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SYMMETRY THEOREMS VIA THE CONTINUOUS STEINER SYMMETRIZATION

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Abstract. Using a new approach due to F. Brock called the Steiner symmetrization, we show first that if u is a solution of an overdetermined problem in the divergence form satisfying the Neumann and non-constant Dirichlet boundary conditions, then Ω is an N-ball. In addition, we show that we can relax the condition on the value of the Dirichlet boundary condition in the case of superharmonicity. Finally, we give an application to positive solutions of some semilinear elliptic problems in symmetric domains for the divergence case.

1. Introduction

This paper deals with an overdetermined boundary value problem and an application to positive solutions of some semilinear elliptic problems in symmetric domains. In Section 1A we describe the first eigenvalue problem concerning a free membrane. This problem was resolved recently by Henrot and Philippin [3] who applied Brock's [1] continuous Steiner symmetrization and the domain derivative due to F. Murat and J. Simon [6], J. Simon [8], and J. Sokolowski and J. P. Zolesio [9]. Assuming that $\phi > 0$, that $\psi > 0$ is an increasing function of r, and λ is the first eigenvalue of the Laplacian, they showed that if the first eigenvector u satisfies

$$\Delta u + \lambda\phi(r)u = 0 \quad \text{on } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

$$\frac{\partial u}{\partial n} = \psi(r^2) \quad \text{on } \partial\Omega, \tag{3}$$

then Ω is an N-ball, where $N \geq 3$ and $r := |x|$. In this section, we generalize this problem to more general operators while using the same technique. We formulate this generalized overdetermined problem, which we will denote (P_ϵ), as follows:

$$-\operatorname{Div}(a(u, |\nabla u|)\nabla u) = \phi(r)F(x, u) \quad \text{in } \Omega, \tag{4}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{5}$$

$$\frac{\partial u}{\partial n} = \epsilon\psi_\epsilon(r^2) \quad \text{on } \partial\Omega. \tag{6}$$

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Using the Steiner symmetrization method and two fundamental theorems of F. Brock [2], we prove that the problem cited above (with appropriate conditions on a and F) is solvable only if Ω is an N-ball. Next we show that the theorem holds for some other choices of the function ψ. To be more precise, using the lemma of Mitidieri [5] concerning a radial positive function ψ which satisfies

$$\Delta\psi \le 0 \quad \text{in } \Omega, \tag{7}$$

we prove the same result without assuming that ψ is increasing. We show that some conditions for Steiner symmetrization of functions can be relaxed when the functions are subharmonic (respectively superharmonic), radial, and positive (resp. negative). We need to recall some properties of Steiner symmetrization of functions as well as those of the domain derivative so that we can use them for our proof. We begin with the domain derivative following [8].

1a. Differentiability and integrability of functions with respect to the domain

We denote by S the set of all bounded, open, connected, and regular domains of $\mathbb{R}^N$. We will assume that J^*, J_*, G, and J are defined on S. Using variational calculus, we compute the differentiability of A and B (defined below) with respect to the domain Ω, and find an Ω realizing $\inf_\Omega J(\Omega)$ where $J(\Omega)$ is a functional domain defined by:

$$J(\Omega) := \int_\Omega a(u, |\nabla u|^2)|\nabla u|^2 - \phi(r)F(x,u)\,dx.$$

To begin, suppose that u is a solution of the following boundary value problem

$$A(u) := 0 \quad \text{in } \Omega, \tag{8}$$

$$B(u) := 0 \quad \text{on } \Gamma := \partial\Omega. \tag{9}$$

We start with the directional derivative of J at Ω. For this, we make a variation of Ω by considering a continuous one-parameter family of domains Ω_t defined by

$$\Omega_t := \{x + tV(x), x \in \Omega, t > 0\}$$

where V is any vector field in $C^2(\mathbb{R}^N, \mathbb{R}^N)$ defined by

$$V(x) := \frac{\partial H}{\partial t}(x,0) := x'(0). \tag{10}$$

We observe that V can be understood as a field of deformation for Ω. At the same time we introduce a real parameter t and a map H such that $\Omega \to \Omega_t$, $x \to x_t := H(x,t) = x + tV(x)$, $\Omega_0 := \Omega$, and $H(.,0) := I$ in Ω. We note that the application $Id + tV$ is a perturbation of the identity which will be a C^2 diffeomorphism for sufficiently small t. If we think of t as time, V is the deformation speed at the origin of the open set Ω.
Consequently, we obtain the new map described below:

$$\Omega_t \to y_t := y_{\Omega_t} \to J(t) := J(\Omega_t).$$

In fact, the maps J and H are defined by $J : O \to \mathbb{R}$, $H : \tilde{O} \to O$, where O is the set of all domains Ω_t, $t > 0$, and $\tilde{O}$ is the set of all domains $\bar{\Omega}$ for which Ω is a bounded domain of $\mathbb{R}^N$.

In this way, the directional derivative of J at Ω in the V direction is equivalent to the derivative of J evaluated at $t = 0$:

$$dJ(\Omega; V) := J'(0). \tag{11}$$

Our first problem is equivalent to finding a t which realizes $\inf_t J(t)$. In practice, Ω will usually depend on one parameter u. In this case, we assume that $\Omega = \Omega_u$ is the image of a fixed domain $\bar{\Omega}$ under a map which depends on u: $H : \bar{\Omega} \to \Omega_u := H(\bar{\Omega}, u)$, $\bar{x} \to x_u := H(\bar{x}, u)$, and we define $\bar{J}$ from $\mathbb{R}$ to $\mathbb{R}$ by $\bar{J}(x) := J(\Omega_x)$. By putting $\Omega_t := \Omega_{u+tv}$, so that $\Omega_0 := \Omega_u$, we can formulate the derivatives of $\bar{J}$ in terms of the function J as $d\bar{J}(u; V) := dJ(\Omega_u; V)$. Returning to our goal, we can now deduce the necessary properties of the domain derivative:

$$\frac{\partial A}{\partial u}\,\frac{\partial u}{\partial V} = 0 \quad \text{in } \Omega, \tag{12}$$

$$\frac{\partial B}{\partial u}\,\frac{\partial u}{\partial V} + V \cdot n \frac{\partial B}{\partial n} = 0 \quad \text{on } \Gamma. \tag{13}$$

Here n denotes the outward normal on Γ and

$$\frac{\partial A}{\partial u} V := \frac{\partial A(u + tV)}{\partial t}\Big|_{t=0}.$$

We close this subsection with some integral derivatives with respect to the domain Ω. Let J^* and J_* be given domain functionals defined by

$$J^* := \int_\Omega f(\Omega)\, d\mathbf{x}, \tag{14}$$

$$J_* := \int_{\partial\Omega} g(\Omega)\, d\mathbf{s}, \tag{15}$$

where f and g are positive C^2 functions on $\bar{\Omega}$. We can compute their integral domain derivatives:

$$dJ^*(\Omega; V) := \int_\Omega f'(\Omega)\, d\mathbf{x} + \int_{\partial\Omega} f(\Omega) V \cdot n\, d\mathbf{s} \tag{16}$$

$$dJ_*(\Omega; V) := \int_{\partial\Omega} g'(\Omega)\, d\mathbf{s} + \int_{\partial\Omega} (N-1) K g(\Omega) V \cdot n\, d\mathbf{s} + \int_{\partial\Omega} \frac{\partial g(\Omega)}{\partial n} V \cdot n\, d\mathbf{s}, \tag{17}$$

where K denotes the mean curvature of the boundary of Ω.

1B. Steiner symmetrization for functions in $\mathbb{F}(\mathbb{R}^N)$

As defined in [1] and [2], the set $\mathbb{F}(\mathbb{R}^N)$ is the set of real symmetrizable functions. We say that $u \in \mathbb{F}(\mathbb{R}^N)$ if and only if u is measurable on $\mathbb{R}^N$ and for every $c > \inf\, u$ the level sets $\{x \in \mathbb{R}^N : u(x) > c\}$ have finite Lebesgue measure. We denote by

$\{u > c\}^t$ the continuous symmetrization of the set $\{u > c\}$. For more details concerning the continuous symmetrization and its properties, we refer to the work of F. Brock [2].

(1) Let $u \in \mathbb{F}(\mathbb{R}^N)$ and let $m_u(c)$ be the corresponding distribution function defined by

$$m_u(c) := |\left\{x \in \mathbb{R}^N \;:\; u(x) > c\right\}|.$$

The inverse function is denoted by $u^* \in \mathbb{F}(\mathbb{R})$ and is called the *symmetrization of* u. The function u^* satisfies the relations

$$c = u^*(x) = u^*(-x),\; c > \inf\, u.$$

(2) Let $u \in \mathbb{F}(\mathbb{R}^N)$, $N \geq 2$ and $y := x_N$ where $x := (x', x_N)$, $x' := (x_1, ..., x_{N-1})$. For almost every $x' \in \mathbb{R}^{N-1}$ there exists a distribution function

$$m_u(x', c) := |\{y \in \mathbb{R} \;:\; u(x', y) > c\}|,\; c > \inf\, u.$$

u^* is called the *symmetrization of* u *with respect to* y.

1c. Continuous Steiner symmetrization for functions in $\mathbb{F}(\mathbb{R}^N)$

Let $u \in \mathbb{F}(\mathbb{R}^N)$. The set of functions u^t, $t \in \mathbb{R}^+$, defined by the relations

$$\begin{aligned}\left\{u^t > c\right\} &= \{u > c\}^t,\; c > \inf\, u,\\ \left\{u^t = \inf\, u\right\} &= \mathbb{R}^N \setminus \bigcup\nolimits_{c > \inf\, u} \{u > c\}^t,\ \text{and}\\ \left\{u^t = \infty\right\} &= \mathbb{R}^N \setminus \bigcap\nolimits_{c > \inf\, u} \{u > c\}^t,\end{aligned}$$

is called the *continuous Steiner symmetrization of* u *with respect to* y in the case $N \geq 2$ and the *continuous symmetrization* in the case $N = 1$. The set $\{u^t > c\}$ is the set of all $x \in \Omega$ such that $u^t > c$, where c is a constant. For $u \in \mathbb{F}(\mathbb{R}^N)$ take $u := u^0$ and $u^\infty := u^*$ (the Steiner symmetrization of u with respect to y). Before citing some properties of the continuous Steiner symmetrization defined above, it is necessary for us to recall some definitions for Steiner symmetrization of sets. (For more details see [1]).

Hardy-Littlewood Inequality: Let $u, v \in \mathbb{F}(\mathbb{R}^N)$ and let t be a real positive parameter. Then,

$$\int_{\mathbb{R}^N} u(\mathbf{x})v(\mathbf{x})\, d\mathbf{x} \leq \int_{\mathbb{R}^N} u^t(\mathbf{x})v^t(\mathbf{x})\, d\mathbf{x}.$$

2. The Main Theorem

We assume the following smoothness conditions to ensure uniqueness of the solution of the problem (4), (5), and (6) in divergence form and convergence of the integrals (in particular convergence of the convex functional J). Note that the uniform ellipticity condition corresponds to these inequalities in the special case $p = 2$. For more details concerning the uniqueness theorem, see [4]. Let u be a positive solution of the overdetermined boundary value problem (4)-(6) and let a be a real valued function defined on $\mathbb{R}\times\mathbb{R}^+$ which satisfies the following conditions:

$$a(u,s)\;\ge\;k_1s^{p-2}, \tag{18}$$

$$a(u,s)+sa_s(u,s)\;\ge\;k_2s^{p-2},\;\text{and} \tag{19}$$

$$|a(u,s)|+|a_u(u,s)|+sa_s(u,s)\;\le\;k_3(s^{p-2}+1) \tag{20}$$

for every $s\in\mathbb{R}^+$, every $x\in\Omega$, and for some appropriate positive constants k_1,k_2,k_3. We denote the partial derivative of a with respect to u and s by a_u and a_s respectively. The number p appearing in (18), (19), and (20) ranges over the interval $(1,+\infty)$.

We assume that the function ϕ in (4) is continuous, positive, and satisfies the following condition:

$$\int_\Omega \phi^{\frac{N}{2}}(\mathbf{x})\,d\mathbf{x}\;<\;\infty. \tag{21}$$

Main Theorem.
Let Ω be a convex domain in $\mathbb{R}^N$ and $u=u(\mathbf{x})$ be a nonnegative solution of (4)-(6). We assume that $F(\mathbf{x},u)$ satisfies the following conditions:

(1) F is measurable on $\mathbb{R}^N\times(\mathbb{R}^+\cup\{0\})$;

(2) F is differentiable with respect to u, and $\dfrac{\partial F}{\partial u}$ is even and nonincreasing in x_N; and

(3) $\psi_\epsilon:=\psi$ is a given positive continuous nondecreasing function on $\mathbb{R}^N$ for $\epsilon\ne+1$ and $\epsilon\ne-1$.

Also we suppose that ψ_ϵ satisfies one of the two following conditions:

(3a) ψ_ϵ is a given positive (resp. negative) continuous function on $\mathbb{R}^N$ which is superharmonic in $\mathbb{R}^N$, for $N=2$ and $\epsilon=1$ (respect. $\epsilon=-1$).

(3b) $|\nabla u|^2:=\ln r^{2(N-2)}\psi$ on the boundary $\partial\Omega$ of Ω where ψ is a given positive continuous function, which is superharmonic in $\mathbb{R}^N$ for $N\ge3$.

Then we conclude that Ω must be an N-ball.

For the proof of this theorem, we recall a lemma due to Mitidieri [5] and two important theorems of F. Brock [2].

Lemma 1. *(Mitidieri)*
If $\psi\in C^2(\mathbb{R}^N)$, $N\ge3$, is positive, radial, and superharmonic (i.e. $\Delta\psi\le0$ in $\mathbb{R}^N$), then for every $r\in(0,\infty)$ we have (in the obvious notation)

$$r\psi'(r)+(N-2)\psi(r)\;\ge\;0. \tag{22}$$

The first theorem of F. Brock [2] concerns some properties of continuous Steiner symmetrization of functions.

Theorem 1. *(F. Brock)*
Let $u\in\mathbb{F}^+(\mathbb{R}^N)$ and let $F:=F(\mathbf{x},u)$ be measurable on $\mathbb{R}^N\times(\mathbb{R}^+\cup\{0\})$. Further, assume that F is differentiable with respect to u and that the function $F_u(\mathbf{x},u)$ (the

first derivative of F with respect to u) is even and nonincreasing in y. Then it follows that for every $t \in [0, +\infty]$,

$$\int\limits_{\mathbb{R}^N} F(\mathbf{x}, u)\, d\mathbf{x} \le \int\limits_{\mathbb{R}^N} F(\mathbf{x}, u^t)\, d\mathbf{x}. \tag{23}$$

The second theorem of F. Brock [2] uses one more important condition which is the *local symmetry* of the positive solution u as defined in [2].

Theorem 2. *(F. Brock)*
Let Ω be a bounded convex domain and let u be a positive function in $H_0^1(\Omega) \cap C_{loc}^1(\Omega)$ that is locally symmetric in every direction - i.e. such that

$$\lim_{t\to 0} \frac{1}{t} \left\{ \int\limits_{\Omega_t} |\nabla u^t|^2\, d\mathbf{x} - \int\limits_{\Omega_t} |\nabla u|^2\, d\mathbf{x} \right\} < 0 \tag{24}$$

holds for every hyperplane H through the origin. Then u has the following form:

$$u(\mathbf{x}) = f_k(|\mathbf{x} - \mathbf{x}_k|) \tag{25}$$

in $C_k := \left\{ \mathbf{x} \in \mathbb{R}^N \mid r_k < |\mathbf{x} - \mathbf{x}_k| < R_k \right\}$, $k = 1, ..., m$, and is piecewise constant in G, where G and C_k are disjoint subsets of Ω such that $\Omega = \bigcup_{k=1}^m C_k \cup G$. Here $C_k := C_k(\mathbf{x}_k, r_k, R_k)$ are m ($\le \infty$) disjoint ring-shaped regions centered at $\mathbf{x}_k$ with interior and exterior radii $0 \le r_k < R_k$ and G is the subset of critical points of u.

Now we will prove our main theorem.

Proof of Main Theorem.

To begin, we suppose that the conditions (1)-(3) of the theorem are realized and we show that Ω is an N-ball.
We argue by contradiction, assuming that Ω is not a ball and constructing a deformation field v such that $dF(\Omega; v) \le 0$, where $F(\Omega)$ is the domain functional defined by:

$$\Omega_t := \left\{ (\mathbf{x}', y) \in \mathbb{R}^N \mid \mathbf{x}' \in \Omega',\, y_1(\mathbf{x}') - t\bar{y}(\mathbf{x}') < y < y_2(\mathbf{x}') - t\bar{y}(\mathbf{x}') \right\}, \tag{26}$$

with

$$\bar{y}(\mathbf{x}') := \frac{1}{2}[y_1(\mathbf{x}') + y_2(\mathbf{x}')]. \tag{27}$$

The corresponding Lipschitz (because Ω is convex) Steiner deformation field

$$V^* := (0, -\bar{y}(\mathbf{x}')), \quad \forall \mathbf{x}' \in \Omega', \tag{28}$$

generated by the continuous Steiner symmetrization, is constant on every straight line perpendicular to T. Furthermore, this justifies the use of the Hadamard formulas (see (38) below). We compute $d\,F(\Omega; V^*)$ explicitly:

$$d\,F(\Omega; V^*) \;:=\; \frac{d}{dt} F(\Omega_t)\,|_{t=0}, \tag{29}$$

$$d\,F(\Omega; V^*) = \int\limits_{P_r(\Omega)} d\mathbf{x}' \int\limits_{y_1(\mathbf{x}';t)}^{y_2(\mathbf{x}';t)} \psi(|\mathbf{x}'|^2 + y^2)\, dy\,|_{t=0}, \text{ and} \tag{30}$$

$$d\,F(\Omega; V^*) \;=\; \int\limits_{P_r(\Omega)} d\mathbf{x}' \int\limits_{y_1(\mathbf{x}') - t\bar{y}(\mathbf{x}')}^{y_2(\mathbf{x}') - t\bar{y}(\mathbf{x}')} \psi(|\mathbf{x}'|^2 + y^2)\, dy\,|_{t=0}. \tag{31}$$

Hence

$$d\,F(\Omega;V^*)\;:=\;\int\limits_{P_r(\Omega)}\quad -\bar{\mathbf{y}}(\mathbf{x}')[\psi(|\mathbf{x}'|^2+{y_1}^2(\mathbf{x}'))-\psi(|\mathbf{x}'|^2+{y_2}^2(\mathbf{x}'))]\,d\mathbf{x}', \qquad (32)$$

where $P_r(\Omega) := \{\mathbf{x}' \in \mathbb{R}^{N-1} \,|\, (\mathbf{x}', y) \in \Omega\}$ is the projection of Ω in $\mathbb{R}^{N-1}$. Since ψ is nondecreasing, $d\,F(\Omega;V^*)\;\le\;0$.
Now we show that the following inequality holds:

$$\int\limits_{\Omega}\;\phi\,(r)\,F(\mathbf{x},u)\,d\mathbf{x}\;\le\;\int\limits_{\Omega}\phi\,(r)\,F(\mathbf{x},u^t)\,d\mathbf{x}. \qquad (33)$$

Since the function F satisfies the conditions of Theorem 10 of [2], we deduce that:

$$\int\limits_{\mathbb{R}^N}\;F(\mathbf{x},u)\,d\mathbf{x}\;\le\;\int\limits_{\mathbb{R}^N}\;F(\mathbf{x},u^t)\,d\mathbf{x}. \qquad (34)$$

Using the Hardy-Littlewood inequality and the fact that $\psi(r)$ is a nonincreasing function independent of Ω, we conclude the desired result. To complete our proof we will combine the two results (24) and (32). From (4) we can derive the following inequality:

$$\{\int\limits_{\Omega}a(u,|\nabla\,u|^2)\,d\mathbf{x}-\int\limits_{\Omega_t}a(u^t,|\nabla\,u^t\,|^2)\,d\mathbf{x}\}\;\le\;0. \qquad (35)$$

Good functions and the Density Theorem

Definition 3: A function u is called *Good* if u, defined on $\mathbb{R}^N$, is positive, piecewise smooth with compact support, and for every $(x_1,...,x_{N-1}) \in \mathbb{R}^{N-1}$ and $c > 0$ the equation

$$u(x_1,...,x_{N-1},y)=c \qquad (36)$$

has only a finite (even) number of solutions $y = y_k$, $(k = 1,...,2m)$ and

$$\inf\{|\frac{\partial u(x)}{\partial y}|:\frac{\partial u(x)}{\partial y}\quad\text{exists and is non-zero}\}\;>\;0.$$

Inequality (35), based on the continuous Steiner symmetrization, is essential for our overdetermined problem in view of the following theorem. We consider a positive solution u in $W^{1,p}(\mathbb{R}^N)$, $1 \le p < +\infty$, and Remark 2 of [2]. The following lemma, which is a key step in the proof, summarizes the cited remark of [2].

Lemma 2. *Good functions are dense in $W_+^{1,p}(\mathbb{R}^N)$ in the $W^{1,p}(\mathbb{R}^N)$ norm.*

For the sake of completeness, we cite the important theorem due to F. Brock which allows us to establish our main inequality - Lemma 3.

Theorem 3. *Let u be a Good function. We assume: the functions $F(\mathbf{x}',u,z)$, $a(\mathbf{x}',y,z)$, $a_i(\mathbf{x}',u), i = 1,...,n-1$, $(\mathbf{x}' \in \mathbb{R}^{N-1}, u, z \in (\mathbb{R}^+ \cup \{0\}))$, are nonnegative* *and continuous in all arguments, $a(\mathbf{x}', y, z)$ is even and convex in y and $F(\mathbf{x}', u, z)$ is monotone, nondecreasing, and convex in z. Then,*

$$\int_{\mathbb{R}^N} (F(\mathbf{x}', u, \{a^2(\frac{\partial u}{\partial n})^2 + \Sigma_{i=1}^{N-1} a_i^2(\frac{\partial u}{\partial x_i})^2\}^{\frac{1}{2}})\, d\mathbf{x}$$
$$\geq \int_{\mathbb{R}^N} (F(\mathbf{x}', u, \{\tilde{a}^2(\frac{\partial u^t}{\partial n})^2 + \Sigma_{i=1}^{N-1} \tilde{a}_i^2(\frac{\partial u^t}{\partial x_i})^2\}^{\frac{1}{2}})\, d\mathbf{x},$$

for every $t \in [0, +\infty]$, where for simplicity we wrote
$u = u(\mathbf{x}), u^t = u^t(\mathbf{x})$, $a = a(\mathbf{x}, u(\mathbf{x}))$, $\tilde{a} = a(\mathbf{x}', u^t(\mathbf{x}))$ $a_i = a_i(\mathbf{x}', u(\mathbf{x}))$, and $\tilde{a}_i = a_i(\mathbf{x}', u^t(\mathbf{x}))$, $i = 1, ..., N-1$.

We are now able to prove our inequality by assuming that $a(u, |\nabla u|)$ satisfies the following conditions:
(i) a is nonnegative and continuous in all the arguments, and
(ii) a is monotone, nondecreasing, and convex in $|\nabla u|$.

Lemma 3. *We suppose that (i) and (ii) are satisfied by the function a. Then:*

$$\int_{\mathbb{R}^N} a(u, \{(\frac{\partial u}{\partial n})^2 + \Sigma_{i=1}^{N-1}(\frac{\partial u}{\partial x_i})^2\}^{\frac{1}{2}})\, d\mathbf{x} \geq \int_{\mathbb{R}^N} a(u, \{(\frac{\partial u^t}{\partial n})^2 + \Sigma_{i=1}^{N-1}(\frac{\partial u^t}{\partial x_i})^2\}^{\frac{1}{2}})\, d\mathbf{x}.$$

Proof. By a suitable choice of a_i, a, $\tilde{a}_i$, $\tilde{a}$ we see that this is an immediate consequence of Theorem 3 above. To complete the argument, it is necessary to apply the well known theorem of Ladyzhenskaya and Uralt'sceva [4] on $W_0^{1,p}(\Omega)$ which says that if the function a satisfies the smoothness conditions (18), (19), and (20) then the following elliptic problem,

$$-\operatorname{Div}(a(u, |\nabla u|)\nabla u) = \phi(r)F(x, u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

has as a unique solution u which is characterized as the minimum of $J(\Omega)$ where

$$J(\Omega) := \int_{\Omega} a(u, |\nabla u|^2)|\nabla u|^2 - \phi(r)F(x, u))\, d\mathbf{x}.$$

Let us define a new functional G as follows:

$$G(\Omega) := \int_{\Omega} \psi(r^2)\, \mathrm{d}\mathbf{x}. \tag{37}$$

Since ψ does not depend on Ω, the derivative of $G(\Omega)$ with respect to the deformation field v is given by the classical Hadamard formula [7], [8], [9]:

$$dG(\Omega, v) := \int_{\partial\Omega} \psi(r^2) v \cdot n\, d\mathbf{x}. \tag{38}$$

As mentioned in [3], the derivative of $G(\Omega)$ with respect to v is given by:

$$d\,G(\Omega; v^*) = \int_{P_r(\Omega)} \bar{\mathbf{y}}(\mathbf{x}')[\psi(|\mathbf{x}'|^2 + {y_1}^2(\mathbf{x}')) - \psi(|\mathbf{x}'|^2 + {y_2}^2(\mathbf{x}'))]\, \mathrm{d}\mathbf{x}'.$$

It is clear that $d\,G \leq 0$ since ψ is nondecreasing. Finally, we claim the positive solution u is locally symmetric in the sense of Brock [2]. Using (31) and (37) we conclude:

Lemma 4. *-$dG = dF$ where G and F are defined by (37) and (4).*

By Theorem (2) and Lemma (2),

$$\left\{ \int_{\Omega} a(u, |\nabla u|^2)\, d\mathbf{x} - \int_{\Omega_t} a(u^t, |\nabla u^t|^2)\, d\mathbf{x} \right\} \leq \int_{\mathbb{R}^N} F(\mathbf{x}, u)\, d\mathbf{x} - \int_{\mathbb{R}^N} F(\mathbf{x}, u^t)\, d\mathbf{x}. \tag{39}$$

Multiplying both sides of (39) by $\frac{1}{t}$ and letting t tend to zero, we obtain,

$$d\,J(\Omega, v) := \left\{ \int_{\Omega} a(u, |\nabla u|^2)\, d\mathbf{x} - \int_{\Omega_t} a(u^t, |\nabla u^t|^2)\, d\mathbf{x} \right\} \leq 0.$$

Consequently, Lemma (3) gives

$$0 \leq \left\{ \int_{\Omega} a(u, |\nabla u|^2)\, d\mathbf{x} - \int_{\Omega_t} a(u^t, |\nabla u^t|^2)\, d\mathbf{x} \right\} \leq 0.$$

Since the hyperplane is arbitrary, we have proved that u is locally symmetric in every direction. We complete the proof of the Main Theorem using the famous Theorem 2 of F. Brock. We observe that G does not reach the boundary $\partial\Omega$ as $|\nabla u|^2 := \psi(r^2)$ is positive on this boundary by (3) (third condition of the Theorem 1) . Moreover, the ring-shaped subsets C_k are locally finite near the boundary. Indeed, let $x \in \partial\Omega$ and assume that there exists a sequence of disjoint subsets C_{x_k, r_k, R_k} such that $x_k \to x$, $r_k \to 0$, and $R_k \to 0$ as $k \to \infty$. Select two points ξ_k, $\eta_k \in C_k$ such that $x_k = \frac{1}{2}(\xi_k + \eta_k)$. From (28) we have $\nabla u_k = -\nabla u_k$. This implies

$$\nabla u(x) = \lim_k \nabla u(\xi_k) = -\lim_k \nabla u(\eta_k) = -\nabla u(x) = 0,$$

in contradiction to $|\nabla u|^2 = \psi(r^2)$. Since Ω is convex it is clear that $\partial\Omega$ coincides with the exterior boundary of a single ring-shaped subset C_k. The proof is as above for both of the possible conditions (a) and (b).

An application to positive solutions of some semilinear elliptic problems in symmetric domains

In this section, we give an application to this new alternative approach to symmetric domains. This application generalizes Theorem 14 of F. Brock [2]. In fact, we will prove the following theorem.

Theorem. *Let $u \in {W_0}^{1,p}(\Omega)$ $(1 < p < +\infty)$ be a positive solution of the problem P_ϵ in a bounded domain Ω which is symmetric with respect to $\{y = 0\}$, convex in the y-direction and such that $F := \phi(r) f(x, u)$. We assume that ϕ is defined and real valued on $\mathbb{R}^+$ and that f is a bounded, even function, defined in $\Omega \times (\mathbb{R}^+ \cup \{0\})$ and taking real values which are monotonicly nonincreasing in y. We assume that $u \in C(\bar{\Omega})$. Then, (i) u is locally symmetric in the y direction. Further, if f is strictly monotonicly decreasing in y for $y > 0$, then (ii) u is symmetric and decreasing in y. Finally, in the case of an N-ball ($\Omega = B_R$) with a positive radius*

R and with $f = f(|x|, u)$ and $F = \phi(r)f(|x|, u)$ monotonously nonincreasing in $|x|$, then (iii) u is locally symmetric in every direction.

Proof:
(i) Since u is an element of ${W_0}^{1,p}(\Omega)$, then u^t is an element of ${W_0}^{1,p}(\Omega)$ and by (4), we obtain,

$$\int_\Omega a(u, |\nabla u|)|\nabla (u^t - u)|\, d\mathbf{x} \geq \int_\Omega \phi(r)f(x,u)(u^t - u)\, d\mathbf{x},$$

for all $t \in [0, +\infty]$. Using the inequality

$$\int_\Omega \phi(r)f(x,u)\, d\mathbf{x} \leq \int_{\Omega^t} \phi(r)f(x,u^t)\, d\mathbf{x},$$

we conclude that

$$\lim_{t\to 0} \frac{1}{t} \int_\Omega a(u^t, |\nabla u^t|)|\nabla (u^t)|^2\, d\mathbf{x} - \int_\Omega a(u, |\nabla u|)|\nabla (u)|^2\, d\mathbf{x} = 0.$$

Consequently, u is locally symmetric, which is the desired result.
(ii) If f is nondecreasing in the positive variable y, we can find

$$x^1 = (x_0', y_1),\ x^2 = (x_0', y_2) \text{ in } \Omega,$$

with

$$y_1 + y_2 \neq 0. \tag{40}$$

By the hypothesis on u, $\frac{\partial u}{\partial y} > 0$ at x^1.
Let U_1 denote the (maximal) connected component of $\Omega \cap \{x : u_y(x) > 0\}$ containing x^1, where $x^2 = (x_0', y_2) \in \Omega$, $y_1 < y_2$,
and $u(x^1) = u(x^2) < u(x_0', y)$ for all y in (y_1, y_2). Then, for all $(x^1, y) \in U_1$, $u(x^1, y) = (x^1, y_1 + y_2 - y) < u(x^1, z)$ for all z in $(y, y_1 + y_2 - y)$.
We put $v(x^1, y) = u(x^1, y_1 + y_2 - y)$ and apply Theorem 13 of Brock to conclude that $v(x) = u(x)$ for all $x \in U \subset V(x^1)$. Using the equation in the statement of the problem, $w := u - v$ which is zero. Then,

$$-\mathrm{Div}(a(w, |\nabla w|)|\nabla w|) = \phi(r)[f(x^1, y, u) - f(x^1, y_1 + y_2 - y, u)] \quad \text{in} \quad U^1,$$

which contradicts (40).
(iii) Let Ω be the ball B_R and $f = f(|x|, u)$. If we associate to x the value ξ defined by $\xi := (\xi^1, \eta)$ for an arbitrary rotation of the coordinate system about the origin, we see that f is not even and nonincreasing in η. By the above considerations, this yields the last assertion of the theorem.

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