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Non-degenerate implicit evolution inclusions *

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Abstract

We prove the existence of solutions for the implicit evolution inclusion

$$(B(t)u(t))' + A(t, u(t)) \ni f(t)$$

under conditions that are easy to verify on the set valued operator $A(t, \cdot)$ and that do not imply the operator is monotone. We also present an example where our existence theorem applies to a time dependent implicit inclusion.

1 Introduction

There are many works which deal in the theory of implicit evolution equations of the form

$$(Bu)' + Au = f.$$

In the case where A is monotone and B is linear, the book by Carroll and Showalter [1], gives many of the best theorems allowing the equation to be replaced by $\ni$ thus including evolution inclusions. Equations of this form have also been discussed by many authors in the case where A is some sort of single valued operator from a Banach space to its dual which may fail to be monotone. See for example, Lions [2], Bardos and Brezis [3], [4], or [5]. More recently these theorems have been generalized to include the case where A may be non monotone and set valued, a recent paper being [6].

The paper [6] includes as a special case the situation where V and W are separable reflexive Banach spaces satisfying

$$V \hookrightarrow W \hookrightarrow W' \hookrightarrow V' \tag{1.1}$$

and V is dense in W along with a family of linear operators, $B(t) \in \mathcal{L}(W, W')$ satisfying

$$\langle B(t)u, v\rangle = \langle B(t)v, u\rangle, \tag{1.2}$$

$$\langle B(t)u, u\rangle \geq 0, \tag{1.3}$$

$$B(t) = B(0) + \int_0^t B'(s)\, ds, \tag{1.4}$$

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for all $u, v \in W$, and $B' \in L^\infty(0,T;\mathcal{L}(W,W'))$.

In the above formulae, $\langle \cdot, \cdot \rangle$ denotes the duality pairing of the Banach space, W, with its dual space. We will use this notation in the present paper, the exact specification of which Banach space being determined by the context in which this notation occurs. Thus in the above, it is clear from context, since $B(t) \in \mathcal{L}(W,W')$, that the Banach space is W. We use this notation throughout the present paper to make the presentation less cluttered with symbols. Occasionally, when it is desired to emphasize which Banach space is meant, we will write the symbol in the form $\langle v, u \rangle_{X',X}$, thus indicating the duality pairing between X and X'. We will also use the notation, $\|u\|_Y$ to denote the norm of u in the space Y where Y is a Banach space or in the form $\|u\|$ when it is clear which space is meant. The symbol, $L^p(0,T;Y)$ denotes the space of strongly measurable Y valued functions, f, for which

$$\int_0^T \|f(t)\|_Y^p \, dt < \infty$$

in the case where $p \geq 1$, and the essential supremum of the function, $t \to \|f(t)\|$ in the case where $p = \infty$. For further discussion of these spaces we refer to [7].

In [6], there is also a set valued operator, A, pseudo-monotone in the sense of [8], which maps elements of a solution space, X, defined below, to $\mathcal{P}(X')$ the power set of X'. The function, $f \in L^{p'}(0,T;V') \equiv \mathcal{V}'$ is also given. The paper, [6] includes existence theorems for the following implicit inclusion,

$$(Bu)' + Au \ni f \text{ in } \mathcal{V}',\ Bu(0) = B(0)u_0,\ u_0 \in W, \tag{1.5}$$

where the prime denotes differentiation in the sense of V' valued distributions. Thus, for $\phi \in C_c^\infty(0,T)$,

$$(Bu)'(\phi) \equiv -\int_0^T B(t)u(t)\phi'(t)\, dt$$

The solution, u, is found in the space of solutions, X where

$$X \equiv \left\{ u \in L^p(0,T;V) \equiv \mathcal{V} : (Bu)' \equiv Lu \in L^{p'}(0,T;V') \equiv \mathcal{V}' \right\} \tag{1.6}$$

and $p \geq 2$ is always assumed. Thus, u is a solution to (1.5) if $u \in \mathcal{V}$, $(Bu)' \in \mathcal{V}'$, and there exists $\xi \in Au \subseteq \mathcal{V}'$ such that along with the initial condition, whose precise meaning is given below, we have the following equation.

$$(Bu)' + \xi = f\,.$$

The meaning of the initial condition is dependent on the following theorem about the space of solutions, X, a special case of one proved in [6]. It is less general because in [6], it is not assumed V is dense in W and more general function spaces are considered. See also [5] for a slightly less general version of the same theorem.

Theorem 1.1 *Let $u, v \in X$, then the following hold.*

1. $t \to \langle B(t)u(t), v(t)\rangle_{W',W}$ *equals an absolutely continuous function a.e. t, denoted by $\langle Bu, v\rangle(\cdot)$.*
2. $\langle Lu(t), u(t)\rangle = \frac{1}{2}\left[\langle Bu, u\rangle'(t) + \langle B'(t)u(t), u(t)\rangle\right]$ *a.e. t.*
3. $|\langle Bu, v\rangle(t)| \leq C\|u\|_X \|v\|_X$ *for some $C > 0$ and for all $t \in [0, T]$.*
4. $t \to B(t)u(t)$ *equals a function in $C(0, T; W')$, a.e. t, denoted by $Bu(\cdot)$.*
5. $\sup\{\|Bu(t)\|_{W'}, t \in [0, T]\} \leq C\|u\|_X$ *for some $C > 0$.*
6. *Let $K : X \to X'$ be given by*
$$\langle Ku, v\rangle_{X',X} \equiv \int_0^T \langle Lu(t), v(t)\rangle dt + \langle Bu, v\rangle(0)\,.$$
Then K is linear, continuous and weakly continuous.
7. $\langle Ku, u\rangle = \frac{1}{2}[\langle Bu, u\rangle(T) + \langle Bu, u\rangle(0)] + \frac{1}{2}\int_0^T \langle B'(t)u(t), u(t)\rangle dt.$

In proving this theorem it is shown that $C^\infty([0, T]\,; V)$, the space of infinitely differentiable functions having values in V is dense in X. Therefore, we also obtained the following formula which is valid for all $u \in X$.

$$Bu(t) = Bu(0) + \int_0^t (Bu)'(s)ds \tag{1.7}$$

Here $Bu(0) \in W'$.

From this theorem we see that we can define a continuous function which equals $B(t)u(t)$ a.e. and therefore, we can give a meaning to the expression $Bu(0) = B(0)u$. The operator A is assumed to be a set valued pseudo-monotone map from X to $\mathcal{P}(X')$. Following [8] these operators are given according to the following definition.

Definition 1.2 *We say $A : X \to \mathcal{P}(X')$ is pseudo-monotone if the following hold.*

1. *The set Au is non-empty, bounded, closed and convex for all $u \in X$.*
2. *If F is a finite dimensional subspace of X, $u \in F$, and if U is a weakly open set in V' such that $Au \subseteq U$, then there exists a $\delta > 0$ such that if $v \in B_\delta(u) \cap F$ then $Av \subseteq U$.*
3. *If $u_i \to u$ weakly in X and $u_i^* \in Au_i$ is such that*
$$\limsup_{i\to\infty} \langle u_i^*, u_i - u\rangle \leq 0, \tag{1.8}$$
then, for each $v \in X$, there exists $u^*(v) \in Au$ such that
$$\liminf_{i\to\infty} \langle u_i^*, u_i - v\rangle \geq \langle u^*(v), u - v\rangle. \tag{1.9}$$

The following lemma shows the second condition in the above list follows as a special case of something which can be proved if it is assumed that A is bounded in addition to Conditions (3) and (1). First we give a definition of what we mean by bounded.

Definition 1.3 *Let $A : X \to \mathcal{P}(X')$ be a set valued map. We say A is bounded if for each bounded set, $G \subseteq X$,*

$$\sup \{\|z\| : z \in Ax, x \in G\} < \infty.$$

Thus A is bounded if the norms of all possible elements of Ax for $x \in G$ are bounded above.

Lemma 1.4 *Let $A : X \to \mathcal{P}(X')$ satisfy conditions (1) and (3) above and suppose A is bounded. Then if $x_n \to x$ in X, and if U is a weakly open set containing Ax, then $Ax_n \subseteq U$ for all n large enough.*

Proof: If this is not true, there exists $x_n \to x$, a weakly open set, U, containing Ax and $z_n \notin Ax_n$, but $z_n \notin U$. Taking a subsequence if necessary, we obtain a sequence which satisfies $z_n \rightharpoonup z \notin U$ in addition to this. Then

$$\limsup_{n\to\infty} \langle z_n, x_n - x\rangle = 0$$

so if $y \in X$ there exists $z(y) \in Ax$ such that $\langle z, x-y\rangle = \liminf_{n\to\infty}\langle z_n, x_n-y\rangle \geq \langle z(y), x - y\rangle$. Letting $w = x - y$, this shows, since $y \in X$ is arbitrary, that the following inequality holds for every $w \in X$.

$$\langle z, w\rangle \geq \langle z(x - w), w\rangle.$$

In particular, we may replace w with $-w$ and obtain

$$\langle z, -w\rangle \geq \langle z(x + w), -w\rangle,$$

which implies

$$\langle z(x - w), w\rangle \leq \langle z, w\rangle \leq \langle z(x + w), w\rangle.$$

Therefore, there exists

$$z_\lambda(y) \equiv \lambda z(x - w) + (1 - \lambda)z(x + w) \in Ax$$

such that $\langle z, w\rangle = \langle z_\lambda(y), w\rangle$. But this is a contradiction to $z \notin Ax$ because if $z \notin Ax$ there exists $w \in X$ such that $\langle z, w\rangle > \langle z_1, w\rangle$ for all $z_1 \in Ax$. Therefore, $z \in Ax$ which contradicts the assumption that z_n and consequently z are not contained in U.

This condition which implies Condition 2 above is known as Upper semicontinuity with respect to the strong topology on X and the weak topology on X'. If A were single valued, this would be called demicontinuity because it would imply that the image of strongly convergent sequences converges weakly.

The importance of this lemma is that it shows Condition 2 is redundant if it is assumed that A satisfies the other two conditions and is bounded.

For $B(t)$ defined above, we may consider the operators, B' and B as maps from $\mathcal{V}$ to $\mathcal{V}'$ according to the definition,

$$\langle B'u, v\rangle \equiv \int_0^T \langle B'(t)u(t), v(t)\rangle dt,\ \langle Bu, v\rangle \equiv \int_0^T \langle B(t)u(t), v(t)\rangle dt$$

and we shall follow this convention throughout the paper. Note the same notation also applies to these operators considered as maps from $L^2(0,T;W)$ to $L^2(0,T;W')$ since $p \geq 2$ and V is assumed to be dense in W.

In [6] an existence theorem is given which contains the following result as a special case.

Theorem 1.5 *Let B, X, W, and V be as defined above, $f \in \mathcal{V}'$, $u_0 \in W$, and suppose $A : \mathcal{V} \to \mathcal{P}(\mathcal{V}')$ is such that A is set valued, bounded as a map from $\mathcal{V}$ to $\mathcal{P}(\mathcal{V}')$, and pseudo-monotone when considered as a map from X to $\mathcal{P}(X')$. Then if $A + \frac{1}{2}B'$ is coercive,*

$$\lim_{\|u\|_{\mathcal{V}}\to\infty} \frac{\langle Au, u\rangle + \frac{1}{2}\langle B'u, u\rangle}{\|u\|_{\mathcal{V}}} = \infty,$$

it follows there exists a solution, $u \in X$ to

$$(Bu)' + Au \ni f,\ Bu(0) = Bu_0. \tag{1.10}$$

Furthermore, u is a solution to (1.10) if and only if u is a solution to the following equation which holds for all $v \in X$.

$$\langle Ku, v\rangle + \langle u^*, v\rangle = \langle f, v\rangle + \langle Bv(0), u_0\rangle \tag{1.11}$$

for some $u^* \in Au$.

While this theorem is very general, even allowing A to depend on the history of the function, u, the hypotheses are sometimes difficult to verify. Therefore, it is important to consider the special case in which the operator, A, is of the form $Au(t) = A(t, u(t))$ and to determine easy to verify conditions on the operators, $A(t,\cdot)$ which will imply the above conditions on A.

A paper by Bian and Webb, [9] gives such convenient conditions in the special case where $B = I$ and $W = H$, a Hilbert space with $H = H'$. Conditions are given on the operators, $A(t,\cdot)$ which make it possible to obtain an existence theorem for the evolution inclusion, (1.5) in the case where $u_0 \in V$. We will demonstrate that under appropriate conditions on B, including the case when $B = I$, their conditions actually imply the operator, A is Pseudo-monotone on the space, X, which makes possible the consideration of more general initial data. In the context of evolution inclusions they considered, it will mean we can take the initial data in H rather than only in V. When this theorem has been proved, we apply it to some existence theorems which follow from it and

Theorem 1.5. Next we consider a model problem for an implicit inclusion in which the nonlinear operators are obtained as a sum of operators considered by Browder [10] and a set valued operator. We conclude by giving a proof of a measurability result. Throughout the paper the symbol $\rightharpoonup$ will mean weak or weak $*$ convergence and the symbol $\to$ will mean strong convergence.

2 Pointwise pseudo-monotone maps

We will need to consider some sort of measurability condition for set valued operators, $S(t,\cdot)$ mapping the Banach space, V, described above, to $\mathcal{P}(V')$. There is quite a well developed theory of set valued maps found in [11], but for our purposes, we will say the operators, $S(t,\cdot)$ are measurable if the following condition holds.

If

$$\emptyset \neq F(t) \equiv \{w \in S(t, x(t)) : \langle w, x(t) - y(t)\rangle \leq \alpha(t)\} \tag{2.1}$$

for α measurable and $x, y \in \mathcal{V}$, then there exists $z \in \mathcal{V}'$ such that $z(t) \in F(t)$ a.e.

The next three conditions are modifications of conditions proposed by Bian and Webb, [9]. We let V and W be the reflexive Banach spaces defined in (1.1).

1. $v \to A(t,v)$ is a set valued pseudo-monotone map from V to $\mathcal{P}(V)$ satisfying conditions 1 and 3 in the definition of pseudo-monotone given in Section 1.

2. There exists $b_1 \geq 0$ and $b_2 \in L^{p'}(0,T)$ such that

$$\|z\|_{V'} \leq b_1 \|u\|_V^{p-1} + b_2(t) \text{ for all } z \in A(t,u)$$

3. There exists $b_3 > 0$, $\alpha \in (0,p)$, $b_4 \geq 0$, and $b_5 \in L^1(0,T)$ such that

$$\inf_{z \in A(t,u)} \langle z, u\rangle + \frac{1}{2}\langle B'(t)u, u\rangle \geq b_3 \|u\|_V^p - b_4 \|u\|_V^\alpha - b_5(t)$$

   and

$$\inf_{z \in A(t,u)} \langle z, u\rangle \geq b_3 \|u\|_V^p - b_4 \|u\|_V^\alpha - b_5(t).$$

   To these three conditions, we append the following.

4. The operators, $A(t,\cdot)$ are measurable in the sense of (2.1).

In Section 5 we show how Condition 4 follows from standard definitions of measurability, in particular, the assumptions in Bian and Webb, [9]. The first inequality of 3. implies we are assuming that $B'(t)$ cannot be too negative and we require both inequalities to hold.

In the case that $A(t,\cdot)$ is single valued, Condition 4 would be satisfied in the context of conditions (1) - (3) if we assumed, as it is reasonable to do, that $t \to A(t,v)$ is measurable. This is because the operators, $A(t,\cdot)$, being pseudo-monotone and bounded would be demicontinuous also. Therefore, if $x \in \mathcal{V}$ we could obtain x as a pointwise limit of simple functions, s_n for which $t \to A(t,s_n(t))$ is measurable and use the demicontinuity of $A(t,\cdot)$ to conclude $t \to A(t,x(t))$ is measurable. Then $F(t) = A(t,x(t))$ and the given estimates would imply $t \to A(t,x(t))$ is in $\mathcal{V}'$.

We will need the following definition.

Definition 2.1 *We define an operator,* $\widehat{A} : \mathcal{V} \to \mathcal{P}(\mathcal{V}')$ *by*

$$\widehat{A}(u) \equiv \{z \in \mathcal{V}' : z(t) \in A(t,u(t)) \ \text{a.e.} t \in [0,T]\}$$

The following is the main theorem in this section.

Theorem 2.2 *Suppose conditions 1 - 4 hold and* $B(t)$ *is one-to-one for a.e.* t. *Then* $\widehat{A}$ *is pseudo-monotone as a map from* X *to* $\mathcal{P}(X')$.

Proof: First we need to verify $\widehat{A}u$ is nonempty closed and convex. It is clear that this is convex and closed. We need to verify this set is nonempty. Let $u, v \in \mathcal{V}$ and let

$$\begin{aligned} &F(t) \\ &\equiv \ \Big\{w \in A(t,u(t)) : \langle w, u(t) - v(t)\rangle \leq \Big[b_1 \|u(t)\|_V^{p-1} + b_2(t)\Big] \|u(t) - v(t)\|\Big\}. \end{aligned}$$

Note that by 2, $F(t)$ is nonempty. Letting

$$\alpha(t) \equiv \Big[b_1 \|u(t)\|_V^{p-1} + b_2(t)\Big] \|u(t) - v(t)\|,$$

it follows from Condition 4 that there exists $z \in \mathcal{V}'$ such that $z(t) \in F(t)$ a.e. Thus $z \in \widehat{A}u$.

Next we must verify the pseudo-monotone limit condition for $\widehat{A}$ on X. Let $u_n \rightharpoonup u$ in X, $z_n \in \widehat{A}u_n$, and suppose

$$\limsup_{n\to\infty} \langle z_n, u_n - u\rangle_{\mathcal{V}',\mathcal{V}} \leq 0. \tag{2.2}$$

We note there is a set of measure zero, Σ_1 such that for $t \notin \Sigma_1$, we have the following holding for all n.

$$\begin{gathered} z_n(t) \in A(t,u_n(t)),\ Bu(t) = B(t)u(t), \\ Bu_n(t) = B(t)u_n(t),\ \text{and } B(t) \text{ is one-to-one.} \end{gathered}$$

First we verify the following claim.

Claim: Let $u_n \rightharpoonup u$ in X and let $t \notin \Sigma_1$. Then

$$\liminf_{n\to\infty} \langle z_n(t), u_n(t) - u(t)\rangle \geq 0.$$

Proof of the claim: Fix $t \notin \Sigma_1$ and suppose to the contrary that

$$\liminf_{n\to\infty} \langle z_n(t), u_n(t) - u(t)\rangle < 0. \tag{2.3}$$

Then there exists a subsequence, n_k such that

$$\lim_{k\to\infty} \langle z_{n_k}(t), u_{n_k}(t) - u(t)\rangle = \liminf_{n\to\infty} \langle z_n(t), u_n(t) - u(t)\rangle < 0. \tag{2.4}$$

Therefore, for all k large enough, the second formula in 3 implies

$$\begin{aligned} b_3\|u_{n_k}(t)\|_V^p - b_4\|u_{n_k}(t)\|_V^\alpha - b_5(t) &< \|z_{n_k}(t)\|_{V'}\|u(t)\|_V \\ &\le (b_1\|u_{n_k}(t)\|_V^{p-1} + b_2(t))\|u(t)\|_V \end{aligned}$$

which implies $\|u_{n_k}(t)\|_V$ and consequently $\|z_{n_k}(t)\|_{V'}$ are bounded. ($\|z_{n_k}(t)\|_{V'}$ is bounded independent of n_k because of the assumption that $A(t,\cdot)$ is bounded and we just showed $\|u_{n_k}(t)\|_V$ is bounded.) Now by (1.7),

$$Bu_{n_k}(t) = Bu_{n_k}(0) + \int_0^t (Bu_{n_k})'(s)ds, \quad Bu(t) = Bu(0) + \int_0^t (Bu)'(s)ds \tag{2.5}$$

where the initial values $Bu_{n_k}(0)$ and $Bu(0)$ are in W'. From Theorem 1.1 $Bu_{n_k}(0)$ is bounded in W'. Also from this theorem, the mapping, $w \to Bw(0)$ is a continuous and linear map from X to W' and so, taking a further subsequence if necessary, we may obtain

$$Bu_{n_k}(0) \rightharpoonup Bu(0) \text{ in } W'$$

and also $u_{n_k}(t) \rightharpoonup \xi$ in V. From (2.5), and the assumption that $u_n \rightharpoonup u$ in X,

$$B(t)u_{n_k}(t) = Bu_{n_k}(t) \rightharpoonup Bu(t) = B(t)u(t) \text{ in } W'.$$

Since $B(t)$ is continuous, it is also closed and hence weakly closed. Therefore, the above implies

$$B(t)u(t) = B(t)\xi \text{ in } W'$$

and since $B(t)$ is one-to-one, $\xi = u(t)$. Now from (2.4), and (2.3),

$$\lim_{k\to\infty} \langle z_{n_k}(t), u_{n_k}(t) - u(t)\rangle < 0,$$

and so the pseudo-monotone limit condition for $A(t,\cdot)$ implies that there exists z_∞ in $A(t, u(t))$ such that

$$\begin{aligned} \liminf_{k\to\infty} \langle z_{n_k}(t), u_{n_k}(t) - u(t)\rangle &\ge \langle z_\infty, u(t) - u(t)\rangle = 0 \\ &> \lim_{k\to\infty} \langle z_{n_k}(t), u_{n_k}(t) - u(t)\rangle, \end{aligned}$$

a contradiction. This proves the claim.

We now continue with the proof of the theorem. It follows from this claim that for a.e. t, in fact any $t \notin \Sigma_1$, we have

$$\liminf_{n\to\infty} \langle z_n(t), u_n(t) - u(t)\rangle \geq 0. \tag{2.6}$$

Now also from the coercivity condition, 3, if $y \in \mathcal{V}$,

$$\begin{aligned}\langle z_n(t), u_n(t) - y(t)\rangle &\geq b_3\|u_n(t)\|_V^p - b_4\|u_n(t)\|_V^\alpha - b_5(t)\\ &\quad -(b_1\|u_n(t)\|^{p-1} + b_2(t))\|y(t)\|_V\,.\end{aligned}$$

Using $p-1 = \frac{p}{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, the right side of the above inequality equals

$$b_3\|u_n(t)\|_V^p - b_4\|u_n(t)\|_V^\alpha - b_5(t) - b_1\|u_n(t)\|^{p/p'}\|y(t)\|_V - b_2(t)\|y(t)\|_V.$$

Now using Young's inequality, we can obtain a constant, $C(b_3, b_4)$, depending on b_3 and b_4 such that

$$b_4\|u_n(t)\|_V^\alpha \leq \frac{b_3}{2}\|u_n(t)\|_V^p + C(b_3, b_4)$$

and another constant, $C(b_1, b_3)$ depending on b_1 and b_3 such that

$$b_1\|u_n(t)\|^{p/p'}\|y(t)\|_V \leq \frac{b_3}{2}\|u_n(t)\|_V^p + C(b_1, b_3)\|y(t)\|_V^p.$$

Letting $k(t) = b_5(t) + C(b_3, b_4)$ and $C = C(b_1, b_3)$, it follows $k \in L^1(0, T)$ and

$$\langle z_n(t), u_n(t) - y(t)\rangle \geq -k(t) - C\|y(t)\|_V^p. \tag{2.7}$$

Letting $y = u$, we may use Fatou's lemma to write

$$\begin{aligned}&\liminf_{n\to\infty}\int_0^T (\langle z_n(t), u_n(t) - u(t)\rangle + k(t) + C\|y(t)\|_V^p)\,dt\\ &\geq \int_0^T \liminf_{n\to\infty}\langle z_n(t), u_n(t) - u(t)\rangle + (k(t) + C\|y(t)\|_V^p)\,dt\\ &\geq \int_0^T (k(t) + C\|y(t)\|_V^p)\,dt\,.\end{aligned}$$

Consequently,

$$\begin{aligned}0 &\geq \limsup_{n\to\infty}\langle z_n, u_n - u\rangle_{\mathcal{V}',\mathcal{V}} \geq \liminf_{n\to\infty}\int_0^T \langle z_n(t), u_n(t) - u(t)\rangle dt\\ &= \liminf_{n\to\infty}\langle z_n, u_n - u\rangle_{\mathcal{V}',\mathcal{V}} \geq \int_0^T \liminf_{n\to\infty}\langle z_n(t), u_n(t) - u(t)\rangle dt \geq 0\end{aligned}$$

showing that

$$\lim_{n\to\infty}\langle z_n, u_n - u\rangle_{\mathcal{V}',\mathcal{V}} = 0\,. \tag{2.8}$$

We need to show that for all $y \in X$ there exists $z(y) \in \widehat{A}u$ such that

$$\liminf_{n\to\infty} \langle z_n, u_n - y\rangle_{\mathcal{V}',\mathcal{V}} \geq \langle z(y), u - y\rangle_{\mathcal{V}',\mathcal{V}}.$$

Suppose to the contrary that for some $y \in X$,

$$\liminf_{n\to\infty} \langle z_n, u_n - y\rangle_{\mathcal{V}',\mathcal{V}} < \langle z, u - y\rangle_{\mathcal{V}',\mathcal{V}}$$

for all $z \in \widehat{A}u$. Taking a subsequence if necessary, we assume

$$\liminf_{n\to\infty} \langle z_n, u_n - y\rangle_{\mathcal{V}',\mathcal{V}} = \lim_{n\to\infty} \langle z_n, u_n - y\rangle_{\mathcal{V}',\mathcal{V}}$$

From (2.7),

$$0 \leq \langle z_n(t), u_n(t) - u(t)\rangle^- \leq k(t) + C\|u(t)\|_V^p .$$

Thanks to (2.6), we know that for a.e. t, $\langle z_n(t), u_n(t) - u(t)\rangle \geq -\varepsilon$ for all n large enough. Therefore, for such n, $\langle z_n(t), u_n(t) - u(t)\rangle^- \leq \varepsilon$ if $\langle z_n(t), u_n(t) - u(t)\rangle < 0$ and $\langle z_n(t), u_n(t) - u(t)\rangle^- = 0$ if $\langle z_n(t), u_n(t) - u(t)\rangle \geq 0$. Therefore, $\lim_{n\to\infty}\langle z_n(t), u_n(t) - u(t)\rangle^- = 0$ and so we may apply the dominated convergence theorem and conclude

$$\lim_{n\to\infty} \int_0^T \langle z_n(t), u_n(t) - u(t)\rangle^- dt = \int_0^T \lim_{n\to\infty} \langle z_n(t), u_n(t) - u(t)\rangle^- dt = 0$$

from (2.6). Now by (2.8) and the above equation,

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T \langle z_n(t), u_n(t) - u(t)\rangle^+ \, dt \\ &= \lim_{n\to\infty} \int_0^T \langle z_n(t), u_n(t) - u(t)\rangle + \langle z_n(t), u_n(t) - u(t)\rangle^- \, dt \\ &= \lim_{n\to\infty} \langle z_n, u_n - u\rangle_{\mathcal{V}',\mathcal{V}} = 0 \,. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \int_0^T |\langle z_n(t), u_n(t) - u(t)\rangle| \, dt = 0$$

so there exists a subsequence, n_k such that

$$\langle z_{n_k}(t), u_{n_k}(t) - u(t)\rangle \to 0 \text{ a.e.} \tag{2.9}$$

Therefore, by the pseudo-monotone limit condition for $A(t,\cdot)$, there exists w_t in $A(t, u(t))$ such that for a.e. t,

$$\alpha(t) \equiv \liminf_{k\to\infty} \langle z_{n_k}(t), u_{n_k}(t) - y(t)\rangle \geq \langle w_t, u(t) - y(t)\rangle.$$

Let

$$F(t) \equiv \{w \in A(t, u(t)) : \langle w, u(t) - y(t)\rangle \leq \alpha(t)\}$$

By the condition on measurability, 4, there exists $z \in \mathcal{V}'$ such that $z(t) \in F(t)$ a.e. Therefore, $z \in \widehat{A}u$ and for a.e. t,

$$\liminf_{k\to\infty} \langle z_{n_k}(t), u_{n_k}(t) - y(t)\rangle \geq \langle z(t), u(t) - y(t)\rangle .$$

From (2.7) and Fatou's lemma,

$$\begin{aligned} &\int_0^T \liminf_{k\to\infty} \{\langle z_{n_k}(t), u_{n_k}(t) - y(t)\rangle + k(t) + C\|y(t)\|_V^p\}\, dt \\ \leq\ & \liminf_{k\to\infty} \int_0^T \{\langle z_{n_k}(t), u_{n_k}(t) - y(t)\rangle + k(t) + C\|y(t)\|_V^p\}\, dt \end{aligned}$$

which implies

$$\begin{aligned} \liminf_{k\to\infty} \langle z_{n_k}, u_{n_k} - y\rangle_{\mathcal{V}',\mathcal{V}} &\geq \int_0^T \liminf_{k\to\infty} \langle z_{n_k}(t), u_{n_k}(t) - y(t)\rangle dt \\ &\geq \int_0^T \langle z(t), u(t) - y(t)\rangle dt \\ &= \langle z, u - y\rangle_{\mathcal{V}',\mathcal{V}} \\ &> \lim_{k\to\infty} \langle z_{n_k}, u_{n_k} - y\rangle_{\mathcal{V}',\mathcal{V}}, \end{aligned}$$

a contradiction that completes the present proof.

3 Existence theorems

In this section we generalize the main existence theorem of Bian and Webb, [9] to the case of implicit evolution inclusions and more general initial data. With the result of Section 2, we may state the following corollary of Theorem 1.5 which generalizes the main result of [9].

Theorem 3.1 *Let $B, \widehat{A}, X, W,$ and V be as defined above, $f \in \mathcal{V}'$, $u_0 \in W$, and suppose $\widehat{A}$ is given by Definition 2.1 where $A(t,\cdot)$ satisfies the conditions 1 - 4 of Section 2. Then if $B(t)$ is one to one a.e. t, there exists a solution, $u \in X$ to*

$$(Bu)' + \widehat{A}u \ni f,\ Bu(0) = Bu_0$$

Proof: Theorem 2.2 implies $\widehat{A}$ is pseudo-monotone as a map from X to $\mathcal{P}(X')$ and so the estimates 3 and 2 of Section 2 apply to give $\widehat{A} + \frac{1}{2}B'$ is bounded and coercive. Therefore, Theorem 1.5 implies the desired conclusion.

There is an easy generalization to Theorem 3.1 which we state next. Suppose B satisfies (1.2) - (1.4) and we modify the conditions 1 - 4 of Section 2. For some $\lambda \geq 0$, we have

1. $u \to \lambda B_\lambda(t)u + A(t,u)$ is a set valued pseudo-monotone map satisfying Conditions 1 and 3 in the definition of pseudo-monotone given in Section 1 where
$$B_\lambda(t)u = B(t)e^{-\lambda t}u.$$

2. There exists $b_1 \geq 0$ and $b_2 \in L^{p'}(0,T)$ such that
$$\|z\|_{V'} \leq b_1 \|u\|_V^{p-1} + b_2(t) \text{ for all } z \in A(t,u)$$

3. There exists $b_3 > 0$, $\alpha \in (0,p)$, $b_4 \geq 0$, and $b_5 \in L^1(0,T)$ such that
$$\inf_{z \in \lambda B_\lambda(t)u + A(t,u)} \langle z, u \rangle + \frac{e^{-\lambda t}}{2} \langle B'(t)u, u \rangle \geq b_3 \|u\|_V^p - b_4 \|u\|_V^\alpha - b_5(t).$$
and
$$\inf_{z \in \lambda B_\lambda(t)u + A(t,u)} \langle z, u \rangle \geq b_3 \|u\|_V^p - b_4 \|u\|_V^\alpha - b_5(t)$$

4. The operators, $\lambda B + A(t,\cdot)$ are measurable in the sense of the condition on measurability, (2.1).

Corollary 3.2 *Let $B, \widehat{A}, X, W$, and V be as defined above, $f \in \mathcal{V}'$, $u_0 \in W$, and suppose $\widehat{A}$ is given by Definition 2.1 where $A(t,\cdot)$ and B satisfy the conditions 1 - 4 above. Then if B is one-to-one, there exists a solution, $u \in X$ to*
$$(Bu)' + \widehat{A}u \ni f,\ Bu(0) = Bu_0. \tag{3.1}$$

Proof: We define a new dependent variable, y, by $u(t) = e^{\lambda t}y(t)$. Then the evolution inclusion of this corollary, written in terms of y becomes
$$(By)' + \lambda By + \widehat{A}_\lambda(y) \ni g, By(0) = Bu_0, \tag{3.2}$$
where $g(t) = e^{-\lambda t}f(t)$ and
$$A_\lambda(t,v) \equiv e^{-\lambda t}A(t, e^{\lambda t}v).$$

It is almost immediate that Condition 2 of Section 2 holds for $\lambda B + A_\lambda$. Condition 3 above implies Condition 3 of Section 2 for $\lambda B + \widehat{A}_\lambda$ with modified b_i.
$$\begin{aligned} &\{w \in \lambda Bx(t) + A_\lambda(t,x(t)) : \langle w, x(t) - y(t)\rangle \leq \alpha(t)\} \\ &= \left\{w \in \lambda Bv(t) + A(t,v(t)) : \langle w, v(t) - e^{\lambda t}y(t)\rangle \leq e^{\lambda t}\alpha(t)\right\}, \end{aligned}$$
where $v(t) = \exp(\lambda t)x(t)$. So the measurability condition (2.1) holds for $\lambda B + A_\lambda$. It is also clear that $\lambda B + A_\lambda$ is set valued pseudo-monotone if Condition 1 above holds. Therefore, we apply Theorem 3.1 to the inclusion (3.2).

4 An example

We give a simple example in this section, a modification of that in [9]. Let Ω be a bounded open set in $\mathbb{R}^3$ having Lipschitz boundary and let V be a closed subspace of $H^1(\Omega), W \equiv L^6(\Omega)$ and $b : [0,T] \times \Omega \to \mathbb{R}$ is Borel measurable and for a.e. t,

$$b(t,x) > 0 \text{ a.e } x,$$
$$b(t,x) = b(0,x) + \int_0^t b_t(s,x)\, ds$$

where $b(0,x) \in L^{3/2}(\Omega)$ and

$$\sup_{s\in[0,T]} \int_0^T |b_t(s,x)|^{3/2}\, dx < \infty$$

Then we define an operator, $B(t) : W \to W'$ by $B(t)u(x) \equiv b(t,x)u(x)$. Thus $B(t)$ is one-to-one for a.e. t. We define a time dependent operator, $A(t,\cdot)$ mapping V to V' as follows.

$$\langle A(t,u), v\rangle \equiv \int_\Omega (\sum_{i=1}^3 a_i(t,x,u,\nabla u)\partial_i v) + a_0(t,x,u,\nabla u)v\, dx\,.$$

We make the following assumptions on the functions a_i.

1. $(t,x) \to a_i(t,x,\mathbf{z})$ is measurable while $\mathbf{z} \to a_i(t,x,\mathbf{z})$ is continuous.
2. There exist constants, $C_1, C_2, C_3 > 0$ and functions $k_1 \in L^2([0,T]\times\Omega)$ and $k_2 \in L^1([0,T]\times\Omega), k_3 \in L^1(\Omega)$, such that
$$|a_i(t,x,u,\mathbf{p})| \le C(|\mathbf{p}| + |u|) + k_1(t,x),$$
and for some λ,
$$\begin{aligned} &\lambda e^{-\lambda t} b(t,x)u^2 + \sum_{i=1}^3 a_i(t,x,u,\mathbf{p})p_i + a_0(t,x,u,\mathbf{p})u + \frac{e^{-\lambda t}}{2} b_t(t,x)u^2 \\ &\ge \quad C_2(|\mathbf{p}|^2 + |u|^2) - k_2(t,x) \end{aligned}$$
and
$$\lambda e^{-\lambda t} b(t,x)u^2 + \sum_{i=1}^3 a_i(t,x,u,\mathbf{p})p_i + a_0(t,x,u,\mathbf{p})u \ge C_3 |\mathbf{p}|^2 - k_3(x)$$
3. For all $(u,\mathbf{p})$ and $(u,\widehat{\mathbf{p}})$, if $\mathbf{p} \neq \widehat{\mathbf{p}}$
$$\sum_{i=1}^3 (a_i(t,x,u,\mathbf{p}) - a_i(t,x,u,\widehat{\mathbf{p}}))(p_i - \widehat{p}_i) > 0\,.$$

Now we define a set valued operator, $G : [0,T] \times \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ as follows.

1. $G(t,x,u) = [g_1(t,x,u), g_2(t,x,u)]$ where $(t,u) \to G(t,x,u)$ is upper semicontinuous, meaning that for each x,
$$G(s,x,v) \subseteq [g_1(t,x,u) - \varepsilon, g_2(t,x,u) + \varepsilon]$$
whenever $|v-u| + |s-t|$ is small enough. Also assume $g_i : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is Borel measurable.

2. There exist $C_3 > 0$ and $C_4 \in [0, C_2), k_3 \in L^2(\Omega)$, and $k_4 \in L^1(\Omega)$ such that G satisfies the estimates
$$\begin{gathered} |G(t,x,u)| = \max\{|g_1(t,x,u)|, |g_2(t,x,u)|\} \le C_3 |u| + k_3(x), \\ \inf\{ug_1(t,x,u), ug_2(t,x,u)\} \ge -C_4 |u|^2 - k_4(x). \end{gathered}$$
We define $w \in G(t,u)$ if and only if $w(x) \in G(t,x,u(x))$ a.e. so $G : V \to \mathcal{P}(H) \subseteq \mathcal{P}(V')$ where $H \equiv L^2(\Omega)$.

We will consider the abstract evolution inclusion
$$(Bu)' + \widehat{A+G} \ni 0,\ Bu(0) = Bu_0,\ u_0 \in L^6(\Omega)\,. \tag{4.1}$$
Solutions to (4.1) yield weak solutions to the nonlinear differential inclusion
$$\begin{gathered} (b(t,x)u)_t - \sum_{i=1}^{3} \partial_i(a_i(t,x,u,\nabla u)) + a_0(t,x,u,\nabla u) \in -G(t,x,u) \\ u(0,x) = u_0(x), \end{gathered}$$
along with appropriate boundary conditions depending on the choice of V. We need to verify the hypotheses of the main existence theorem. First we deal with the measurability issue for the function, G.

Lemma 4.1 *If conditions 1 and 2 hold for G, then for all closed convex subset P of H,*
$$G^-(P) \equiv \{(t,u) \in [0,T] \times V : G(t,u) \cap P \neq \emptyset\}$$
is a closed set.

Proof: Let $(t_n, u_n) \in G^-(P)$ and suppose $(t_n, u_n) \to (t,u)$ in $[0,T] \times H$. Taking a subsequence, still denoted by (t_n, u_n), we may also assume $u_n(x) \to u(x)$ for a.e. x. By the assumption that $(t_n, u_n) \in G^-(P)$, there exists $w_n \in H \cap P$ such that for a.e. x,
$$g_1(t_n, x, u_n(x)) \le w_n(x) \le g_2(t_n, x, u_n(x)).$$

By the given estimates, we see w_n is bounded in H and so we may take a further subsequence, still denoted by w_n such that $w_n \rightharpoonup w \in H$. Now let

$$\mathfrak{B}_\varepsilon \equiv \{x \in \Omega : w(x) \le g_1(t, x, u(x)) - \varepsilon\}.$$

We know from Estimate 2 given above for G that

$$\begin{aligned}
&\int_{\mathfrak{B}_\varepsilon} -C_3 |u_n(x)| - k_3(x) dx \\
&\quad \ge \int_{\mathfrak{B}_\varepsilon} [(g_1(t_n, x, u_n(x)) - (C_3 |u_n(x)| + k_3(x))) - w_n(x)]\, dx\,.
\end{aligned}$$

Adding $3 \int_{\mathfrak{B}_\varepsilon} C_3 |u_n(x)| + k_3(x) dx$ to both sides in order to get both integrands positive, we obtain

$$\begin{aligned}
&2 \int_{\mathfrak{B}_\varepsilon} C_3 |u_n(x)| + k_3(x) dx \\
&\quad \ge \int_{\mathfrak{B}_\varepsilon} [(g_1(t_n, x, u_n(x)) - w_n(x)) + 2(C_3 |u_n(x)| + k_3(x))]\, dx\,.
\end{aligned}$$

Taking the lim inf of both sides and using Fatou's lemma,

$$\begin{aligned}
&2 \int_{\mathfrak{B}_\varepsilon} C_3 |u(x)| + k_3(x) dx \\
&\quad \ge \int_{\mathfrak{B}_\varepsilon} \liminf_{n\to\infty} (g_1(t_n, x, u_n(x)) - w_n(x)) dx + 2 \int_{\mathfrak{B}_\varepsilon} (C_3 |u(x)| + k_3(x))
\end{aligned}$$

and so from the upper semi-continuity of G,

$$\begin{aligned}
0 &\ge \int_{\mathfrak{B}_\varepsilon} \liminf_{n\to\infty} (g_1(t_n, x, u_n(x)) - w_n(x)) dx \\
&= \int_{\mathfrak{B}_\varepsilon} (g_1(t, x, u(x)) - w(x)) dx \ge \int_{\mathfrak{B}_\varepsilon} \varepsilon dx
\end{aligned}$$

showing that $m(\mathfrak{B}_\varepsilon) = 0$. Thus, since $\varepsilon > 0$ is arbitrary, it follows $w(x) \ge g_1(t, x, u(x))$ a.e. Similar reasoning shows $w(x) \le g_2(t, x, u(x))$ a.e. Since P is a closed and convex subset of H, $w \in P$ as well. Thus $(t, u) \in G^-(P)$, and this shows this set is closed as claimed.

We just showed that

$$\{(t, v) : G(t, v) \cap P \ne \emptyset\} \tag{4.2}$$

is a Borel set whenever P is convex and closed.

Lemma 4.2 *For G defined above, G is measurable in the sense of (2.1).*

Proof: This follows from Lemma 5.3 which is proved in the next section because we have just shown the hypothesis of Lemma 5.3 are satisfied in 4.2. Since $A(t,\cdot)$ is single valued, it follows the sum $A(t,\cdot)+G(t,\cdot)$ satisfies the measurability condition, (2.1).

Next we must consider the question of whether the operators are pseudo-monotone. That $\lambda B_\lambda(t)+A(t,\cdot)$ is pseudo-monotone follows from results of Browder [10]. Therefore, we must verify $G(t,\cdot)$ is set valued pseudo-monotone. It remains to verify Conditions 1 and 3 in the list of conditions for Pseudo-monotone because the assumed estimates imply $G(t,\cdot)$ is bounded. Since weak convergence in V implies strong convergence in $L^2(\Omega)\equiv H$, it is easy to verify that $G(t,u)$ is a closed and convex subset of $H\subseteq V'$. From estimates on g_i it follows $g_i(t,\cdot,u(\cdot))\in L^2(\Omega)$ and so $G(t,u)\neq\emptyset$. It remains to verify condition 3, the limit condition for pseudo-monotone. Suppose $u_n\rightharpoonup u$ in V. We show for each $v\in V$, there exists $u_\infty^*\in G(t,u)$ such that

$$\liminf_{n\to\infty}\langle u_n^*,u_n-v\rangle\geq\langle u_\infty^*,u-v\rangle.$$

The weak convergence of u_n in V implies $u_n\to u$ in H and taking a subsequence, we may assume that $u_n(x)\to u(x)$ a.e. Let $u_n^*\in G(t,u_n)\subseteq H$. By the estimates, u_n^* is bounded in $H\equiv L^2(\Omega)$ and so, taking a subsequence, we may assume that u_n^* converges weakly in H to u_∞^*. By the compactness of the embedding of V into H, it follows that the embedding of H into V' is also compact and so u_n^* converges strongly to u_∞^* in V'. We need to verify $u_\infty^*\in G(t,u)$. Define

$$\mathfrak{B}_\varepsilon\equiv\{x:u_\infty^*(x)\geq g_2(t,x,u(x))+\varepsilon\}.$$

We know

$$\begin{aligned}&-\int_{\mathfrak{B}_\varepsilon}C_3\,|u_n(x)|+k_3(x)dx\\&\quad\leq\int_{\mathfrak{B}_\varepsilon}-u_n^*(x)dx-\int_{\mathfrak{B}_\varepsilon}\left[C_3\,|u_n(x)|+k_3(x)-g_2(t,x,u_n(x))\right]dx\end{aligned}$$

so taking the lim sup of both sides using the assumed weak convergence of u_n^* to u_∞^*,

$$\begin{aligned}&-\int_{\mathfrak{B}_\varepsilon}C_3\,|u(x)|+k_3(x)dx\\&\quad\leq\int_{\mathfrak{B}_\varepsilon}-u_\infty^*(x)dx-\liminf\int_{\mathfrak{B}_\varepsilon}\left[C_3\,|u_n(x)|+k_3(x)-g_2(t,x,u_n(x))\right]dx\,.\end{aligned}$$

Using Fatou's lemma as in the proof of Lemma 4.1 and the upper semi-continuity of $(t,u)\to G(t,x,u)$, we obtain

$$-\int_{\mathfrak{B}_\varepsilon}C_3\,|u(x)|+k_3(x)dx$$

$$\begin{aligned}
&\le \int_{\mathfrak{B}_\varepsilon} -u_\infty^*(x)dx - \int_{\mathfrak{B}_\varepsilon} \left[C_3 \left|u(x)\right| + k_3(x) - \limsup g_2(t,x,u_n(x))\right] dx \\
&\le \int_{\mathfrak{B}_\varepsilon} -u_\infty^*(x)dx - \int_{\mathfrak{B}_\varepsilon} \left[C_3 \left|u(x)\right| + k_3(x) - g_2(t,x,u(x))\right] dx,
\end{aligned}$$

which implies

$$0 \le \int_{\mathfrak{B}_\varepsilon} g_2(t,x,u(x)) - u_\infty^*(x)dx \le -\varepsilon m(\mathfrak{B}_\varepsilon)$$

so $m(\mathfrak{B}_\varepsilon) = 0$. Therefore, letting ε be replaced by $\frac{1}{n}$, taking the union of $\mathfrak{B}_{1/n}$, we see that $u_\infty^*(x) \le g_2(x,u(x))$ a.e. and a similar argument shows $u_\infty^*(x) \ge g_1(x,u(x))$ a.e. We have shown that if $u_n \rightharpoonup u$ in V, there exists a subsequence, u_{n_k} such that $u_{n_k}^* \rightharpoonup u_\infty^* \in G(t,u)$ in H. Therefore, by the strong convergence of u_n^* to u_∞^* in V' and the weak convergence of u_{n_k} to u in V, it follows that for any $v \in V$,

$$\lim_{k\to\infty} \langle u_{n_k}^*, u_{n_k} - v\rangle_{V',V} = \langle u_\infty^*, u - v\rangle_{V',V}$$

If for some $v \in V$, there exists a sequence, $u_n \rightharpoonup u$ in V such that

$$\liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle_{V',V} < \langle u^*, u - v\rangle_{V',V},$$

for all $u^* \in G(u)$, this is a contradiction to what was just shown. Therefore, we have verified the pseudo-monotone limit condition.

It follows $\lambda B_\lambda(t) + A(t,\cdot) + G(t,\cdot)$, being a sum of pseudo-monotone operators is pseudo-monotone [8]. The assumed estimates give the rest of the hypotheses of Corollary 3.2. Therefore, there exists a solution to (4.1).

5 Measurability

In this section we demonstrate that the conditions on measurability given by Bian and Webb imply Condition 4 of Section 2. The following definition of measurability, used in [9], is the one referred to as strong measurability in [11].

Definition 5.1 We say $S : [0,T] \times V \to \mathcal{P}(V')$ is measurable if whenever, $\mathfrak{H} \subseteq V'$ is a closed set, the set $\{(t,v) : S(t,v) \cap \mathfrak{H} \neq \emptyset\}$ is a Borel set in $[0,T] \times V$.

The following lemma is an immediate consequence of the above definition.

Lemma 5.2 *If S is measurable as just described and if $x : [0,T] \to V$ is measurable, then for $\mathfrak{H}$ a closed set in V', the set $\{t \in [0,T] : S(t,x(t)) \cap \mathfrak{H} \neq \emptyset\}$ is measurable.*

Now we verify the following lemma which is the main result of this section.

Lemma 5.3 *Suppose S satisfies the condition of Definition 5.1 for all $\mathfrak{H}$ a closed convex set where $S(t,v)$ equals a closed convex nonempty subset of V'. Then $S(t,\cdot)$ is measurable in the sense of (2.1).*

Proof: We show that if α is real valued and measurable, and x, y are strongly measurable V valued functions, then $F : [0,T] \to \mathcal{P}(V')$ given by

$$F(t) \equiv \{w \in S(t, x(t)) : \langle w, x(t) - y(t)\rangle \leq \alpha(t)\}$$

has the property that $\{t : F(t) \cap U \neq \emptyset\}$ is measurable whenever U is an open set. We define

$$F_n^m(t) \equiv \left\{w \in S(t, x(t)) : \langle w, x_n(t) - y_n(t)\rangle \leq \alpha_n(t) + \frac{1}{m}\right\}$$

where here x_n, y_n, α_n are simple functions converging pointwise to x, y, and α respectively. Thus there exist disjoint measurable subsets of $[0,T]$, $\{E_i^n\}_{i=1}^{m_n}$ such that each of x_n, y_n, and α_n are constant on E_i^n. If $\mathfrak{H}$ is a closed convex set in V'

$$\{t \in [0,T] : F_n^m(t) \cap \mathfrak{H} \neq \emptyset\} = \cup_{i=1}^{m_n} \{t \in E_i^n : F_n^m(t) \cap \mathfrak{H} \neq \emptyset\}. \tag{5.1}$$

On the set, E_i^n denote the values of x_n, y_n and α_n as x_n^i, y_n^i, and α_n^i respectively. Then

$$F_n^m(t) \cap \mathfrak{H} = S(t, x(t)) \cap \mathfrak{H} \cap C_n^i$$

where

$$C_n^i \equiv \left\{w \in V' : \langle w, x_n^i - y_n^i\rangle \leq \alpha_n^i + \frac{1}{m}\right\}$$

and is a closed set. Therefore,

$$\{t \in E_i^n : F_n^m(t) \cap \mathfrak{H} \neq \emptyset\} = \left\{t \in E_i^n : S(t, x(t)) \cap \widetilde{\mathfrak{H}} \neq \emptyset\right\}$$

where $\widetilde{\mathfrak{H}} \equiv \mathfrak{H} \cap C_n^i$, a closed convex set. Therefore, by our hypotheses, the set $\{t \in E_i^n : F_n^m(t) \cap \mathfrak{H} \neq \emptyset\}$ is measurable and so it follows from (5.1) that the set $\{t \in [0,T] : F_n^m(t) \cap \mathfrak{H} \neq \emptyset\}$ is also measurable.

Claim: Let $\mathfrak{H}$ be a closed ball. Then

$$\{t : F(t) \cap \mathfrak{H} \neq \emptyset\} = \cap_{m=1}^{\infty} \cup_{k=1}^{\infty} \cap_{n \geq k} \{t : F_n^m(t) \cap \mathfrak{H} \neq \emptyset\}. \tag{5.2}$$

Proof of the claim: If $t \in \{t : F(t) \cap \mathfrak{H} \neq \emptyset\}$, then there exists $w \in S(t, x(t)) \cap \mathfrak{H}$ such that $\langle w, x(t) - y(t)\rangle \leq \alpha(t)$. Therefore, for that w it follows that for each m,

$$\langle w, x_n(t) - y_n(t)\rangle \leq \alpha_n(t) + \frac{1}{m}$$

for all n large enough. Therefore, t is an element of the right side of (5.2).

Now let t be an element of the right side. Then for all m, there exists w_n^m in $S(t, x(t)) \cap \mathfrak{H}$ such that

$$\langle w_n^m, x_n(t) - y_n(t)\rangle \leq \alpha_n(t) + \frac{1}{m}$$

for all n large enough. Since $S(t,x(t))\cap\mathfrak{H}$ is closed and bounded and convex, we can take a subsequence, $w_{n_k}^m$ which converges weakly to $w^m\in S(t,x(t))\cap\mathfrak{H}$. Therefore, taking a limit as $k\to\infty$, using the strong convergence of $x_n(t)$ and $y_n(t)$ to $x(t)$ and $y(t)$ respectively, we obtain

$$\langle w^m, x(t)-y(t)\rangle \le \alpha(t)+\frac{1}{m}.$$

Now take another subsequence, w^{m_k} converging weakly to $w\in S(t,x(t))\cap\mathfrak{H}$ and take a limit as $k\to\infty$ to obtain

$$\langle w, x(t)-y(t)\rangle \le \alpha(t).$$

It follows t is an element of the left side of (5.2), proving the claim.

Now if U is an arbitrary open set, we know that since V and consequently, V' are separable, U is the union of countably may closed balls, $U=\cup_{k=1}^{\infty}\mathfrak{H}_k$ and

$$\{t: F(t)\cap U\neq\emptyset\}=\cup_{k=1}^{\infty}\{t: F(t)\cap\mathfrak{H}_k\neq\emptyset\},$$

a measurable set.

We have just verified that for all U open, $\{t: F(t)\cap U\neq\emptyset\}$ is a measurable set. This is a sufficient condition for the existence of a measurable, selector, $z(t)\in F(t)$. [12], [13]. From the assumed estimates, it follows $z\in\mathcal{V}'$ whenever $x\in\mathcal{V}$.

We did not gain any generality by only requiring the closed set, $\mathfrak{H}$ to be convex. To see this, note that our argument is concluded by verifying that the set $\{t: F(t)\cap U\neq\emptyset\}$ is measurable for all U open. It is known [12], [13] this is equivalent under certain conditions, including the case where Lebesgue measure is used on the Lebesgue measurable sets of $[0,T]$ to the set $\{t: F(t)\cap\mathfrak{H}\neq\emptyset\}$ being measurable for all $\mathfrak{H}$ closed or even Borel. Nevertheless, it is easier to verify the measurability condition for closed convex sets than for arbitrary closed sets.

Summary. The paper has given an existence theorem for implicit evolution inclusions of the form $(Bu)'+Au\ni f$ under assumptions that $B(t)$ is one-to-one a.e. It would be very interesting to obtain similar theorems involving reasonable pointwise conditions on the operators $A(t,\cdot)$ using different methods, and also to include the case where $B(t)$ could be a degenerate operator as in [6].

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