(\epsilon)$ such that if*

$$c^k(x,t) \geq c_0(\epsilon) > 0, k = 1, 2.$$

then the solution to

$$\mathcal{L}\begin{pmatrix}\zeta\\ \chi\end{pmatrix}=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \tag{3.13}$$

with $\zeta=\chi$ at $t=0$ and $\zeta=\chi=0$ on Σ , satisfies the estimate,

$$\|\zeta\|_{L^2(Q)}+\|\chi\|_{L^2(Q)}\le\epsilon(\|\alpha\|_{L^2(Q)}+\|\beta\|_{L^2(Q)}).$$

Proof: We multiply the first equation in (3.13) by ζ and the second equation by χ. Integrating over Q and using the coercivity of the parabolic operators L_1, L_2, we get,

$$\begin{aligned}\frac{1}{2}\int_{\Omega\times T}(\zeta^2+\chi^2)+\theta\int_Q(|\nabla\zeta|^2+|\nabla\chi|^2)+c_0\int_Q(\zeta^2+\chi^2)&\\ \le\ -\int_Q u\zeta\int_Q c\chi-\int_Q v\chi\int_Q d\zeta+\int_Q\alpha\zeta+\int_Q\beta\chi&\\ -\int_Q b_i^1\zeta_{x_i}\zeta-\int_Q b_i^2\chi_{x_i}\chi.&\end{aligned} \tag{3.14}$$

Now we use the L^∞ bounds of u,v,c,d and the ϵ - Cauchy inequality to estimate the right hand side of the above inequality.

(i) $-\int_Q u\zeta\int_Q c\chi\le C\int_Q\zeta\int_Q\chi\le C\int_Q\zeta^2+C\int_Q\chi^2$.
(ii) $-\int_Q v\chi\int_Q d\zeta\le C\int_Q\zeta^2+C\int_Q\chi^2$.
(iii) $\int_Q\alpha\zeta\le\epsilon\int_Q\alpha^2+C_\epsilon\int_Q\zeta^2$.
(iv) $\int_Q\beta\chi\le\epsilon\int_Q\beta^2+C_\epsilon\int_Q\chi^2$.
(v) $\int_Q b_i^1\zeta_{x_i}\zeta\le\frac{\theta}{2}\int_Q|\nabla\zeta|^2+C_\theta\int_Q\zeta^2$.
(vi) $\int_Q b_i^2\chi_{x_i}\chi\le\frac{\theta}{2}\int_Q|\nabla\chi|^2+C_\theta\int_Q\chi^2$.

Choosing $c0$ sufficiently large and estimating the right hand side of (3.14) by the above estimates, we arrive at our conclusion. $\diamondsuit$

We now we prove the existence of first derivatives of u,v with respect to the controls. These derivatives satisfy a system with operator $\mathcal{L}$ from (3.12).

Proposition 3.2 *For c_0 sufficiently large, the mapping*

$$c\mapsto(u(c,d),v(c,d))\in V^2$$

is differentiable in the sense

$$\frac{u(c+\beta\overline{c},d)-u(c,d)}{\beta}\to\zeta\quad\text{weakly in } V, \tag{3.15}$$

$$\frac{v(c+\beta\overline{c},d)-v(c,d)}{\beta}\to\chi\quad\text{weakly in } V, \tag{3.16}$$

as $\beta\to0$, for any $(c,d)\in[C_\Gamma]^2$ and $\overline{c}\in L^\infty(Q)$ such that $c+\beta\overline{c}\in C_\Gamma$.
Also $(\zeta,\chi)\overset{\text{def}}{=}((\zeta(c,d;\overline{c},0),(\chi(c,d;\overline{c},0))$ is the unique solution of

$$\mathcal{L}\begin{pmatrix}\zeta\\ \chi\end{pmatrix}=-\begin{pmatrix}u(c,d)\int_Q\overline{c}v(c,d)\\ 0\end{pmatrix}. \tag{3.17}$$

with $\zeta=\chi=0$ at $t=0$ and $\zeta=\chi=0$ on Σ.

Proof: Let (u_β, v_β) be the solution of the state system corresponding to the controls $(c+\beta\overline{c}, d)$.Then multipling the first equation of the state system by $(u_\beta - u)$ and the second equation by $(v_\beta - v)$ and then integrating over Q, we get
$\int_Q (u_\beta - u)_t(u_\beta - u) + \int_Q [a^1_{ij}(u_\beta - u)_{x_i}(u_\beta - u)_{x_j} + \int_Q b^1_i(u_\beta - u)_{x_i}(u_\beta - u) + \int_Q c^1(u_\beta - u)^2]$

$$= -\int_Q u_\beta(u_\beta - u)\int_Q (c+\beta\overline{c})v_\beta + \int_Q u(u_\beta - u)\int_Q cv \tag{3.18}$$

and

$$\begin{gathered}\int_Q (v_\beta - v)_t(v_\beta - v) + \int_Q a^2_{ij}(v_\beta - v)_{x_i}(v_\beta - v)_{x_j} + \int_Q b^2_i(v_\beta - v)_{x_i}(v_\beta - v) \\ + \int_Q c^2(x,y,t,\tau)(v_\beta - v)^2(y,\tau)\,dy\,d\tau \\ = \\ -\int_Q v_\beta(v_\beta - v)\int_Q du_\beta + \int_Q v(v_\beta - v)\int_Q du. \end{gathered} \tag{3.19}$$

After standard manipulations on the left hand side of (3.18) (using coercivity of the a^1_{ij}'s and applying ϵ - Cauchy inequality to separate $\int_Q b^1_i(u_\beta - u)_{x_i}(u_\beta - u)$)

$$\begin{aligned}\frac{\theta}{2}\int_Q |\nabla(u_\beta - u)|^2 + c_0\int_Q (u_\beta - u)^2 \quad &\leq \quad -\int_Q u_\beta(u_\beta - u)\int_Q (c+\beta\overline{c})v_\beta \\ &+ \quad \int_Q u(u_\beta - u)\int_Q cv.\end{aligned}$$

Adding and subtracting $\int_Q u_\beta(u_\beta - u)\int_Q (c+\beta\overline{c})v$, we get

$$\begin{aligned}\frac{\theta}{2}\int_Q |\nabla(u_\beta - u)|^2 + c_0\int_Q (u_\beta - u)^2 \quad &\leq \quad -\int_Q (u_\beta(u_\beta - u)\int_Q (c+\beta\overline{c})(v_\beta - v) \\ &- \quad \int_Q (u_\beta - u)^2\int_Q cv \\ &- \quad \int_Q u_\beta(u_\beta - u)\int_Q \beta\overline{c}v.\end{aligned}$$

Again using the apriori bounds of u, v, c ,
$\frac{\theta}{2}\int_Q |\nabla(u_\beta - u)|^2 + c_0\int_Q (u_\beta - u)^2$

$$\leq C\{\|(u_\beta - u)\|^2_{L^2(Q)} + \|(v_\beta - v)\|^2_{L^2(Q)}\} + C\beta^2\|\bar{c}\|^2_{L^2(Q)}. \tag{3.20}$$

Similarly (3.19) yields
$\frac{\theta}{2}\int_Q |\nabla(v_\beta - v)|^2 + c_0\int_Q (v_\beta - v)^2$

$$\leq C\{\|(u_\beta - u)\|^2_{L^2(Q)} + \|(v_\beta - v)\|^2_{L^2(Q)}\} \tag{3.21}$$

Combining (3.20) and (3.21), using c_0 large, and dividing across by β, we get

$$\left\|\frac{u_\beta - u}{\beta}\right\|_V + \left\|\frac{v_\beta - v}{\beta}\right\|_V \le C.$$

Since bounded sets in V are weakly compact, we arrive at the required weak limits. In the weak formulation the system satisfied by

$$\frac{u_\beta - u}{\beta}, \frac{v_\beta - v}{\beta}$$

is, for test functions $\phi, \psi \in V$

$$\begin{gathered}\int_Q \left(\frac{(u_\beta - u)}{\beta}\right)_t \phi + \int_Q a^1_{ij}\left((\frac{u_\beta - u)}{\beta}\right)_{x_i}\phi_{x_j} + \int_Q b^1_i \left(\frac{(u_\beta - u)}{\beta}\right)_{x_i}\phi \\ + \int_Q c^1 \left(\frac{(u_\beta - u)}{\beta}\right)\phi \\ = \\ -\frac{1}{\beta}\int_Q u_\beta \phi \int_Q (c + \beta\overline{c})v_\beta + \frac{1}{\beta}\int_Q u\phi \int_Q cv \end{gathered} \tag{3.22}$$

and

$$\begin{gathered}\int_Q \left(\frac{(v_\beta - v)}{\beta}\right)_t \psi + \int_Q a^2_{ij}\left((\frac{v_\beta - v)}{\beta}\right)_{x_i}\psi_{x_j} + \int_Q b^2_i \left(\frac{(v_\beta - v)}{\beta}\right)_{x_i}\psi \\ + \int_Q c^2 \left(\frac{(v_\beta - v)}{\beta}\right)\psi \\ = \\ -\frac{1}{\beta}\int_Q v_\beta \psi \int_Q du_\beta + \frac{1}{\beta}\int_Q u\psi \int_Q du. \end{gathered} \tag{3.23}$$

Letting $\beta \to 0$ and noting $u_\beta \to u$, $v_\beta \to v$ we get

$$\mathcal{L}\begin{pmatrix}\zeta \\ \chi\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix} = -\int_Q (\phi, \psi)\begin{pmatrix} u(c,d)\int_Q \overline{c}v(c,d) \\ 0\end{pmatrix}. \tag{3.24}$$

$\diamondsuit$

We have a similar result for the directional derivative of u, v with respect to the control d.

Proposition 3.3 *For c_0 sufficiently large, the mapping*

$$d \mapsto (u(c,d), v(c,d)) \in V^2$$

is differentiable in the sense

$$\frac{u(c, d + \beta\bar{d}) - u(c,d)}{\beta} \to \xi \quad \textit{weakly in } V, \tag{3.25}$$

$$\frac{v(c,d+\beta\bar{d})-v(c,d)}{\beta} \to \sigma \quad \textit{weakly in } V, \tag{3.26}$$

as $\beta \to 0$, *for any* $(c,d) \in [C_\Gamma]^2$ *and* $\bar{d} \in L^\infty(Q)$ *such that* $d+\beta\bar{d} \in C_\Gamma$.
Also $(\xi,\sigma) \stackrel{\text{def}}{=} ((\xi(c,d;0,\bar{d}),(\sigma(c,d;0,\bar{d}))$ *is the unique solution of*

$$\mathcal{L}\begin{pmatrix}\xi\\ \sigma\end{pmatrix} = -\begin{pmatrix}0\\ v(c,d)\int_Q \overline{d}u(c,d)\end{pmatrix}. \tag{3.27}$$

with $\xi=\sigma=0$ *at* $t=0$ *and* $\xi=\sigma=0$ *on* Σ.

We present next the result for the second dervatives of u,v with respect to the controls.

Proposition 3.4 *The mapping*

$$c \mapsto (u(c,d),v(c,d)) \in V^2$$

admits second derivatives with respect to c in the sense

$$\frac{\zeta(c+\beta\overline{c},d;\overline{c},0)-\zeta(c,d;\overline{c},0)}{\beta} \to \tau \quad \textit{weakly in } V, \tag{3.28}$$

$$\frac{\chi(c+\beta\overline{c},d;\overline{c},0)-\chi(c,d;\overline{c},0)}{\beta} \to \eta \quad \textit{weakly in } V, \tag{3.29}$$

as $\beta \to 0$, *for any* $(c,d) \in [C_\Gamma]^2$ *and* $\overline{c} \in L^\infty(Q)$ *such that* $c+\beta\overline{c} \in C_\Gamma$.
Also $(\tau,\eta) \stackrel{\text{def}}{=} ((\tau(c,d;\overline{c},0;\overline{c},0),\eta(c,d;\overline{c},0;\overline{c},0))$ *is the unique solution of*

$$\mathcal{L}\begin{pmatrix}\tau\\ \eta\end{pmatrix} = -2\begin{pmatrix}\zeta\int_Q c\chi + u\int_Q \overline{c}\chi + \zeta\int_Q \overline{c}v\\ \chi\int_Q d\zeta\end{pmatrix}. \tag{3.30}$$

with $\tau=\eta=0$ *at* $t=0$ *and* $\tau=\eta=0$ *on* Σ.

Proof: We denote by $\zeta_\beta,\chi_\beta,\zeta,\chi$ the solutions of system (3.17) corresponding to
$(c+\beta\overline{c},d;\overline{c},0)$ and $(c,d;\overline{c},0)$ respectively. Using test functions $(\zeta_\beta-\zeta,\chi_\beta-\chi)$ we subtract the (ζ,χ) system from the (ζ_β,χ_β) system

$$\begin{aligned}
&\int_Q [(\zeta_\beta-\zeta)_t(\zeta_\beta-\zeta)+a^1_{ij}(\zeta_\beta-\zeta)_{x_i}(\zeta_\beta-\zeta)_{x_j}+b^1_i(\zeta_\beta-\zeta)_{x_i}(\zeta_\beta-\zeta)+c^1(\zeta_\beta-\zeta)^2]\\
&\quad+\int_Q [(\chi_\beta-\chi)_t(\chi_\beta-\chi)+a^2_{ij}(\chi_\beta-\chi)_{x_i}(\chi_\beta-\chi)_{x_j}]\\
&\quad+\int_Q [b^2_i(\chi_\beta-\chi)_{x_i}(\chi_\beta-\chi)+c^2(\chi_\beta-\chi)^2]\\
&= \qquad (3.31)\\
&-\int_Q(\zeta_\beta-\zeta)\zeta_\beta\int_Q c(v_\beta-v)-\int_Q(\zeta_\beta-\zeta)^2\int_Q cv-\int_Q(\zeta_\beta-\zeta)u_\beta\int_Q c(\chi_\beta-\chi)
\end{aligned}$$

$$-\int_Q(\zeta_\beta-\zeta)(u_\beta-u)\int_Q c\chi-\int_Q(\zeta_\beta-\zeta)\zeta_\beta\int_Q\beta\overline{c}v_\beta-\int_Q(\zeta_\beta-\zeta)u_\beta\int_Q\beta\overline{c}\chi_\beta$$
$$-\int_Q(\zeta_\beta-\zeta)u_\beta\int_Q\overline{c}(v_\beta-v)-\int_Q(\zeta_\beta-\zeta)(u_\beta-u)\int_Q\overline{c}v-\int_Q(\chi_\beta-\chi)^2\int_Q du_\beta$$
$$-\int_Q(\chi_\beta-\chi)v_\beta\int_Q d(\zeta_\beta-\zeta)-\int_Q(\chi_\beta-\chi)(v_\beta-v)\int_Q d\zeta-\int_Q(\chi_\beta-\chi)\chi\int_Q d(u_\beta-u).$$

We illustrate the estimates for a term with the kernel;

$$\begin{aligned}
&\left|\int_Q(\zeta_\beta-\zeta)\zeta_\beta\int_Q c(v_\beta-v)\right| \\
&= \left|\int_Q\left((\zeta_\beta-\zeta)\zeta_\beta\int_Q c(v_\beta-v)(y,\tau)\,dy\,d\tau\right)(x,t)\,dx\,dt\right| \qquad (3.32)\\
&\le C\int_Q(\zeta_\beta-\zeta)^2(x,t)\,dx\,dt+C\int_Q\left(\zeta_\beta\int_Q(v_\beta-v)(y,\tau)\,dy\,d\tau\right)^2dx\,dt \\
&\le C\int_Q(\zeta_\beta-\zeta)^2(x,t)\,dx\,dt+C\left(\int_Q\zeta_\beta^2\,dx\,dt\right)\left(\int_Q(v_\beta-v)^2\,dy\,d\tau\right) \qquad (3.33)
\end{aligned}$$

Notice how the specific form of the non-local term was used to derive (3.33) from (3.32). Other such terms are estimated as below:

$$-\int_Q(\zeta_\beta-\zeta)^2\int_Q cv\le 0.$$

$$\int_Q(\zeta_\beta-\zeta)u_\beta\int_Q c(\chi_\beta-\chi)\le C\int_Q(\zeta_\beta-\zeta)^2+C\int_Q(\chi_\beta-\chi)^2.$$

$$\int_Q(\zeta_\beta-\zeta)(u_\beta-u)\int_Q c\chi\le C\int_Q(\zeta_\beta-\zeta)^2+C\int_Q(u_\beta-u)^2.$$

$$\int_Q(\zeta_\beta-\zeta)\zeta_\beta\int_Q\beta\overline{c}v_\beta\le C\int_Q(\zeta_\beta-\zeta)^2+C\beta^2\int_Q\zeta_\beta^2.$$

All the terms of the form above $\int_Q(\zeta_\beta-\zeta)^2$ and $\int_Q(\chi_\beta-\chi)^2$ can be combined with the $c_1\int_Q(\zeta_\beta-\zeta)^2$ and $c_2\int_Q(\chi_\beta-\chi)^2$ in equation (3.31). Terms above with ζ_β^2, χ_β^2 are estimated as follows:

$$\begin{aligned}
&\left(\int_Q\zeta_\beta^2\right)\left(\int_Q(v_\beta-v)^2\right)+\left(\int_Q\chi_\beta^2\right)\left(\int_Q(u_\beta-u)^2\right)\\
&\le \left(\|u_\beta-u\|^2_{L^2(Q)}\|+\|v_\beta-v\|^2_{L^2(Q)}\|\right)\left(\|\zeta_\beta\|^2_{L^2(Q)}+\|\chi_\beta\|^2_{L^2(Q)}\right).
\end{aligned}$$

Other terms include

$$C\beta^2\int_Q\zeta_\beta^2,\ C\beta^2\int_Q\chi_\beta^2$$

and

$$C\int_Q (v_\beta - v)^2,\ C\int_Q (u_\beta - u)^2.$$

Now use the estimate in equation (3.20) to get

$$\left(\|u_\beta - u\|^2_{L^2(Q)} + \|v_\beta - v\|^2_{L^2(Q)}\right) \le C\beta^2\|\overline{c}\|^2_{L^2(Q\times Q)}.$$

Using proposition (3.1) with $\alpha = \bar{c}$ and $\beta = 0$ we derive

$$\left(\|\zeta_\beta\|^2_{L^2(Q)} + \|\chi_\beta\|^2_{L^2(Q)}\right) \le C\|\overline{c}\|^2_{L^2(Q\times Q)}.$$

The above estimates provide an *a priori* bound for the second derivative which proves (3.28) and (3.29). These weak convergences of the quotients justify that η and τ satisfy the weak formulation of the system (3.30). ♢

Remark 1. The estimates in the above Proposition also give us uniform L^2 bounds for the second derivatives of τ and η. Namely,

$$\|\tau\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \le C\|\overline{c}\|^2_{L^2(Q\times Q)}.$$

Remark 2. From the proof of previous proposition,

$$\begin{aligned}
\frac{u_\beta - u}{\beta} &\to \zeta \quad \text{strongly in } L^2(Q),\\
\frac{v_\beta - v}{\beta} &\to \chi \quad \text{strongly in } L^2(Q),\\
\frac{\zeta_\beta - \zeta}{\beta} &\to \tau \quad \text{strongly in } L^2(Q),\\
\frac{\chi_\beta - \chi}{\beta} &\to \eta \quad \text{strongly in } L^2(Q).
\end{aligned} \tag{3.34}$$

We have a similar result for the second derivatives of u and v with respect to the control d.

Proposition 3.5 *The mapping*

$$d \mapsto (u(c,d), v(c,d)) \in V^2$$

admits second derivatives with respect to c in the sense

$$\frac{\zeta(c, d+\beta\overline{d}, 0; \overline{d}) - \zeta(c, d; 0, \overline{d})}{\beta} \to \kappa \quad \textit{weakly in } V, \tag{3.35}$$

$$\frac{\chi(c, d+\beta\overline{d}, 0; \overline{d}) - \chi(c, d; 0, \overline{d})}{\beta} \to \delta \quad \textit{weakly in } V, \tag{3.36}$$

as $\beta \to 0$, *for any* $(c,d) \in [C_\Gamma]^2$ *and* $\overline{d} \in L^\infty(Q)$ *such that* $d + \beta\overline{d} \in C_\Gamma$. *Also* $(\kappa, \delta) \stackrel{\text{def}}{=} ((\kappa(c, d; 0, \overline{d}; 0, \overline{d}), \delta(c, d; 0, \overline{d}; 0, \overline{d}))$ *is the unique solution of*

$$\mathcal{L}\begin{pmatrix} \kappa \\ \delta \end{pmatrix} = -2\begin{pmatrix} \xi\int_Q d\sigma + v\int_Q \overline{d}\sigma + \xi\int_Q \overline{d}u \\ \sigma\int_Q c\xi \end{pmatrix}. \tag{3.37}$$

with $\kappa = \delta = 0$ *at* $t = 0$ *and* $\kappa = \delta = 0$ *on* Σ.

Proposition 3.6 *For a fixed $d \in C_\Gamma$, the mapping $c \in C_\Gamma \mapsto \mathcal{J}(c,d)$ is strictly convex.*

Proof: As mentioned earlier, it suffices to show that $J''(\nu) > 0$ for $\nu \in [0,1]$.

The justification for differentiating J is a consequence of the above established first and second derivatives of u, v with respect to the control variable c and the strong convergence noted in the previous proposition. Now, for $0 \le \nu \le 1$,

$$J(\nu) = \mathcal{J}(\overline{c} + \nu(c - \overline{c}), d).$$

The directional derivative is now in the direction $c - \overline{c}$.
Denoting

$$\begin{aligned}
u &= u(\overline{c} + \nu(c - \overline{c}), d),\\
v &= v(\overline{c} + \nu(c - \overline{c}), d),\\
\zeta &= \zeta(\overline{c} + \nu(c - \overline{c}), d; c - \overline{c}, 0),\\
\chi &= \chi(\overline{c} + \nu(c - \overline{c}), d; c - \overline{c}, 0),\\
\tau &= \chi(\overline{c} + \nu(c - \overline{c}), d; c - \overline{c}, 0; c - \overline{c}, 0),\\
\eta &= \eta(\overline{c} + \nu(c - \overline{c}), d; c - \overline{c}, 0; c - \overline{c}, 0),
\end{aligned}$$

$$J(\nu) = \int_Q \left\{K[u - \tilde{u}]^2 - L[v - \tilde{v}]^2\right\} + \int_Q \int_Q \left\{N(\overline{c} + \nu(c - \overline{c}))^2 - Md^2\right\}.$$

Differentiating twice with respect to c

$$\begin{aligned}
J''(\nu) \quad &= \quad \int_Q (K\zeta^2 + K[u - \tilde{u}]\tau - L[v - \tilde{v}]\eta - L\chi^2)\\
&\quad + \int_Q \int_Q N(c - \overline{c})^2.
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
J''(\nu) \quad \ge \quad & -K\|u\|_{L^2(Q)}\|\tau\|_{L^2(Q)} - K\|\tilde{u}\|_{L^2(Q)}\|\tau\|_{L^2(Q)}\\
- \quad & L\|\chi\|^2_{L^2(Q)} - L\|v\|_{L^2(Q)}\|\eta\|_{L^2(Q)}\\
- \quad & L\|\tilde{v}\|_{L^2(Q)}\|\eta\|_{L^2(Q)} + N\|c - \overline{c}\|^2_{L^2(Q)}.
\end{aligned}$$

From proposition 3.4, for $\epsilon > 0$ there exists $c_0(\epsilon)$ such that for $c_k > c_0$, $k = 1, 2$,

$$\|\tau\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \le \epsilon\|c - \overline{c}\|^2_{L^2(Q\times Q)}$$

and

$$\|\chi\|^2_{L^2(Q)} \le \epsilon\|c - \overline{c}\|^2_{L^2(Q\times Q)}.$$

Combining these estimates with above we get

$$J''(\nu) > (N - \tilde{\epsilon})\|c - \overline{c}\|^2_{L^2(Q\times Q)}.$$

$\diamondsuit$

Proposition 3.7 *For a fixed $d \in C_\Gamma$ the mapping $c \in C_\Gamma \mapsto \mathcal{J}(c,d)$ is lower semicontinuous in the weak topology on $L^2(Q \times Q)$.*

Proof: It is enough to show for every $\alpha \in \mathbf{R}$,

$$S(d,\alpha) = \{h | h \in C_\Gamma, \mathcal{J}(h,d) \leq \alpha\}$$

is closed in the weak topology of $L^2(Q \times Q)$. Let $d \in C_\Gamma$ and $\alpha \in \mathbf{R}$ be fixed such that

$$\mathcal{J}(c_n, d) \leq \alpha$$

and

$$c_n \to \hat{c} \text{ weakly in } L^\infty(Q \times Q).$$

Using the state systems, $u_n(c_n,d), v_n(c_n,d)$ satisfy

$$\|u_n\|_V, \|v_n\|_V \leq C.$$

Then, using compactness results [7, Chapter 4, Prop. 4.2] , we can find subsequences such that

$$u_n \to u \qquad \text{weakly in } V, \text{ strongly in } L^2(Q)$$

$$v_n \to v \qquad \text{weakly in } V, \text{ strongly in } L^2(Q).$$

Standard continuity arguments show that

$$u = u(\hat{c}, d) \qquad v = v(\hat{c}, d).$$

Also using a generalization of Fatou's lemma,

$$\liminf \mathcal{J}(c_n, d) \geq \mathcal{J}(\hat{c}, d).$$

◇

Theorem 3.1 *If c_0 is large enough, there exists a unique saddle point (c^*, d^*).*

Proof: Combining Propostion 3.6 and Proposition 3.7 and the existence of saddle point result from Ekeland and Temam [1, Chap.6, Propositions 1.5 and 2.1], we conclude that the cost functional admits a unique saddle point. ◇

4 The Optimality System

The solution of the optimality system, consisting of the two state equations and two suitably chosen adjoint equations, will be used to characterize the saddle point of the game.

Theorem 4.1 *If $(c,d) \in [C_\Gamma]^2$ is the saddle point and c_0 is sufficiently large, then there exists $(u,v,p,q) \in V^4$ satisfying:*

$$\begin{aligned} L_1 u + u \textstyle\int_Q cv &= f \\ L_2 v + v \textstyle\int_Q du &= g \\ L_1^* p + p \textstyle\int_Q cv - \int_Q d^T vq &= K(u-\tilde{u}) \\ L_2^* q + q \textstyle\int_Q du - \int_Q c^T up &= -L(v-\tilde{u}) \quad in\ \ Q \end{aligned} \tag{4.39}$$

$$\begin{aligned} u(x,t) = u^0, v(x,t) &= v^0 \quad on\ \ \Omega \times \{0\} \\ u(x,t) = v(x,t) &= 0 \qquad\qquad on\ \ \Sigma \\ p(x,T) = q(x,T) &= 0 \qquad on\ \ \Omega \times T \\ p(x,t) = q(x,t) &= 0 \qquad\qquad on\,\Sigma, \end{aligned} \tag{4.40}$$

where

$$L_k^* u = -u_t - (a_{ij}^k u_{x_j})_{x_i} - (b_i^k u)_{x_i} + c^k u, \;\; k = 1,2.$$

Moreover on Q,

$$c(x,y,t,\tau) = \min(\frac{p^+(x,t)u(x,t)v(y,\tau)}{N}, \Gamma),$$

$$d(x,y,t,\tau) = \min(\frac{q^+(x,t)u(y,\tau)v(x,t)}{M}, \Gamma),$$

where $p^+ = max(p,0)$.

Proof: Let $(c,d) \in [C_\Gamma]^2$ be a saddle point. Choose $\bar{c} \in L^\infty(Q \times Q)$ in such a way that for $\beta > 0$ arbitrarily small, $c + \beta\bar{c}$ lies in the set C_Γ. Since (c,d) is a saddle point,

$$\lim_{\beta \to 0} \frac{\mathcal{J}(c+\beta\bar{c},d) - \mathcal{J}(c,d)}{\beta} \geq 0. \tag{4.41}$$

Substituting the explicit form of the cost functional in (4.41), dividing by β, and noting $\frac{u_\beta - u}{\beta} \to \zeta$ strongly in $L^2(Q)$, and $u_\beta \to u$, we get

$$\int_Q (K(u-\tilde{u})\zeta - L(v-\tilde{v})\chi)\,dx\,dt + \int_Q\int_Q N c\bar{c}\,dxdy\,dtd\tau \geq 0. \tag{4.42}$$

We introduce a new notation:

$$d^T(x,y,t,\tau) := d(y,x,\tau,t),\; c^T(x,y,t,\tau) := c(y,x,\tau,t).$$

Now we define an operator $\mathcal{L}^*$ such that **formally**

$$(p,q)\mathcal{L}\begin{pmatrix} \zeta \\ \chi \end{pmatrix} = (\zeta,\chi)\mathcal{L}^*\begin{pmatrix} p \\ q \end{pmatrix}.$$

We define the adjoint functions (p, q) as the solutions in V of

$$\begin{aligned}
L_1^* p + p\int_Q cv - \int_Q d^T vq &= K(u-\tilde{u}), && \text{in } Q\\
L_2^* q + q\int_Q du - \int_Q c^T up &= -L(v-\tilde{v}), && \\
p(x,T) = q(x,T) &= 0, && \text{on } \Omega\times T\\
p(x,t) = q(x,t) &= 0, && \text{on } \Sigma.
\end{aligned} \tag{4.43}$$

The solution of the above system, after a change of variable $\hat{p}(x,t) = p(x, T-t)$ and $\hat{q}(x,t) = q(x, T-t)$, is constructed in a manner similar to that of the solution of the original state system. Substituting (4.43) in (4.42), we get

$$\int_Q (\zeta,\chi)\mathcal{L}^* \begin{pmatrix} p \\ q \end{pmatrix} + \int_Q\int_Q Nc\overline{c} \geq 0$$

Now, from (3.27)

$$\int_Q (p,q)\mathcal{L} \begin{pmatrix} \zeta \\ \chi \end{pmatrix} + \int_Q\int_Q Nc\overline{c} = \int_Q (p,q) \begin{pmatrix} -u\int_Q \overline{c}v \\ 0 \end{pmatrix} + \int_Q\int_Q Nc\overline{c} \geq 0$$

This implies

$$\int_Q\int_Q (Nc - puv)\overline{c} \geq 0. \tag{4.44}$$

Since we can choose $\bar{c}$ non-negative and arbitrary, this implies

$$Nc - puv \geq 0.$$

On the set $\{(x,y,t,\tau) | c(x,y,t,\tau) = 0\}$ we get $puv \leq 0$ which gives

$$p^+ = 0.$$

On the set $\{(x,y,t,\tau) | 0 < c(x,y,t,\tau)\Gamma\}, \bar{c}$ has arbitrary sign, which implies $p \geq 0$ and

$$c = \frac{p^+ uv}{N}.$$

On the set $\{(x,y,t,\tau) | c(x,y,t,\tau) = \Gamma\}, \bar{c}$ must be non-positive, which gives

$$Nc - puv \leq 0.$$

Combining these results we get

$$c(x,y,t,\tau) = \min\Big(\frac{p^+(x,t)u(x,t)v(y,\tau)}{N}, \Gamma\Big)$$

.

Similarly,

$$d(x,y,t,\tau) = \min\Big(\frac{q^+(x,t)u(y,\tau)v(x,t)}{M}, \Gamma\Big).$$

$\diamondsuit$

Theorem 4.2 *For c_0 sufficiently large, bounded solutions of the optimality system:*

$$
\begin{aligned}
L_1 u + u \int_Q min(\tfrac{p^+ uv}{N}, \Gamma) v &= f \\
L_2 v + v \int_Q min(\tfrac{q^+ uv}{M}, \Gamma) u &= g \\
L_1^* p + p \int_Q min(\tfrac{p^+ uv}{N}, \Gamma) v - \int_Q min(\tfrac{q^+ uv}{M}, \Gamma)^T vq &= K(u - \tilde{u}) \\
L_2^* q + q \int_Q min(\tfrac{q^+ uv}{M}, \Gamma) u - \int_Q min(\tfrac{p^+ uv}{N}, \Gamma)^T up &= -L(v - \tilde{u}) \quad in\ \ Q
\end{aligned}
\tag{4.45}
$$

$$
\begin{aligned}
u(x,t) = u^0, v(x,t) &= v^0 \quad on\ \ \Omega \times 0 \\
u(x,t) = v(x,t) &= 0 \qquad on\ \ \Sigma \\
p(x,T) = q(x,T) &= 0 \quad on\ \ \Omega \times T \\
p(x,t) = q(x,t) &= 0 \qquad on\ \ \Sigma.
\end{aligned}
\tag{4.46}
$$

exist and are unique in the solution space $[V]^4$.

Proof: The existence of the saddle point implies the existence of u, v and then the existence of p, q. The optimality system for a strictly convex-concave functional completely characterizes its saddle points ♢

5 Summary

We have proved that a two person zero sum game described by a system of parabolic equations with competitive interactions can be controlled via the non-local kernels of the interacting terms, and the saddle point can be represented in terms of the optimality system.

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