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Existence results for singular anisotropic elliptic boundary-value problems *

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Abstract

We establish the existence of a positive solution for anisotropic singular quasilinear elliptic boundary-value problems. As an example of the problems studied we have

$$u^a u_{xx} + u^b u_{yy} + \lambda(u+1)^{a+r} = 0$$

with zero Dirichlet boundary condition, on a bounded convex domain in $\mathbb{R}^2$. Here $0 \le b \le a$, and λ, r are positive constants. When $0 < r < 1$ (sublinear case), for each positive λ there exists a positive solution. On the other hand when $r > 1$ (superlinear case), there exists a positive constant λ^* such that for λ in $(0, \lambda^*)$ there exists a positive solution, and for $\lambda^* < \lambda$ there is no positive solution.

1 Introduction

In this paper we study a class of singular anisotropic elliptic problems for which the singularity occurs at the boundary of the domain. Moreover we consider a nonlinear source term that has either sublinear or superlinear growth rate relative to that of the leading elliptic coefficient.

Such problems naturally arise, for example, in the study of transonic flow. Čanić and Keyfitz [1, 2] showed that the unsteady transonic small disturbance equation gives rise to a nonlinear problem of Keldysh type which has both singular and smooth solutions. Their problems came from a conservation law system in two space variables. At the same time, Choi, Lazer and McKenna used different techniques to obtain existence results for the anisotropic singular problem $u^a u_{xx} + u^b u_{yy} + p(x,y) = 0$ subject to zero Dirichlet boundary condition in a convex and smooth domain, where a and b are positive constants, see [5]. Later, Choi and McKenna [6] extended their results to higher dimensional problems and got some partial results about non-convex domains. Zheng [15] established existence results for another degenerate quasilinear problem which came from gas dynamics.

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Motivated by these problems, Choi and the author [3, 4] studied more general classes of anisotropic quasilinear problems where the source term contains additional nonlinearities. Although, these problems were not degenerate, it opened the door for studying singular anisotropic cases.

In this paper, we present existence results for a class of singular quasilinear anisotropic boundary value problems where the positive source term has either sublinear or superlinear growth rate. We obtain results qualitatively similar to those found in [7, 12] for semilinear elliptic equations. To be precise, we consider

$$\textstyle\sum_{i=1}^{N} A_i(u)\frac{\partial^2 u}{\partial x_i^2} + \lambda f(\mathbf{x}, u) = 0 \quad \text{in} \quad \Omega\,, \tag{1}$$

$$u\big|_{\partial\Omega} = 0\,, \tag{2}$$

where λ is a positive constant.

First we list some assumptions about Ω, the coefficients A_i, and f. Different theorems will require different combinations of such assumptions.

D. $\Omega \subset R^N$ is a bounded convex domain with smooth boundary $\partial\Omega$ of class $C^{2+\gamma}$ where $0 < \gamma < 1$.

C. For each $1 \leq i \leq N$, $A_i : [0, \infty) \to [0, \infty)$, is C^1 and non-decreasing.

C1. There exist constants $0 < c_i \leq d_i < \infty$, and $a_i \geq 0$ such that $c_i t^{a_i} \leq A_i(t) \leq d_i t^{a_i}$ for all $t \geq 0$ and $i = 1, \ldots, N$.
Without loss of generality, we let $a_1 \equiv \max_{1 \leq i \leq N} a_i > 0$.

C2. For all $t > 0$, $(A_i(t)/A_1(t))' \leq 0$, $i = 2, \ldots, N$.

F. $f : \Omega \times \mathbf{R} \to \mathbf{R}$, is C^1, and there exists an $m > 0$ such that for all $\mathbf{x} \in \overline{\Omega}$, $f(\mathbf{x}, \cdot) \geq m > 0$.

F1. (Sublinear f) $\lim_{t\to\infty} f(\mathbf{x}, t)t^{-(a_1+1)} = 0$ uniformly.

F2. (Superlinear f) $\lim_{t\to\infty} f(\mathbf{x}, t)t^{-(a_1+1)} = \infty$ uniformly.

We will establish the following theorems.

Theorem 1 (Sublinear Case) *Assume* **D, C, C1, F** *and* **F1** *hold. Then for any $\lambda > 0$, equations (1) and (2) have at least one positive solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

Theorem 2 (Superlinear Case) *Assume* **D, C, C1, C2, F** *and* **F2** *hold. Then there exists a positive constant λ^* such that*

1. *for each $\lambda \in (0, \lambda^*)$, equations (1) and (2) have at least one positive solution u in $C^2(\Omega) \cap C(\overline{\Omega})$.*

2. *for $\lambda > \lambda^*$, no positive solution exists.*

One of the major difficulties of our problem is the loss of uniform ellipticity. Using recent results about related non-degenerate problems [3], we approximate this problem with a sequence of non-singular cases.

The main technique used in this paper is the upper-lower solutions method. Since the diffusion coefficients are solution-dependent and are anisotropic, we cannot employ the usual techniques for isotropic problems, and need to construct appropriate comparison principles for sublinear and superlinear cases differently. In the sublinear case, we modify the *comparison lemma* from [5] for our nonlinear case. For the superlinear case, we employ the *monotone iteration lemma* in [3] together with the modified *comparison lemma*. Our comparison theorems allow us to control the sequence of approximate solutions allowing us to extract a solution via a compactness argument.

This paper consists of five sections. We state the modified comparison lemma in section 2. We establish Theorem 1 in section 3, and Theorem 2 in section 4. In the fifth section we present numerical results for some remaining open questions.

2 A Comparison Lemma

In this section, we state the *comparison lemma*, another version of Lemma 1 in [5]. This allows us to obtain a nontrivial lower bound and an upper barrier function for this class of equations. Since the proof is similar to that of Lemma 1 in [5], it will be skipped.

Consider the following problem for a nonlinear operator $\mathcal{F}$

$$\mathcal{F}(u) \equiv \sum_{i=1}^{N} f_i(\mathbf{x},u)\frac{\partial^2 u}{\partial x_i^2} = -g(\mathbf{x},u) \tag{3}$$
$$u\big|_{\partial\Omega} = u_0\,,$$

where each $f_i \in C(\Omega \times [0,\infty))$ is a positive and non-decreasing function of $u \in [0,\infty)$ for $1 \le i \le N$, and $g \in C(\Omega \times [0,\infty))$ is positive.

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a positive solution of (3) with

$$\sup u \le t_1 \tag{4}$$

for some constant $t_1 > 0$. For notational convenience, we write
$M \equiv \sup_{t\in[0,t_1]} g(\cdot,t)$ and $m \equiv \inf_{t\in[0,t_1]} g(\cdot,t) > 0$.

An UPPER SOLUTION of (3) is a function $\psi \in C^2(\Omega)\cap C(\overline{\Omega})$ satisfying $\mathcal{F}(\psi) \le -(M+1)$ with $\psi_{x_ix_i} \le 0$ for all $1 \le i \le N$, and $\psi\big|_{\partial\Omega} \ge u_0$.

A LOWER SOLUTION is a function $\varphi \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying $\mathcal{F}(\varphi) \ge -\frac{m}{2}$ with $\varphi_{x_ix_i} \le 0$ for all $1 \le i \le N$, and $\varphi\big|_{\partial\Omega} \le u_0$.

Then the comparison lemma is as follows.

Lemma 1 *Let Ω be as in Theorem 1, and let u be a positive solution of (3) with (4). If φ is a non-negative lower solution and ψ is an upper solution with $\varphi \le \psi$, then $\varphi \le u \le \psi$ on $\overline{\Omega}$.*

Remark: Note that the inequality $u \leq \psi$ holds if we assume only the existence of an upper solution. And similarly, $u \geq \varphi$ holds if we assume only a lower solution.

3 Sublinear Case.

We establish Theorem 1 in this section. As discussed before, we lose uniform ellipticity because of the singularity of the problem. To overcome this difficulty, we approximate our equations by a sequence of non-singular problems. In section 3.1, we construct positive solutions to this sequence of equations. In section 3.2, because of the existence of a uniform bound obtained by the appropriate comparison lemma, we can show that the corresponding sequences of solutions have a limit, which will solve equations (1) and (2).

We fix λ and write $f(\mathbf{x},u) \equiv \lambda f(\mathbf{x},u)$ in section 3.1 and 3.2.

3.1 An Approximation

Rewrite equation (1) and define G as follows:

$$G(u) \equiv u_{x_1x_1} + \textstyle\sum_{i=2}^{N} \frac{A_i(u)}{A_1(u)} u_{x_ix_i} + \frac{f(\mathbf{x},u)}{A_1(u)} = 0\,, \qquad u\big|_{\partial\Omega} = 0\,. \tag{5}$$

Let $0 < \varepsilon < 1$. We approximate problem (5) by the nonsingular problem

$$G_\varepsilon(u) \equiv u_{x_1x_1} + \textstyle\sum_{i=2}^{N} \frac{A_i(u+\varepsilon)}{A_1(u+\varepsilon)} u_{x_ix_i} + \frac{f(\mathbf{x},u)}{A_1(u+\varepsilon)} = 0\,, \qquad u\big|_{\partial\Omega} = 0\,. \tag{6}$$

Since Ω is bounded, without loss of generality, it can be translated so that it lies in the interior of the strip $[0,L] \times \mathbf{R}^{N-1}$ for some $L > 0$.

By Theorem 2 in [11], we know that there exists a unique positive solution $\psi \in C^2(0,L) \cap C[0,L]$ of the problem

$$\psi_{x_1x_1} + \frac{M}{\psi^{a_1}} = 0\,, \tag{7}$$

$$\psi(0) = \psi(L) = 0\,, \tag{8}$$

where $M > 0$.

Let η_1 be the first eigenfunction of the problem

$$\eta_{1\,x_1x_1} + \beta_1\eta_1 = 0 \tag{9}$$

with corresponding first eigenvalue $\beta_1 > 0$, in the extended domain $[-1, L+1]$, with zero boundary condition. We define

$$\delta \equiv \min_{x\in[0,L]} \eta_1(x) > 0. \tag{10}$$

Then the following lemma holds.

Lemma 2 *Assume the same condition as in Theorem 1 and let ψ be the function in (7). There exists a $K_1 \geq 1$ such that $\Psi \equiv K_1\eta_1 + \psi$ is an upper solution of (5).*

Proof: Since condition **F1** holds, there exists a large positive constant t_o such that

$$\frac{f(\mathbf{x},t)}{t^{a_1+1}} \leq \frac{\beta_1 c_1}{2} \tag{11}$$

for all $t \geq t_o$ and all $\mathbf{x} \in \overline{\Omega}$ where c_1 is the positive constant in condition **C1**. With

$$M_o \equiv \sup_{t\in[0,t_o]} f(t),$$

we get

$$\frac{f(\mathbf{x},t)}{t^{a_1}} \leq \frac{\beta_1}{2}t + \frac{M_o}{t^{a_1}} \tag{12}$$

for all $t > 0$ and all $\mathbf{x} \in \overline{\Omega}$.

Let $M \equiv M_o/c_1$, and let ψ be the positive solution of (7) with this constant M. Let $K_1 \geq 1$, and define

$$\Psi \equiv K_1\eta_1 + \psi.$$

Let $\mu \equiv \beta_1/2$. Then we have

$$\begin{aligned}
\Psi_{x_1x_1} + \frac{f(\mathbf{x},\Psi)}{A_1(\Psi)} &\leq \Psi_{x_1x_1} + \frac{f(\mathbf{x},\Psi)}{c_1\Psi^{a_1}} \\
&\leq \Psi_{x_1x_1} + \mu\Psi + \frac{M}{\Psi^{a_1}} \\
&= \{K_1\eta_1+\psi\}_{x_1x_1} + \mu\{K_1\eta_1+\psi\} + \frac{M}{\{K_1\eta_1+\psi\}^{a_1}} \\
&= K_1\eta_{1_{x_1x_1}} + \mu K_1\eta_1 + \mu\psi + \left[\frac{M}{\{K_1\eta_1+\psi\}^{a_1}} - \frac{M}{\psi^{a_1}}\right].
\end{aligned}$$

Since $[\frac{M}{\{K_1\eta_1+\psi\}^{a_1}} - \frac{M}{\psi^{a_1}}] < 0$, by choosing K_1 sufficiently large in the line above,

$$\begin{aligned}
\Psi_{x_1x_1} + \frac{f(\mathbf{x},\Psi)}{A_1(\Psi)} &\leq -\beta_1K_1\eta_1 + \mu K_1\eta_1 + \mu\psi \\
&\leq -(\beta_1/2)K_1\delta + \mu\|\psi\|_{L^\infty} \\
&< 0\,.
\end{aligned} \tag{13}$$

Therefore, Ψ is an upper solution for (5). ◇

By a similar calculation, it is easy to check that Ψ satisfies $G_\varepsilon(\Psi) \leq 0$ for each $\varepsilon \in (0,1)$. With this Ψ, define the set

$$S \equiv \{u \in C(\overline{\Omega}) |\ 0 \leq u \leq \Psi \quad \text{on} \quad \overline{\Omega}, \quad u\big|_{\partial\Omega} = 0\},$$

which is closed, bounded and convex. Using the Schauder fixed point theorem, we establish the existence of positive solution to equation (6) in the following lemma. The proof is similar to that of Theorem 1 in [3].

Lemma 3 *Let the assumptions of in Theorem 1 hold. Then for each* $0 < \varepsilon < 1$*, equation (6) has a positive solution* $u \leq \Psi$ *with* $u \in C^{2,\alpha}(\overline{\Omega})$ *for some* $0 < \alpha < 1$.

Proof: Fix ε. Define a map T for each $u \in S$, by $Tu = w$, where w satisfies

$$w_{x_1x_1} + \textstyle\sum_{i=2}^N \frac{A_i(u+\varepsilon)}{A_1(u+\varepsilon)} w_{x_ix_i} + \frac{f(\mathbf{x},u)}{A_1(w+\varepsilon)} = 0\,, \tag{14}$$
$$w\big|_{\partial\Omega} = 0\,.$$

This equation is uniformly elliptic by condition **C1** since $a_1 \geq a_i$ for $i = 2, \ldots, N$. By condition **F1**, we get

$$f(\cdot, t) \leq \frac{\beta_1}{2} t^{a_1+1} + M_o, \tag{15}$$

with the same positive eigenvalue β_1 from equation (9) and the positive constant M_o as in equation (12). Using the bound (15), condition **C1**, $u \in S$, and inequality (13) in Lemma 1, we know that

$$\begin{aligned} \Psi_{x_1x_1} + \frac{f(\mathbf{x},u)}{A_1(\Psi+\varepsilon)} &\leq \Psi_{x_1x_1} + \frac{\frac{\beta_1}{2}u^{a_1+1} + M_o}{c_1\Psi^{a_1}} \\ &\leq \Psi_{x_1x_1} + \frac{\frac{\beta_1}{2}\Psi^{a_1+1} + M_o}{c_1\Psi^{a_1}} \\ &= \Psi_{x_1x_1} + \mu\Psi + \frac{M}{\Psi^{a_1}} \\ &\leq 0. \end{aligned}$$

Hence, with Ψ an upper solution and zero a lower solution, the existence of the solution $w \in W^{2,q}(\Omega) \cap C(\overline{\Omega})$ for any $q > N$ is obtained using Lemma 1 of [6]. Moreover, $w \in S$ by construction. The uniqueness of the solution to (6) is a consequence of the maximum principle. Thus T is well defined and T maps S into S.

Applying a calculation similar to Theorem 1 in [3], we prove that T is compact and continuous as well.

Therefore, by applying the Schauder fixed point theorem, we obtain u as a fixed point of T in the set S. Since $u \in W^{2,q}(\Omega)$ for any $q > N$, then u is in $C^{1,\alpha}(\overline{\Omega})$ for $\alpha = 1 - N/q$. A bootstrap argument using Schauder estimates [8], we have $u \in C^{2,\alpha}(\overline{\Omega})$. This completes the proof of this lemma. ♢

Therefore, for each $\varepsilon \in (0,1)$, we know that (6) has a corresponding positive solution u_ε.

3.2 Proof of Theorem 1 (Sublinear Case).

We first construct a nonzero lower solution so that in each compact subset in Ω, we have a uniform ellipticity condition which is independent of ε. Rewrite (6) as follows:

$$\textstyle\sum_{i=1}^N A_i(u+\varepsilon)\frac{\partial^2 u}{\partial x_i^2} + f(\mathbf{x},u) = 0\,, \tag{16}$$
$$u\big|_{\partial\Omega} = 0\,.$$

For any $\varepsilon \in (0,1)$, by Lemma 3, a solution $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$, and satisfies $0 \le u_\varepsilon \le \Psi$ on $\overline{\Omega}$.

We now construct a uniform positive lower bound for u_ε in any compact subset of Ω. We use a similar idea to [5, 6].

For any $\mathbf{x}_0 \in \Omega$, we can take an open ball centered at $\mathbf{x}_0$, which lies inside Ω. Without loss of generality, we can consider $\mathbf{x}_0$ as the origin and let the ball be B_R where radius $R \le 1$. With $\delta > 0$, we define

$$\varphi \equiv \delta(R^2 - |\mathbf{x}|^2)$$

where $|\mathbf{x}| = \sqrt{\sum_{i=1}^N x_i^2}$. Then using condition **C** and **F**, we get

$$\begin{aligned} \sum_{i=1}^N A_i(\varphi+\varepsilon)\frac{\partial^2\varphi}{\partial x_i^2} + f(\mathbf{x},\varphi) &\ge -2\delta\sum_{i=1}^N d_i(\delta+1)^{a_i} + \frac{m}{2} \\ &\ge 0\,, \end{aligned}$$

by choosing a sufficiently small $\delta \equiv \delta(m, a_i, d_i)$. Applying the comparison lemma in section 2 in B_R,

$$0 < \varphi \le u_\varepsilon \quad \text{inside of} \quad B_R, \tag{17}$$

and we know that $u_\varepsilon \le \Psi$. By condition **F**, there exist positive constants m and M such that for $t \in [0, \|\Psi\|_\infty]$,

$$m \le f(\cdot, t) \le M.$$

With φ serving as a lower bound for u_ε and the above L^∞ bound of $f(\cdot, u_\varepsilon(\cdot))$, using an argument similar to Theorem 1 in [6], we can extract a positive u from sequence $\{u_\varepsilon\}$ on every compact subset of Ω. To be precise, we first apply the Hölder estimates in [8] and find that $\|u_\varepsilon\|_{C^\alpha(B_{R/2})} \le C$, where $\alpha \in (0,1)$ and C are independent of ε. With this estimate of the coefficients of (16), using condition **C** and the boundedness of u_ε, we apply the standard interior Schauder estimates [8] and get $\|u_\varepsilon\|_{C^{2,\alpha}(B_{R/4})} \le C_1$. By the standard compactness argument (Arzela-Ascoli theorem), there exists a $C^{2,\alpha'}(B_{R/4})$-convergent subsequence for any $\alpha' < \alpha$. Now for any compact subset Ω' of Ω, a covering argument immediately gives a $C^{2,\alpha'}(\Omega')$-convergent subsequence. With a diagonalization argument if necessary, we obtain a subsequence of u_ε which converges in $C^2_{loc}(\Omega)$ to a limit $u \in C^2(\Omega)$ and u satisfies (1) in Ω. Moreover, $0 < u \le \Psi$ in Ω.

To complete the proof, we need to prove that u satisfies the zero Dirichlet boundary condition and is continuous up to boundary. We use an argument similar to Theorem 1 in [6]. Since Ω is convex, for any $x \in \partial\Omega$, there is a tangent line T_x passing through x with Ω lying on one side of T_x. We can also find T'_x parallel to T_x such that Ω is contained in the semi-infinite strip enclosed by T_x and T'_x. Without loss of generality, we let x be the origin. Take orthogonal coordinate axes $y_1, \dots, y_{N-1}$ and y so that the y-axis is perpendicular to T_x. Let β_i be the angle that the y-axis makes with the x_i-axis for $i = 1, \dots, N-1$, and β be the angle that the y-axis makes with the the x_N-axis. So, we can write $y = \sum_{i=1}^{N-1} x_i \cos\beta_i + x_N \cos\beta$.

We consider two cases.

Case1: Suppose $\beta \neq \pi/2$, $\beta \neq 3\pi/2$. Let $\psi = \psi(y)$ be a positive solution of

$$\begin{gathered} \psi_{yy} + \frac{M+1}{c_N \psi^{a_N}} = 0\,, \\ \psi\big|_{T_x} = \psi\big|_{T'_x} = 0\,, \end{gathered} \tag{18}$$

where c_N is the positive constant from condition **C1** and M is an upper bound of f. We know that ψ exists, $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$, $\psi > 0$, and $\psi_{yy} \leq 0$ (see equation (7)). Then define $\psi(x) \equiv \psi(y(x))/\cos^2\beta$. It is easy to check that for $i = 1, \dots, N-1$, $\psi_{x_i x_i} = \psi_{yy} \frac{\cos^2\beta_i}{\cos^2\beta} \leq 0$, and $\psi_{x_N x_N} = \psi_{yy} \leq 0$. By conditions **C** and **C1**, we get

$$\begin{aligned} &\sum_{i=1}^{N-1} A_i(\psi+\varepsilon)\psi_{x_i x_i} + A_N(\psi+\varepsilon)\psi_{x_N x_N} + (M+1) \\ &\leq \quad A_N(\psi+\varepsilon)\left(\psi_{yy} + \frac{M+1}{c_N(\psi+\varepsilon)^{a_N}}\right) \\ &\leq \quad A_N(\psi+\varepsilon)\left(\psi_{yy} + \frac{M+1}{c_N \psi^{a_N}}\right) \\ &\leq \quad 0. \end{aligned}$$

Hence by the comparison lemma in section 2, this ψ is an upper solution of u_ε on $\overline{\Omega}$, and $\psi \geq u_\varepsilon$. We can take the limit as $\varepsilon \to 0$ to obtain $0 < u \leq \psi$ on Ω.

Case2: Suppose $\beta = \pi/2$ or $3\pi/2$. Then define $\psi = \psi(x)$ to be a positive solution of

$$\begin{gathered} \psi_{x_1 x_1} + \tfrac{M+1}{c_1 \psi^{a_1}} = 0\,, \\ \psi\big|_{T_x} = \psi\big|_{T'_x} = 0\,, \end{gathered} \tag{19}$$

where M is the same constant as above, and c_1 and a_1 are from condition **C1**. Then an argument similar to Case1 implies

$$A_1(\psi+\varepsilon)\psi_{x_1 x_1} + \sum_{i=2}^{N} A_i(\psi+\varepsilon)\psi_{x_i x_i} + (M+1) \leq 0.$$

Therefore, the inequality $u \leq \psi$ holds.

Now in both cases, we define $u\big|_{\partial\Omega} = 0$. Take any $y \in \Omega$. Since

$$|u(y) - u(x)| = u(y) \leq \psi(y) \to 0$$

as $y \to x$, u is continuous at $x \in \partial\Omega$. Since x can be chosen as an arbitrary point on $\partial\Omega$, u is in $C^2(\Omega) \cap C(\overline{\Omega})$, and u satisfies equations (1) and (2). This completes the proof. $\diamondsuit$

4 Superlinear Case.

We establish Theorem 2 in this section. In addition to a singularity in Theorem 1, we also have superlinear growth for the source term. Hence, obtaining a uniform bound is more difficult in this case. Before we discuss the main proof, in section 4.1, we describe a construction in [3]. This method will allow us to approximate equations (1) and (2) by a sequence of nonsingular problems. Moreover, the solutions to these nonsingular problems will have a monotonicity property so that their limit will satisfy equations (1) and (2). In section 4.2, we prove existence results for small λ and show the second part of Theorem 2. Finally, in section 4.3, we complete the proof of Theorem 2.

4.1 Monotone Iteration Lemma

We state the monotone iteration lemma for quasilinear problems. The proof of this result can be found in Lemma 4 and Theorem 2 in [3].

Consider the boundary value problem

$$\begin{gathered} Q(u) \equiv \textstyle\sum_{i=1}^{N} A_i(u) \frac{\partial^2 u}{\partial x_i^2} + f(\mathbf{x}, u) = 0\,, \\ u\big|_{\partial\Omega} = u_0 > 0\,, \end{gathered} \tag{20}$$

where u_0 is a constant.

An UPPER SOLUTION to this problem is a function $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying $Q(\psi) \leq 0$ and $\psi\big|_{\partial\Omega} \geq u_0$.

A LOWER SOLUTION is a function $\phi \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying $Q(\phi) \geq 0$, and $\phi\big|_{\partial\Omega} = u_0$.

Then the monotone iteration lemma is as follows.

Lemma 4 *Let the conditions in Theorem 2 hold, let ϕ be a non-negative lower solution and ψ be an upper solution, with $\phi \leq \psi$ on Ω. Assume both are in $C^2(\overline{\Omega})$. Then there exists a solution $u \in C^2(\overline{\Omega})$ to the boundary value problem (20) with $\phi \leq u \leq \psi$.*

Proof: Write $v \equiv u - u_0$ so that (20) becomes $Q(u) = Q(v + u_0) = \sum A_i(v + u_0)v_{x_i x_i} + f(\mathbf{x}, v + u_0) = 0$ and $v\big|_{\partial\Omega} = 0$. Thus by $u_0 > 0$ and assumption **C**, we obtain strict ellipticity of Q. Therefore, we apply the same proof as in [3] to establish the lemma. ♢

Remark: In [3], we established this lemma by using an auxiliary parabolic problem with a lower solution as an initial condition. It was required to have a compatibility of zero order, that is, $u_0 = \phi$ on $\partial\Omega$, to obtain the result. This accounts for the equality sign in the boundary condition in the definition of lower solutions.

Define $L \equiv \{\lambda \in [0,\infty) :$ (1) and (2) have a positive solution in $C^2(\Omega) \cap C(\overline{\Omega})\}$, and define Q to be the operator:

$$Q(u,\lambda) \equiv \sum_{i=1}^{N} A_i(u) \frac{\partial^2 u}{\partial x_i^2} + \lambda f(\mathbf{x}, u). \tag{21}$$

We will show that L is bounded in the following section.

4.2 Boundedness of L

Lemma 5 *Assume the same hypothesis as in Theorem 2. Then L is non-empty.*

Proof: Let $\varepsilon_1 \equiv 1$. By Theorem 2 in [3], we know that there exists a Λ such that for each $0 \le \lambda < \Lambda$, there exists a positive solution $u \in C^2(\overline{\Omega})$ of

$$Q(u,\lambda) = 0, \quad \text{and} \quad u\big|_{\partial\Omega} = \varepsilon_1 > 0. \tag{22}$$

We can see this by denoting $w \equiv u - \varepsilon_1$. Then equation (22) becomes $Q(w + \varepsilon_1, \lambda) = 0$ with the boundary condition $w\big|_{\partial\Omega} = 0$ so that a positive solution w exists for $\lambda < \Lambda$. Hence $u > \varepsilon_1$.

Let $0 < \lambda_1 < \Lambda$ and u_1 be the corresponding minimal solution associated with the construction. Let $0 < \varepsilon_2 < \varepsilon_1$. Then it is easy to check that u_1 satisfies $Q(u_1, \lambda_1) = 0$, and $u_1\big|_{\partial\Omega} \ge \varepsilon_2$, which is an upper solution, and $\phi \equiv \varepsilon_2$ satisfies $Q(\phi, \lambda_1) \ge 0$, $\phi\big|_{\partial\Omega} = \varepsilon_2$, which is a lower solution. Clearly, $u_1 > \varepsilon_2$. Hence, we can apply lemma 4, the monotone iteration lemma, to obtain a solution u_2 of $Q(u_2, \lambda_1) = 0$ and $u_2\big|_{\partial\Omega} = \varepsilon_2$, satisfying

$$u_1(\mathbf{x}) \ge u_2(\mathbf{x}) \ge \varepsilon_2 > 0.$$

For any decreasing sequence $\{\varepsilon_n\}$, a repetition of the argument yields to a sequence of functions u_n which satisfy $Q(u_n, \lambda_1) = 0$, and $u_n\big|_{\partial\Omega} = \varepsilon_n$, and $u_1 \ge u_2 \ge \ldots \ge u_n \ge \varepsilon_n > 0$ in $\overline{\Omega}$. The usual regularity argument ensures that $u_n \in C^{2+\alpha}(\overline{\Omega})$.

Since $f(\cdot, u_n(\cdot))$ has a uniform L^∞ bound, then by virtue of the non-trivial lower solution $\varphi = \delta(R^2 - |\mathbf{x}|^2)$ as in Theorem 1 and Lemma 1, we can apply the same compactness argument to extract a limit u from a subsequence of $\{u_n\}$. Moreover $u > 0$ on every compact subset of Ω, and $Q(u, \lambda_1) = 0$ in Ω. Define u to be zero on the boundary of Ω.

We now need to prove that u is continuous up to the boundary. Using an argument similar to Theorem 1, we construct an upper barrier function. Let $y = \sum_{i=1}^{N-1} x_i \cos\beta_i + x_N \cos\beta$, and consider two cases as in Theorem 1.

The first case is when $\beta \ne \pi/2$, $\beta \ne 3\pi/2$. Fix β and define $M \equiv \sup_{t\in[0,\|u_1\|_\infty]} f(\cdot, t)$ to be a uniform upper bound. We let $\psi_1 = \psi_1(y)$ be a positive solution of equation (18) with the constant of the source term chosen to be $\lambda_1 M + 1$. Define $\psi(x) \equiv \psi_1(y)/\cos^2\beta$. Define $w_n \equiv u_n - \varepsilon_n$. Then w_n is a positive solution of

$$\sum_{i=1}^{N-1} A_i(w_n+\varepsilon_n)w_{n x_i x_i} + A_N(w_n+\varepsilon_n)w_{n x_N x_N} + \lambda_1 f(\mathbf{x}, w_n+\varepsilon_n) = 0 \quad (23)$$
$$w_n\big|_{\partial\Omega} = 0\,.$$

In (23), applying the comparison lemma to ψ and w_n, we get $\psi \geq w_n = u_n - \varepsilon_n > 0$ for each n. Hence take the limit as $\varepsilon_n \to 0$ to obtain $0 < u \leq \psi$ on Ω.

In the second case, when $\beta = \pi/2$ or $3\pi/2$, we also define $\psi = \psi(x)$ to be a positive solution of equation (19) with the constant $\lambda_1 M + 1$ for the source term. Then by the same calculation, $0 < u \leq \psi$ on Ω.

In both cases, ψ is an upper barrier function for u. Therefore $u \in C(\overline{\Omega})$. Hence we have established a positive solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of equations (1) and (2) for $\lambda = \lambda_1$. $\diamondsuit$

This lemma implies that for each $\lambda \leq \lambda_1$, $\lambda \in L$. Hence, we get $(0, \lambda_1] \subset L$.

Lemma 6 *Assume the hypothesis of Theorem 2 hold. Then L is bounded.*

Proof: Let u be a positive solution in $C^2(\Omega) \cap C(\overline{\Omega})$ of

$$u_{x_1x_1} + \textstyle\sum_{i=2}^{N} \frac{A_i(u)}{A_1(u)} \frac{\partial^2 u}{\partial x_i^2} + \lambda \frac{f(\mathbf{x},u)}{A_1(u)} = 0\,, \quad (24)$$
$$u\big|_{\partial\Omega} = 0\,,$$

where $\lambda \geq \lambda_1/2$. We note that for each compact subset $B \subset\subset \Omega$, for given $0 < R < 1$, there exist a finite open covering $B_R(\mathbf{x}_i)$ and a $\delta > 0$ such that (17) holds. Hence for given $B \subset\subset \Omega$, we can choose a $\delta > 0$ small such that $u \geq \delta$ on ∂B. Note that δ is independent of u. Moreover, $u \geq \delta$ in $\overline{B}$. Now fix B and δ.

Let $\phi \in C^2(\overline{B})$, positive, and $\phi = 0$ on ∂B. Multiplying equation (24) by ϕ, and applying integration by parts on B, we have

$$\begin{aligned} 0 &= \int_B \phi \left\{ u_{x_1x_1} + \sum_{i=2}^{N} \frac{A_i(u)}{A_1(u)} u_{x_ix_i} + \lambda \frac{f(\mathbf{x},u)}{A_1(u)} \right\} \\ &= \int_{\partial B} \phi u_{x_1} \cdot n_1 + \int_{\partial B} \sum_{i=2}^{N} \phi \frac{A_i(u)}{A_1(u)} u_{x_i} \cdot n_i - \int_B \phi_{x_1} u_{x_1} \\ &\quad - \int_B \sum_{i=2}^{N} \phi_{x_i} \frac{A_i(u)}{A_1(u)} u_{x_i} - \int_B \sum_{i=2}^{N} \phi \left(\frac{A_i(u)}{A_1(u)} \right)' u_{x_i}^2 + \int_B \lambda \phi \frac{f(\mathbf{x},u)}{A_1(u)} \end{aligned}$$

where $\mathbf{n}$ is the unit outward normal to ∂B. Since $(A_i(u)/A_1(u))' \leq 0$ by condition **C2**, $\phi = 0$ on ∂B, and by applying integration by parts once more, we get

$$0 \;\geq\; -\int_B \phi_{x_1} u_{x_1} - \int_B \sum_{i=2}^{N} \phi_{x_i} \frac{A_i(u)}{A_1(u)} u_{x_i} + \int_B \lambda \phi \frac{f(\mathbf{x},u)}{A_1(u)}$$

$$\begin{aligned} &= -\int_{\partial B} \phi_{x_1} u \cdot n_1 - \int_{\partial B} \sum_{i=2}^{N} \phi_{x_i} \frac{A_i(u)}{A_1(u)} u \cdot n_i \\ &\quad + \int_B \left(\phi_{x_1 x_1} + \sum_{i=2}^{N} \left\{ \phi_{x_i} \frac{A_i(u)}{A_1(u)} \right\}_{x_i} \right) u + \int_B \lambda \phi \frac{f(\mathbf{x}, u)}{A_1(u)}. \qquad (25) \end{aligned}$$

By condition **C1**, for every $t \geq \delta$, we get

$$\frac{A_i(t)}{A_1(t)} \leq \frac{d_i}{c_1 t^{a_1 - a_i}} \leq \frac{d_i}{c_1 \delta^{a_1}} \equiv M < \infty,$$

for all $2 \leq i \leq N$. We now choose $\phi > 0$ such that it is the first eigenfunction satisfying

$$\begin{gathered} \phi_{x_1 x_1} + \textstyle\sum_{i=2}^{N} \left(\frac{A_i(u) \phi_{x_i}}{A_1(u)} \right)_{x_i} + \beta \phi = 0\,, \\ \phi\big|_{\partial B} = 0\,, \end{gathered} \qquad (26)$$

with eigenvalue $\beta > 0$. Thus

$$\begin{aligned} \beta &\equiv \min_{\phi \in H_0^1(B)} \frac{\int_B \phi_{x_1}^2 + \int_B \sum_{i=2}^{N} \frac{A_i(u)}{A_1(u)} \phi_{x_i}^2}{\int_B \phi^2} \\ &\leq \min_{\phi \in H_0^1(B)} \frac{\int_B \phi_{x_1}^2 + M \sum_{i=2}^{N} \int_B \phi_{x_i}^2}{\int_B \phi^2} \equiv \beta_M < \infty \end{aligned}$$

where β_M is independent of u.

By conditions **C1**, **F** and **F2**, we can find a small $k > 0$ such that

$$\frac{f(\mathbf{x}, t)}{A_1(t)} \geq kt$$

for all $t > 0$ and for all $\mathbf{x} \in \overline{\Omega}$.

Since $\phi > 0$ on B, zero on the boundary, and $\phi_{x_i} \cdot n_i \leq 0$, $u > 0$ and $A_i(u)/A_1(u) > 0$ on the boundary of B for each i, we get

$$\int_{\partial B} \phi_{x_1} u \cdot n_1 \leq 0,$$

and

$$\int_{\partial B} \phi_{x_i} \frac{A_i(u)}{A_1(u)} u \cdot n_i \leq 0,$$

for each $i = 2, \ldots, N$. Hence inequality (25) becomes

$$0 \quad \geq \quad -\beta \int_B \phi u + \lambda k \int_B \phi u.$$

Thus we get $\lambda \leq \beta/k \leq \beta_M/k$ which completes the present proof. $\diamondsuit$

Define $\lambda^* \equiv \sup_\lambda L$. Then by this lemma, we know that such λ^* exists and no solution exists for $\lambda > \lambda^*$.

Now, we establish the existence result for each $\lambda < \lambda^*$. We have established existence for $\lambda \in (0, \lambda_1]$ in Lemma 5, and by the definition of λ^*, $\lambda_1 \leq \lambda^*$. In the following section, we prove the existence result for $\lambda < \lambda^*$.

4.3 Completion of the Proof of Theorem 2 (Superlinear Case).

We first assume that for a given $\hat{\lambda} > 0$, equations (1) and (2) have a positive solution $\hat{u}$, that is, $\hat{\lambda} \in L$. We claim that for each $\lambda < \hat{\lambda}$, equations (1) and (2) have a positive solution.

Consider $\lambda < \hat{\lambda}$. Given $\varepsilon > 0$, define $\Omega_\varepsilon \equiv \{\mathbf{x} \in \Omega | dist(\mathbf{x}, \partial\Omega) \geq \varepsilon\}$. From inequality (17) (by taking $R = \varepsilon$), we know that for any $0 < \varepsilon_0 \leq 1$ there exists a $\delta > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, $\hat{u} \geq \delta\varepsilon > 0$ on $\partial\Omega_\varepsilon$. We now consider

$$Q(u, \lambda) = 0 \quad \text{in} \quad \Omega_\varepsilon \quad \text{with} \quad u\big|_{\partial\Omega_\varepsilon} = \delta\varepsilon\,. \tag{27}$$

Since $\hat{u}$ satisfies $Q(\hat{u}, \lambda) \leq 0$ and $\hat{u}\big|_{\partial\Omega_\varepsilon} \geq \delta\varepsilon$, it is an upper solution of (27) in Ω_ε. It is easy to check that $\delta\varepsilon$ is a lower solution of (27) in Ω_ε. Moreover, $\hat{u} \geq \delta\varepsilon$. Thus, applying the monotone iteration lemma, we get a solution u_ε which solves (27) and satisfies

$$\delta\varepsilon \leq u_\varepsilon \leq \hat{u}\,.$$

Using the same argument as in Theorem 1, we extract a solution u from a subsequence of $\{u_\varepsilon\}$, which is bounded away from zero in any compact subset of Ω. Moreover, $Q(u, \lambda) = 0$ in Ω, and $u \leq \hat{u}$.

Define $u = 0$ on the boundary of Ω. We now want to show that $u \in C(\overline{\Omega})$. Using an argument similar to Theorem 1, we consider the following upper barrier function.

Let $y = \sum_{i=1}^{N-1} x_i \cos\beta_i + x_N \cos\beta$, and consider two cases as before. For the first case $\beta \neq \pi/2$, $\beta \neq 3\pi/2$, define $M \equiv \sup_{t\in[0,\|\hat{u}\|_\infty]} f(\cdot, t)$ to be a uniform upper bound, and $\psi = \psi(y)$ to be a positive solution of equation (18) with the constant of the source term $M_1 \equiv \lambda M$. Define $\hat{\psi} \equiv k\psi(y)/\cos^2\beta$ where $k \geq 1$.

Then by repeating as calculation similar to Theorem 1 on $\overline{\Omega}_\varepsilon$, we get

$$\begin{aligned} \sum_{i=1}^{N-1} A_i(\hat{\psi})\hat{\psi}_{x_i x_i} + A_N(\hat{\psi})\hat{\psi}_{x_N x_N} + (M_1 + 1) \quad &\leq \quad A_N(\hat{\psi})\left(k\psi_{yy} + \frac{M_1 + 1}{c_N (k\psi)^{a_N}}\right) \\ &\leq \quad 0\,. \end{aligned}$$

Owing to the growth rate of ψ, as given by [11], by taking k large if necessary, we can always ensure that $\hat{\psi}\big|_{\partial\Omega_\varepsilon} \geq \delta\varepsilon$. Note that k does not depend on ε. Hence by the comparison lemma, we obtain the inequality $u_\varepsilon \leq \hat{\psi}$ on $\overline{\Omega}_\varepsilon$. Taking the limit as $\varepsilon \to 0$, we obtain $u \leq \hat{\psi}$ on $\overline{\Omega}_\varepsilon$. Since Ω_ε is arbitrarily close to Ω , it follows that $0 < u \leq \hat{\psi}$ on Ω.

For the second case $\beta = \pi/2$ or $3\pi/2$, the tangent plane is orthogonal. Define $\hat{\psi} = \psi(x)$ to be a positive solution of equation (19) with the same constant M_1 in the source term. Then again with the same calculation, we get $\hat{\psi} \geq u_\varepsilon$ for each ε on Ω_ε.

In both cases, $\hat{\psi}$ is an upper barrier function for u. Applying an argument similar to Theorem 1, we get $u \in C(\overline{\Omega})$. Therefore, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and satisfies equations (1) and (2). This completes the proof of Theorem 2. $\diamondsuit$

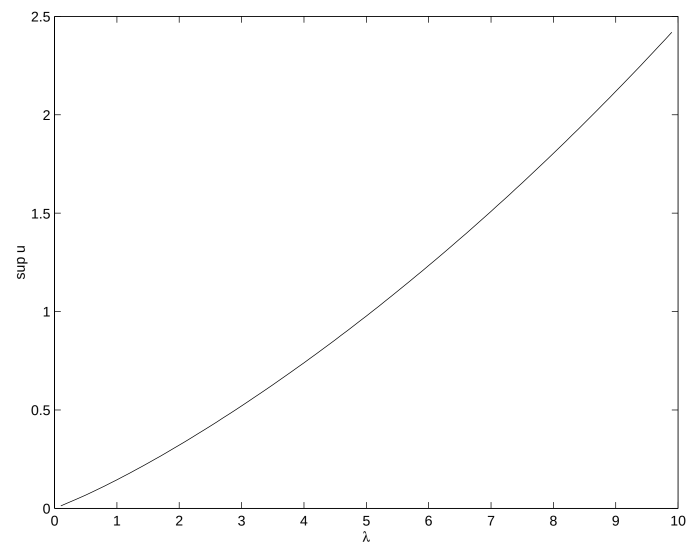

Figure 1: $uu_{xx} + u_{yy} + \lambda(u+1)^{1.5} = 0$

5 Open Problems

We conclude our paper by presenting some numerical results and discussing open questions. Typical examples of sublinear and superlinear anisotropic problems are tested, see Figures 1 and 2. In Figure 3, we compare three bifurcation diagrams, where ε takes the values $\varepsilon = 1$, $\varepsilon = 0.5$ and $\varepsilon = 0$ respectively. This figure well illustrates that $\lambda_1 \approx 1.2$ is smaller than $\lambda^* \approx 2.4$, where λ_1 and λ^* are the bifurcation points when $\varepsilon = 1$ and $\varepsilon = 0$, respectively. Details of the numerical method can be found in [3, 9].

We now pose some open theoretical questions. One would like to study the uniqueness of the solution of the sublinear case. In *Figure 1*, it appears that a unique solution exists for each $\lambda > 0$.

Further investigation of the superlinear case is required. *Figure 2* shows that there exists $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$, multiple solutions appear, and for $\lambda > \lambda^*$, no solution exists. It is an open question to prove the existence of

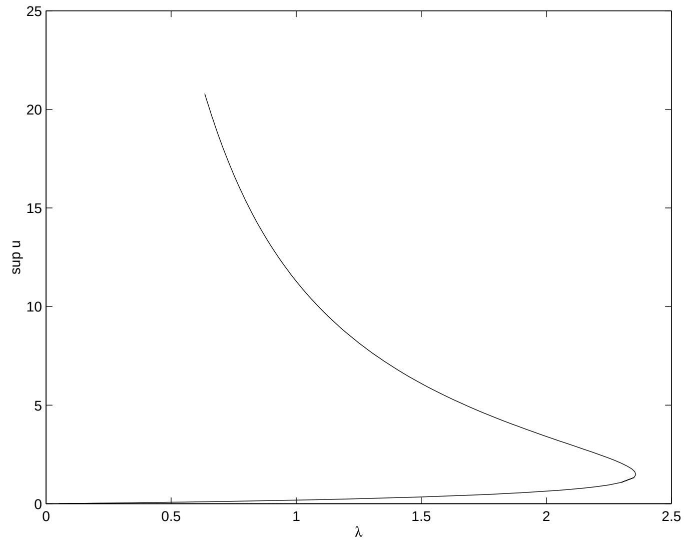

Figure 2: $uu_{xx} + u_{yy} + \lambda(u+1)^3 = 0$

multiple solutions for $0 < \lambda < \lambda^*$.

The next natural question is, how we can extend to more general classes of quasilinear degenerate problems, such as

$$\begin{gathered}\sum_{i,j=1}^{N} A_{i,j}(\mathbf{x}, \bigtriangledown u, u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} B_i(\mathbf{x}, \bigtriangledown u, u)\frac{\partial u}{\partial x_i} + f(\mathbf{x}, u) = 0 \\ u\big|_{\partial\Omega} = 0\,. \end{gathered} \tag{28}$$

Beyond extensions of the class of anisotropic elliptic problems as in [3], the approach in this paper might be useful in studying other types of degenerate elliptic problems. Our interest focused on the singular anisotropic elliptic problem where the singularity occurs at the boundary of the region, and the nonlinear source term has either sublinear or superlinear growth rate relative to the elliptic coefficients.

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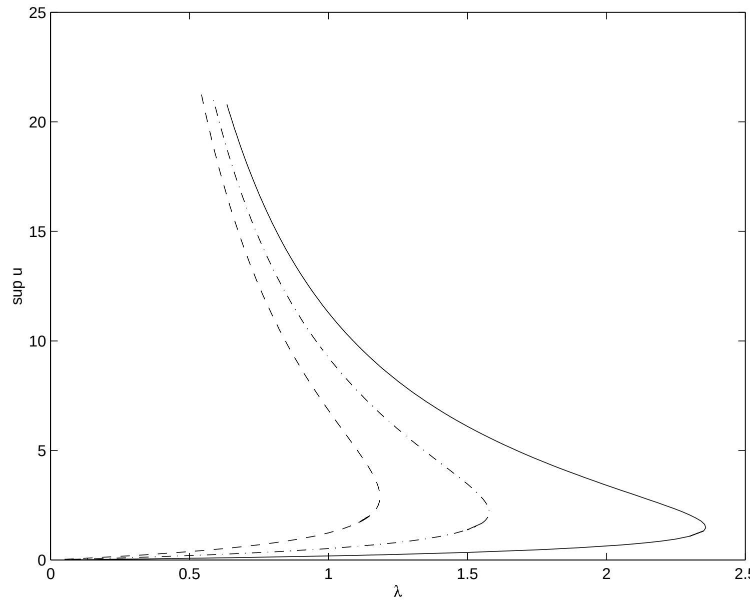

Figure 3: Bifurcations of $(u+\varepsilon)u_{xx}+u_{yy}+\lambda(u+1+\varepsilon)^3=0$ and $u\big|_{\partial\Omega}=0$, where '– –' (left) is the case when $\varepsilon=1$, '-.-' (middle) is $\varepsilon=0.5$, and '—' (right) is $\varepsilon=0$.

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