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Differential operators on equivariant vector bundles over symmetric spaces *

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Abstract

Generalizing the algebra of motion-invariant differential operators on a symmetric space we study invariant operators on equivariant vector bundles. We show that the eigenequation is equivalent to the corresponding eigenequation with respect to the larger algebra of all invariant operators. We compute the possible eigencharacters and show that for invariant integral operators the eigencharacter is given by the Abel transform. We show that sufficiently regular operators are surjective, i.e. that equations of the form $Df = u$ are solvable for all u.

1 Equivariant vector bundles

Let X denote a manifold with a smooth action of a Lie group G. An *equivariant vector bundle* E over X is a smooth vector bundle $\pi : E \to X$ together with a smooth action of G on E such that $\pi(gv) = g\pi(v)$ for $v \in E$ and such that all maps of the fibres $g : E_x \to E_{gx}$, $g \in G$, $x \in X$, are linear. An example is given by $E = X \times V$, where (σ, V) is a finite dimensional representation of G and G acts on E by $g(x, v) = (gx, \sigma(g)v)$.

Now assume X to be a homogeneous space, i.e. $X = G/H$ for a closed subgroup H of G. Given an equivariant bundle E over X we get a representation of H on the fibre over eH, where e denotes the neutral element of G. Conversely given a representation τ of H on a finite dimensional vector space V we let H act from the right on $G \times V$ by $(g, v)h = (gh, \tau(h)^{-1}v)$ and define $E_\tau = (G \times V)/H$. This is a vector bundle over X. This construction gives an equivalence of categories between the category of equivariant vector bundles over X and the category of finite dimensional representations of H. If τ splits as a direct sum $\tau = \tau_1 \oplus \tau_2$ then $E_\tau \cong E_{\tau_1} \oplus E_{\tau_2}$ and vice versa.

Now let G denote a connected semisimple Lie group with finite center and let K denote a maximal compact subgroup. The quotient $X = G/K$ is diffeomorphic to $\mathbb{R}^n$ for some n. Using the Killing form of G one defines a Riemannian metric on X such that the group G acts by isometries. This is the most general symmetric space without compact factors. See [5] for further details.

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2 Differential operators

Let $\hat{K}$ denote the set of isomorphism classes of irreducible unitary representations of the group K. Since K is compact, every $\tau \in \hat{K}$ is finite dimensional. We do not distinguish between a class in $\hat{K}$ and a representative. For $(\tau, V_\tau) \in \hat{K}$ let $E_\tau \to G/K$ denote the vector bundle as in section 1. The group G acts on the space of smooth sections $\Gamma^\infty(E_\tau)$ of the bundle E_τ by $g.s(x) = gs(g^{-1}x)$, $x \in X$, $g \in G$, $s \in \Gamma^\infty(E_\tau)$. Hence G also acts on differential operators by conjugation. Let $\mathcal{D}_\tau$ denote the algebra of G-invariant differential operators on E_τ, i.e. those operators D that satisfy $D(g.s) = g.D(s)$ for any $s \in \Gamma^\infty(E_\tau)$.

Let $C_\tau^\infty(G)$ denote the space of all infinitely often differentiable maps from G to V with $f(kx) = \tau(k)f(x)$ for $k \in K$ and $x \in G$. The group G acts on the space $C_\tau^\infty(G)$ by translations from the right. The map $s \mapsto f_s$ with $f_s(x) = xf_s(x^{-1})$ gives a G-isomorphism of $\Gamma^\infty(E_\tau)$ to $C_\tau^\infty(G)$. We conclude that the algebra $\mathcal{D}_\tau$ acts on $C_\tau^\infty(G)$.

Let the compact group K act on the space $C^\infty(G)$ of smooth functions on G by

$$L_k(x) = f(k^{-1}x), \quad k \in K,\ x \in G.$$

Then there is a decomposition into K-isotypes:

$$C^\infty(G) = \bigoplus_{\tau \in \hat{K}} C^\infty(G)(\tau).$$

The sum here means that finite sums on the right hand side are dense in the Fréchet space $C^\infty(G)$. Further $C^\infty(G)(\tau)$ is the space of functions in $C^\infty(G)$ that transform under L according to τ.

For $(\tau, V) \in \hat{K}$ fix any nonzero v^* in the dual space V^* and for $f \in C_\tau^\infty(G)$ let $Bf(x) = v^*(f(x))$. Then $Bf \in C^\infty(G)(\tau)$ and the map B is a G-isomorphism. We have

$$\Gamma^\infty(E_\tau) \;\cong\; C_\tau^\infty(G) \;\cong\; C^\infty(G)(\tau).$$

Let $\mathfrak{g}$ denote the Lie algebra of G. The universal enveloping algebra $U(\mathfrak{g})$ may be viewed as the algebra of all right invariant differential operators on G, i.e. the algebra of all differential operators D on G such that $R_g D = DR_g$ for any $g \in G$. Here for a function $\varphi \in C^\infty(G)$ and $g \in G$ we have $R_g\varphi(x) = \varphi(xg)$ for all $x \in G$. Let $U(\mathfrak{g})^K$ denote the subalgebra of all differential operators which are K-invariant on the left side, i.e. which satisfy $L_k D = DL_k$ for any $k \in K$. The algebra $U(\mathfrak{g})^K$ leaves stable the decomposition of $C^\infty(G)$ and thus acts on $C^\infty(G)(\tau)$. We therefore get an algebra homomorphism

$$\varphi_\tau : U(\mathfrak{g})^K \to \mathcal{D}_\tau.$$

Proposition 2.1 *The homomorphism φ_τ is surjective and the intersection of all kernels* $\ker(\varphi_\tau)$ *for varying τ is zero.*

Proof: Let the algebra $U(\mathfrak{g}) \otimes \mathrm{End}(V)$ acts on the space $C^\infty(G, V)$ of smooth functions on G with values in V. The group K acts on $U(\mathfrak{g})$ via the adjoint representation and on $\mathrm{End}(V)$ by means of conjugation via τ. Then the algebra of K-invariants, $(U(\mathfrak{g}) \otimes \mathrm{End}(V))^K$ acts on C^∞_τ. The annihilator I of C^∞_τ in $U(\mathfrak{g}) \otimes \mathrm{End}(V)$ is generated by elements of the form $XY \otimes T + X \otimes T\tau(Y)$ with $X \in U(\mathfrak{g})$, $Y \in \mathfrak{k}$ and $T \in \mathrm{End}(V)$. Since K is reductive we have

$$\mathcal{D}_\tau \;\cong\; (U(\mathfrak{g}) \otimes \mathrm{End}(V))/I)^K \;\cong\; (U(\mathfrak{g}) \otimes \mathrm{End}(V))^K / I^K$$

Since τ is irreducible and K is connected it follows that the representation of $U(\mathfrak{k})$ induced by τ also is irreducible. Its image $\tau(U(\mathfrak{k}))$ then may be considered as a von Neumann algebra (it is weakly closed since it is finite dimensional), hence it coincides with its bicommutator. Its commutator, however, is $\mathbb{C}\mathrm{Id}$, hence $\tau(U(k)) = \mathrm{End}(V)$. This implies that any element of $\mathcal{D}_\tau$ can be written in the form $Z \otimes 1$ for some $Z \in U(\mathfrak{g})$. It follows that Z must be in $U(\mathfrak{g})^K$, which implies the surjectivity of φ_τ.

For the second assertion assume X is in the intersection of all kernels. Then, since $C^\infty(G) = \bigoplus_\tau C^\infty(G)(\tau)$, we get that $Xf = 0$ for every $f \in C^\infty(G)$, this gives $X = 0$. □

Corollary 2.2 *$\mathcal{D}_\tau$ is finitely generated as $\mathbb{C}$-algebra.*

Proof: $U(\mathfrak{g})$ has a natural filtration by order. The associated graded version equals the symmetric algebra $S(\mathfrak{g})$ over $\mathfrak{g}$. Since the adjoint action of K preserves the filtration, we have a filtration on $U(\mathfrak{g})^K$ with graded version $S(\mathfrak{g})^K$. The latter is finitely generated by invariant theory, hence the former is, too. □

Examples

For $\tau = 1$ the trivial representation, the algebra $\mathcal{D}_\tau$ is the algebra of G-invariant differential operators on G/K. In this case $\mathcal{D}_\tau$ is isomorphic to the polynomial ring in r generators, where r is the rank of the symmetric space G/K (see [6] II.5).

Let $G = SO(n,1)^+$ the group of motions on the hyperbolic space H_n. For $\tau = \wedge^p(\mathrm{Ad})$ the space $\Gamma^\infty(E_\tau)$ is just the space of p-differential forms on H_n. For $p = 0$ or $p = n$ the algebra $\mathcal{D}_\tau$ is the polynomial ring in one variable, generated by δd and $d\delta$ respectively, where d is the exterior differential and δ its formal adjoint. For n odd and $p = (n \pm 1)/2$ the algebra $\mathcal{D}_\tau$ is generated by $d\delta$ and $*d$ resp. δd and $d*$, where $*$ is the Hodge operator, with the generating relations

$$d\delta * d = 0 = *dd\delta, \quad \delta dd* = d * \delta d = 0.$$

In all other cases $\mathcal{D}_\tau$ is generated by $d\delta$ and δd with the relations $d\delta\delta d = 0 = \delta dd\delta$. Summarizing we get the structure of $\mathcal{D}_\tau$ as:

$$\mathcal{D}_\tau \cong \begin{cases} \mathbb{C}[x,y]/xy & \text{for } 1 \le p \le n-1, \\ \mathbb{C}[x] & \text{for } p = 0, n, \end{cases}$$

These results are proven as follows: First the algebras generated by $d\delta$ and δd give subalgebras of the algebras of invariant operators with the given structure.

One then has to show that these are all, which is done by considering the corresponding graded algebras of principal symbols which are describable by an invariant problem. Explicit calculations show that the dimensions of the graded parts already give the dimensions of the graded parts of the subalgebras.

3 Integral operators

A smooth function $\Phi : G \to \mathrm{End}(V)$ which satisfies $\Phi(kxl) = \tau(k)\Phi(x)\tau(l)$ for $x \in G$, $k, l \in K$ is called *τ-sherical.* The algebra $\mathcal{D}_\tau$ acts on the set of τ-spherical functions. Compactly supported τ-spherical functions form an algebra under convolution:

$$\Phi * \Psi(x) = \int_G \Phi(y)\Psi(y^{-1}x)\, dy\,,$$

where dy denotes a Haar measure on G. This algebra is denoted by $\mathcal{A}_\tau$. The algebra $\mathcal{A}_\tau$ acts on $C_\tau^\infty(G)$ by $L_\Phi f = \Phi * f$ for $\Phi \in \mathcal{A}_\tau$ and $f \in C_\tau^\infty$. The algebra $\mathcal{A}_\tau$ contains an approximate identity.

Let $D_1, \dots, D_n$ be a set of generators of the algebra $\mathcal{D}_\tau$. For $z \in \mathbb{C}^n$ let

$$\mathrm{Eig}(z) = \{f \in C^\infty(G, \mathrm{End}(V))|\ f \text{ is } \tau-\text{spherical and } D_j f = z_j f\}.$$

Proposition 3.1 *For any $z \in \mathbb{C}^n$ we have* $\dim \mathrm{Eig}(z) \le 1$.

Proof: By Lemma 1 in [2] any $f \in \mathrm{Eig}(z)$ is analytic. So let $f \in \mathrm{Eig}(z)$ and $H \in \mathfrak{g}$. If H is small enough then

$$f(\exp(H)) = \sum_{n\ge 0} \frac{1}{n!}\left(\frac{\partial}{\partial t}\right)^n f(\exp(tH))|_{t=0} \ = \sum_{n\ge 0}\frac{1}{n!} H^n f(0).$$

We get

$$\int_K \tau(k) f(\exp(H))\tau(k^{-1})\, dk = \sum_{n\ge 0}\int_K (\mathrm{Ad}(k)H)^n f(e)\, dk\,.$$

Now $\int_K (\mathrm{Ad}(k)H)^n\, dk$ is a right invariant differential operator mapping the space $C_\tau^\infty(G)$ to itself, hence defines an element of the algebra $\mathcal{D}_\tau$. So the values $\int_K (\mathrm{Ad}(k)H)^n f(e)\, dk$ only depend on z and not on f. We conclude that the function $\mathrm{tr} f(x)$ is determined by z up to scalar. But the map $\mathcal{A}_\tau \to C^\infty(G)$, $f \to \mathrm{tr}(f)$ is injective [8]. $\square$

Theorem 3.2 *Suppose $f \in C_\tau^\infty(G)$ is an eigenform for any $D \in \mathcal{D}_\tau$. Then f is an eigenform for every $T \in \mathcal{A}_\tau$ with an eigenvalue only depending on T and the eigenvalues on $\mathcal{D}_\tau$.*

Proof: For $\varphi \in C_\tau^\infty(G)$ and $w \in V$ define $\varphi_w \in C^\infty(G, \mathrm{End}(V))$ by

$$\varphi_w(x)v = \langle v, w\rangle \varphi(x),$$

where $\langle .,. \rangle$ denotes the scalar product on V. Fot $psi : G \to \mathrm{End}(V)$ let

$$M\psi(x) = \int_K \psi(xk)\tau(k^{-1})\, dk\,.$$

Now let f be as in the theorem and assume $f(e) \neq 0$. (Otherwiese replace f by $R_g f$ for a suitable $g \in G$.) Fix some $w \in V$ such that $\mathrm{tr} f_w(e) = 1$. Now $M(f_w)$ lies in $\mathrm{Eig}(z)$ for some z. Let Φ be τ-spherical and compactly supported. Since $L_\Phi M(f_w) = M((L_\Phi f)_w)$ we see that there is a $\lambda \in \mathbb{C}$ such that

$$M((\lambda f - L_\Phi f)_w) = 0,$$

and λ does not depend on f or w. The claim now follows by the proposition. $\square$

Let $\mathfrak{a}$ denote the Lie algebra of the maximal $\mathbb{R}$-split torus A of G. Let $G = KNA$ be a corresponding Iwasawa decomposition. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ the complex dual space of $\mathfrak{a}$ and any $v \in V$ let

$$p_{\lambda,v}(kna) = \tau(k)e^{\lambda(\log(a))}v.$$

This defines $p_{\lambda,v} \in C^\infty_\tau(G)$. Let $P = MAN$ be the minimal parabolic given by A and N. Then M is a closed subgroup of K.

Lemma 3.3 *For every $T \in \mathcal{A}_\tau \oplus \mathcal{D}_\tau$ and every $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ there is a $S_\lambda(T)$ in* $\mathrm{End}(V)$ *such that for all $v \in V$*

$$T(p_{\lambda,v}) = p_{\lambda,S_\lambda(T)v}.$$

We have $S_\lambda(T)\tau(m) = \tau(m)S_\lambda(T)$ for all $m \in M$

Proof: The lemma is clear by group invariance and the fact that A normalizes N. $\square$

For a simultaneous eigenform $f \in C^\infty_\tau(G)$ of $\mathcal{D}_\tau$ let χ_f denote the eigencharacter $\chi_f : \mathcal{A}_\tau \oplus \mathcal{D}_\tau \to \mathbb{C}$ defined by

$$Tf = \chi_f(T)f.$$

Theorem 3.4 *Let f denote a bounded simultaneous eigenform. Then there is a $v \in V$ and a $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ such that $p_{\lambda,v}$ is an eigenform and*

$$\chi_f = \chi_{p_{\lambda,v}}.$$

Proof: The character χ_f is determined by its values on $\mathcal{A}_\tau$. Let p denote the trivial seminorm on G, i.e. $p(g) = 1$ for all $g \in G$. Set

$$\parallel \Phi \parallel_p = \int_G \parallel \Phi(y) \parallel p(y)\, dy,$$

where $\| \,.\, \|$ is the norm on $\mathrm{End}(V)$. Assume $\| f(e) \| = 1$ and $\| f(x) \| \leq M$, $x \in G$. Then we get for $\Phi \in \mathbb{A}_\tau$:

$$\begin{aligned} |\chi_f(L_\Phi)| &= \| \int_G \Phi(y) f(y^{-1})\, dy \| \\ &\leq M \int_G \| \Phi(y) \| \, dy \\ &\leq M \| \Phi \|_p \,. \end{aligned}$$

Thus χ_f is a p-continuous representation of $\mathcal{A}_\tau$. The claim now follows from the theorem of Glover [8], p. 40. □

We now give the computation of the eigencharacters of $\mathcal{A}_\tau$. Let $\rho = \frac{1}{2}\sum_{\alpha>0} m_\alpha \alpha \in \mathfrak{a}^*$ be the usual modular shift, i.e. the sum runs over all positive roots and m_α is the dimension of the root space to the root α.

Theorem 3.5 *Let $S_\lambda(L_\Phi)$ denote the endomorphism of Lemma 3.3. Then with*

$$g_\Phi(a) = a^{-\rho} \int_N \Phi(na)\, dn,$$

(Abel transform), we get

$$S_\lambda(L_\Phi) = \int_A a^{\rho-\lambda} g_\Phi(a)\, da,$$

(Fourier transform on A). Moreover, $g_\Phi(a)$ is in the center of $\tau|_M$ and $g_{\Phi*\Psi} = g_\Phi * g_\Psi$ with A-convolution on the right hand side. The map $\Phi \to g_\Phi$ is injective.

Proof: A calculation using the integral formula of the Iwasawa decomposition gives the first claim. The injectivity is proved in [8], p. 35. □

4 Surjectivity of differential operators

Let θ denote the Cartan involution fixing K pointwise. Let $\Gamma_1, \ldots, \Gamma_r$ denote a complete system of nonconjugate θ-stable Cartan subgroups of G and let $A_i = \Gamma_i \cap \exp(\mathfrak{p})$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denotes the polar decomposition of $\mathfrak{g}$. Let A denote one of the A_i. Let $L = MA$ denote the centralizer of A in G. Let τ_M denote the restriction f τ to $K_M = K \cap M$. Let $C^\infty(MA, \tau)$ denote the set of τ_M-spherical functions on MA. For $g \in C^\infty(MA, \tau_M)$ let $g^\#(kman) = \tau(k)g(ma)$. The set of these functions $g^\#$ is stable under $\mathcal{D}_\tau$ and we get a homomorphism

$$\gamma : \mathcal{D}_\tau \to \mathcal{D}_{\tau_M}(MA) = \mathcal{D}_{\tau_M} \otimes U(\mathfrak{a}).$$

Now every $T \in \mathcal{D}_{\tau_M}$ operates on the finite dimensional space of rapidly decreasing cusp forms ${}^0\mathcal{C}(M, \tau_M)$ as defined in [3]. So $\gamma(D)$ defines an element ${}^0\gamma(D) \in |End({}^0\mathcal{C}(M, \tau_M)) \otimes U(\mathfrak{a})$. Let $\det({}^0\gamma(D)) \in U(\mathfrak{a})$ denote the determinant. Call D *regular* if $\det({}^0\gamma(D)) \neq 0$ for all $A = A_i$.

Theorem 4.1 *If $D \in \mathcal{D}_\tau$ is regular, then D is surjective as a map from C_τ^∞ to C_τ^∞.*

Proof: We are going to formulate a vector bundle version of Holmgren's uniqueness theorem. Let X denote a real analytuc manifold and E an analytic vector bundle over X. Let P be a differential operator on E with analytic coefficients. For a susbset A of X let $N(A)$ denote the set of normal vectors to A in T^*X, (see [7], chap. 8). Consider the determinant of the principal symbol σ_P as a map:

$$\det \sigma_p : T^*X \to \mathbb{C}.$$

Proposition 4.2 *(Holmgren's principle) Let $u \in \Gamma_c^\infty(E)'$ be a distribution with $Pu = 0$. Then we have*

$$\det \sigma_P(N(\mathrm{supp}(u))) = 0.$$

Proof: Since the assertion is local in nature the proof for the trivial bundle ([7], Theorem 8.6.5) carries over to the present case. □

Let P be as above. A point $x \in X$ is called a *singular point* of P if

$$\det \sigma_P(T_x^*X) = 0.$$

Now consider the case when X is a symmetric space G/K and let $E = E_\tau$ be a homogeneous bundle. Assume that P is invariant, i.e. $P \in \mathcal{D}_\tau$. Let ∇ denote the canonical homogeneous connection on E. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ let $p_\lambda(ank) = e^{\lambda(\log(a))}$.

Lemma 4.3 *There is a section γ_P of the bundle $S(\mathfrak{a}) \otimes \mathrm{End}(E)$ such that*

$$P(p_\lambda s) = \gamma_P(\lambda) p_\lambda s,$$

for all parallel sections s of E. For this section γ_P and any parallel s we have

$$\sigma_P(dp_\lambda)s = \gamma_{P,m} s,$$

where $m = \deg(P)$ and $\gamma_{P,m}$ is the principal part of γ_P with respect to the gradation of $S(\mathfrak{a})$.

Proof: The first part is well known, the second follows from a calculation in Iwasawa coordinates. □

A differential operator P on E is called *D-convex*, if for any Weyl group stable compact convex subset $\mathfrak{s}$ of $\mathfrak{a}$ and anny section s of E with compact support and $\mathrm{supp}(Ps) \subset K\exp(\mathfrak{s})K$ we already have $\mathrm{supp}(s) \subset K\exp(\mathfrak{s})K$.

Theorem 4.4 *Any $P \in \mathcal{D}_\tau$ with $\det \sigma_P \neq 0$ is D-convex.*

Proof: Let P be as in the theorem. Since $\det \sigma_P$ is G-invariant, it follows from $\det \sigma_P \neq 0$ that P has no singular points. Let for $x \in X$, $x = k \exp(H)K$, $H \in \mathfrak{a}$,

$$\begin{aligned} \delta(x) &= \inf\{t > 0 | \frac{H}{t} \in \mathfrak{s}\}, \\ \delta(A) &= \sup\{\delta(x) | x \in A\}, \quad \text{for } A \subset X. \end{aligned}$$

Assume $\delta(\mathrm{supp} Ps) = 1$ and $\delta(\mathrm{supp} s) >> 1$. Let $x_0 \in \mathrm{supp} s$ with $\delta(x_0) = \alpha$ and $x_0 = \exp(H)$, H in the positive Weyl chamber of $\mathfrak{a}$. Let $\lambda \in \mathfrak{a}_+^*$. With Kostants convexity theorem we get

$$p_\lambda(x) \ \leq \ p_\lambda(x_0),$$

for all $x \in \mathrm{supp} s$. So $dp_\lambda(x_0) \in N(\mathrm{supp} s)$. Hence $\det \sigma_P(dp_\lambda(x_0)) = 0$ for all $\lambda >> 0$, hence for all λ. By group invariance this gives that x_0 is a singular point for P, a contradiction. This proves Theroem 4.4. □

Now regular operators P satisfy the condition of Theroem 4.4. They further admit fundamental solutions [1]. Now Theorem 4.1 follows as in [4]. □

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It was brought to my knowledge by Werner Hoffmann that the proof of Theorem 4.4 of the paper *Differential operators on equivariant vector bundles over symmetric spaces*, Electron. J. Diff. Eqns., Vol. 2000, No. 59, pp. 1-8 (2000) was lacking. Indeed the principal idea was shortened beyond recognition, so that I have set up this erratum.

Proof of Theorem 4.4: Let $\mathfrak{s} \subset \mathfrak{a}$ be convex compact and stable under the Weyl group W. Let s be a smooth section of E with compact support and $\operatorname{supp}(Ps) \subset K\exp(\mathfrak{s})K = X_{\mathfrak{s}}$. Expanding s as a sum of K-finite vectors we have that s is supported in the K-stable set $X_{\mathfrak{s}}$ if and only if every summand is. Moreover, the operator P may be applied termwise to the sum and hence we see that we may assume s to be K-finite. Then the support of s is K-invariant and it suffices to show that $\operatorname{supp}(s) \cap AK$ is a subset of $\exp(\mathfrak{s})K$.

Assume that $\operatorname{supp}(s) \cap AK$ is not a subset of $\exp(\mathfrak{s})K$. Let

$$\operatorname{supp}_{\mathfrak{a}}(s) = \{Y \in \mathfrak{a} | \exp(Y)K \in \operatorname{supp}(s)\}.$$

Then $\operatorname{supp}_{\mathfrak{a}}(s)$ is not a subset of $\mathfrak{s}$. Let L be the set of all linear maps $\lambda : \mathfrak{a} \to \mathbb{R}$ such that

$$0 < \max_{Y\in\mathfrak{s}} \lambda(Y) < \max_{Y\in\operatorname{supp}_{\mathfrak{a}}(s)} \lambda(Y).$$

By the assumption L is a nonempty open subset of the dual space $\mathfrak{a}^*$.

Recall that by the invariance of P the map $\det\sigma_P : T^*X \to \mathbb{C}$ is G-invariant. We identify

$$\begin{aligned} G\backslash T^*X &\cong K\backslash T^*_{eK}X \\ &\cong K\backslash\mathfrak{p}^* \\ &\cong W\backslash\mathfrak{a}^* \end{aligned}$$

and so $\det\sigma_P$ is given by a W-invariant polynomial map on $\mathfrak{a}^*$. Note that the first identification step is given by mapping a vector $v \in T^*_{xK}$ to $x^{-1}v$.

Let $\lambda \in L$. Since L is invariant under the Weyl group we may choose λ to be antidominant. Let $Y_0 \in \operatorname{supp}_{\mathfrak{a}}(s)$ be a point at which the maximum $\max_{Y\in\operatorname{supp}_{\mathfrak{a}}(s)} \lambda(Y)$ is taken. Then $Y_0 \notin \mathfrak{s}$ and

$$\lambda(Y) \le \lambda(Y_0)$$

for any $Y \in \operatorname{supp}_{\mathfrak{a}}(s)$. Let $a_0 = \exp(Y_0)$ then $a_0K \notin \operatorname{supp}(Ps)$ but $a_0K \in \operatorname{supp}(s)$. Let $g_\lambda : X \to \mathbb{R}$ be defined by

$$g_\lambda(naK) = \lambda(\log(a)).$$

Lemma 4.5 *For any $x \in \operatorname{supp}(s)$ we have $g_\lambda(x) \le g_\lambda(a_0K)$.*

Proof: Write $x = naK$. We have to show that

$$\lambda(\log(a)) \le \lambda(Y_0).$$

For this write $na = k_1 \exp(Z) k_2$ for some $Z \in \mathrm{supp}_{\mathfrak{a}}(s)$ and $k_1, k_2 \in K$. By the convexity theorem of van den Ban, E. [*A convextity theorem for semisimple symmetric spaces.* Pacific J. Math. 124, 21-55 (1986)] we infer that $\log(a)$ is in the convex hull of WZ. Therefore

$$\lambda(\log(a)) \leq \max_{w \in W} \lambda(wZ) \leq \max_{Y \in \mathrm{supp}(s)} \lambda(Y) \leq \lambda(Y_0).$$

$\square$

The lemma implies that $dg_\lambda(a_0 K)$ is normal to $\mathrm{supp}(s)$. Therefore, by the Holmgren principle we get

$$\det \sigma_P(dg_\lambda(a_0)) = 0.$$

The above identification $G \backslash T^* X \cong W \backslash \mathfrak{a}^*$ defines a projection map $T^* X \to W \backslash \mathfrak{a}^*$. Since the group A is abelian and g_λ is linear on A, the vector

$$a^{-1} dg_\lambda(a) \in T^*_{eK} X$$

does not depend on the point $a \in A$. Therefore the image of $dg_\lambda(a_0)$ in $W \backslash \mathfrak{a}$ only depends on λ and not on a_0. The image of the map $L \ni \lambda \mapsto dg_\lambda(a_0) \in W \backslash \mathfrak{a}^*$ is an open set. Therefore the assertion

$$\det \sigma_P(dg_\lambda(a_0)) = 0$$

for any λ leads to $\det \sigma_P = 0$, a contradiction. $\square$