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POSITIVE SOLUTIONS FOR A NONLOCAL BOUNDARY-VALUE PROBLEM WITH INCREASING RESPONSE

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Abstract. We study a nonlocal boundary-value problem for a second order ordinary differential equation. Under a monotonicity condition on the response function, we prove the existence of positive solutions.

1. Introduction

When looking for positive solutions of the equation

$$u''(t) + a(t)f(u(t)) = 0, \ \ t \in [0, 1],$$

associated with various boundary conditions the main assumption on the response function f is the existence of the limits of $f(u)/u$, as u approaches 0 and $+\infty$. Existence of solutions under these conditions has been shown, for instance, in [1, 4, 5, 6, 7, 11, 18]. Such conditions distinguish two cases: The sublinear case when the limits are $+\infty$ and 0, and the superlinear case when the limits are 0 and $+\infty$, respectively. In [16] the authors present a detailed investigation of a twwo-point boundary-value problem under similar limiting conditions and they introduce the meaning of the index of convergence.

In this paper, we discuss a general problem with non-local boundary conditions. We avoid the limits above, and therefore weaken the restriction of the function f. Instead, we assume that there exist real positive numbers u, v such that $f(u) \geq \rho u$ and $f(v) < \theta v$, where ρ, θ are prescribed positive numbers. This is a rather weak condition, but we have to pay for it. Indeed, we assume that the function f is increasing (not necessarily strictly increasing). More precisely, we consider the ordinary differential equation

$$(p(t)x')' + q(t)f(x) = 0, \ \ \text{a.e. } t \in [0, 1] \tag{1.1}$$

with the initial condition

$$x(0) = 0 \tag{1.2}$$

and the non-local boundary condition

$$x'(1) = \int_\eta^1 x'(s)dg(s). \tag{1.3}$$

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Here $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, the real valued functions p, q, g are defined at least on the interval $[0, 1]$ and η is a real number in the open interval $(0, 1)$. Also the integral in (1.3) is meant in the sense of Riemann-Stieljes.

When (1.1) is an equation of Sturm-Liouville type, Il'in and Moiseev [12], motivated by a work of Bitsadze [2] and Bitsadze and Samarskii [3], investigated the existence of solutions of the problem (1.1), (1.2) with the multi-point condition

$$x'(1) = \sum_{i=1}^{m} \alpha_i x'(\xi_i), \tag{1.4}$$

where the real numbers $\alpha_1, \alpha_2, \dots, \alpha_m$ have the same sign. The formed boundary-value problem (1.1), (1.2), (1.4) was the subject of some recent papers (see, e.g. [9, 10]). Condition (1.3) is the continuous version of (1.4) which happens when g is a piece-wise constant function that is increasing and has a finitely many jumps.

The question of existence of positive solutions of the boundary-value problem (1.1)-(1.3) is justified by the large number of papers. For example one can consult the papers [1, 4, 5, 6, 7, 11, 18] which were motivated by Krasnoselskii [17], who presented a complete theory for positive solutions of operator equations. One of the more powerful tools exhibited in [17] is the following general fixed point theorem. This theorem is an extension of the classical Bolzano-Weierstrass sign theorem for continuous real valued functions to Banach spaces, when the usual order is replaced by the order generated by a cone.

Theorem 1.1. *Let $\mathcal{B}$ be a Banach space and let $\mathbb{K}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$, with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let*

$$A : \mathbb{K} \cap (\Omega_2 \setminus \overline{\Omega_1}) \to \mathbb{K}$$

be a completely continuous operator such that either

$$\|Au\| \le \|u\|, \ \ u \in \mathbb{K} \cap \partial\Omega_1, \ \ \|Au\| \ge \|u\|, \ \ u \in \mathbb{K} \cap \partial\Omega_2$$

or

$$\|Au\| \ge \|u\|, \ \ u \in \mathbb{K} \cap \partial\Omega_1, \ \ \|Au\| \le \|u\|, \ \ u \in \mathbb{K} \cap \partial\Omega_2.$$

Then A has a fixed point in $\mathbb{K} \cap (\Omega_2 \setminus \overline{\Omega_1})$.

In the literature, boundary-value problems of the form (1.1)-(1.3) are often solved by using the well known Leray-Schauder Continuation Theorem (see, e.g. [9, 10, 13, 19]), or the Nonlinear Alternative (see, e.g. [8, 15] and the references therein. For another approach see, also, [14]). On the other hand Krasnoselskii's fixed point theorem, when it is applied, it provides some additional properties of the solutions, for instance, positivity (see, e.g. [1, 4, 5, 6, 7, 11, 14]). However, the more information on the solutions the more restrictions on the coefficients are needed.

2. Preliminaries and assumptions

In the sequel we shall denote by $\mathbb{R}$ the real line and by I the interval $[0, 1]$. Then $C(I)$ will denote the space of all continuous functions $x : I \to \mathbb{R}$. Let $C_0^1(I)$ be the space of all functions $x : I \to \mathbb{R}$, whose the first derivative x' is absolutely continuous on I and $x(0) = 0$. This is a Banach space when it is furnished with the norm defined by

$$\|x\| := \sup\{|x'(t)| : t \in I\}, \;\; x \in C_0^1(I).$$

We denote by $L_1^+(I)$ the space of functions $x : I \to \mathbb{R}^+ := [0, +\infty)$ which are Lebesgue integrable on I.

Consider the system (1.1), (1.2) and the nonlocal-value condition (1.3). By a solution of the problem (1.1)-(1.3) we mean a function $x \in C_0^1(I)$ satisfying equation (1.1) for almost all $t \in I$ and condition (1.3).

Before presenting our results we give our basic assumptions:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is an increasing continuous function, with $f(x) \geq 0$, when $x > 0$

(H2) The functions p, q belong to $C(I)$ and they are such that $p > 0$, $q \geq 0$ and $\sup\{q(s) : \eta \leq s \leq 1\} > 0$. Without loss of generality we can assume that $p(1) = 1$.

(H3) The function $g : I \to \mathbb{R}$ is increasing and such that $g(\eta) = 0 < g(\eta+)$.

(H4) $\displaystyle\int_\eta^1 \frac{1}{p(s)} dg(s) < 1$

To search for solutions to problem (1.1)-(1.3), we first re-formulate the problem as an operator equation of the form $x = Ax$, for an appropriate operator A. To find this operator consider the equation (1.1) and integrate it from t to 1. Then we derive

$$x'(t) = \frac{1}{p(t)} x'(1) + \frac{1}{p(t)} \int_t^1 q(s) f(x(s)) ds. \tag{2.1}$$

Taking into account the condition (1.3) we obtain

$$x'(1) = \int_\eta^1 x'(s) dg(s) = x'(1) \int_\eta^1 \frac{1}{p(s)} dg(s) + \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta) f(x(\theta)) d\theta dg(s)$$

and so

$$x'(1) = \alpha \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta) f(x(\theta)) d\theta dg(s),$$

where

$$\alpha := \left(1 - \int_\eta^1 \frac{1}{p(s)} dg(s)\right)^{-1}.$$

Then, from (2.1), we get

$$x(t) = \alpha \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta) f(x(\theta)) d\theta dg(s) \int_0^t \frac{1}{p(s)} ds + \int_0^t \frac{1}{p(s)} \int_s^1 q(\theta) f(x(\theta)) d\theta ds.$$

(Notice that $x(0) = 0$.)

This process shows that solving the boundary-value problem (1.1)-(1.3) is equivalent to solve the operator equation $x = Ax$ in $C_0^1(I)$, where A is the operator defined by

$$Ax(t) := \alpha P(t) \int_\eta^1 \Phi(f(x))(s) dg(s) + \int_0^t \Phi(f(x))(s) ds, \tag{2.2}$$

where we have set

$$P(t) := \int_0^t \frac{1}{p(s)} ds, \;\; t \in I$$

and

$$(\Phi y)(t) := \frac{1}{p(t)} \int_t^1 q(s)y(s)ds, \;\; t \in I, \;\; y \in C(I).$$

It is clear that A is a completely continuous operator. We set

$$b_0 = g(\eta+)(>0).$$

The following lemma is the basic tool in the proof of our main result.

Lemma 2.1. *If $y \in C(I)$ is a nonnegative and increasing function, then it holds*

$$\int_\eta^1 \Phi(y)(s)dg(s) \geq \lambda b \int_0^1 q(s)y(s)ds, \;\; b \in [0, b_0],$$

where

$$\lambda := \frac{\int_\eta^1 q(s)ds}{\int_0^1 q(s)ds} \left(\sup_{s \in I} p(s) \right)^{-1}.$$

Proof. Since the function g is increasing, for every $b \in (0, b_0]$ we have

$$g(s) \geq b, \;\; s \in (\eta, 1]. \tag{2.3}$$

Hence it follows that

$$\begin{aligned} \int_0^1 q(s)y(s)ds &= \int_0^\eta q(s)y(s)ds + \int_\eta^1 q(s)y(s)ds \\ &\leq y(\eta) \int_0^\eta q(s)ds + \int_\eta^1 q(s)y(s)ds \\ &\leq \frac{\int_0^\eta q(s)ds}{\int_\eta^1 q(s)ds} \int_\eta^1 q(s)y(s)ds + \int_\eta^1 q(s)y(s)ds \\ &= \frac{\int_0^1 q(s)ds}{\int_\eta^1 q(s)ds} \int_\eta^1 q(s)y(s)ds. \end{aligned}$$

Now we use assumption (H_3) and relation (2.3) to obtain that

$$\begin{aligned} \int_0^1 q(s)y(s)ds &\leq b^{-1} \frac{\int_0^1 q(s)ds}{\int_\eta^1 q(s)ds} \int_\eta^1 q(s)y(s)g(s)ds \\ &= -b^{-1} \frac{\int_0^1 q(s)ds}{\int_\eta^1 q(s)ds} \int_\eta^1 d\left(\int_s^1 q(\theta)y(\theta)d\theta \right) g(s) \\ &= b^{-1} \frac{\int_0^1 q(s)ds}{\int_\eta^1 q(s)ds} \int_\eta^1 \int_s^1 q(\theta)y(\theta)d\theta dg(s) \\ &\leq (\lambda b)^{-1} \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta)y(\theta)d\theta dg(s). \end{aligned}$$

The proof is complete. $\square$

For convenience we set
$$D := \int_\eta^1 \Phi(P)(s)dg(s), \quad H := \int_\eta^1 \Phi(1)(s)dg(s)$$
and we observe the following:

Lemma 2.2. *Let b be a fixed real number such that*
$$0 < b \le \min\left\{\frac{H}{\alpha\lambda|D\eta p(0) - H|}, b_0\right\}.$$
Then $\sigma\eta \le H$, where $\sigma := \dfrac{\alpha\lambda b p(0)}{\alpha\lambda b + 1}D$.

Proof. Obviously $b \le \frac{H}{\alpha\lambda|D\eta p(0)-H|}$. If $D\eta p(0) - H > 0$, by a simple calculation we have the result. Also, if $D\eta p(0) - H < 0$, then
$$\sigma\eta = \frac{\alpha\lambda b p(0)\eta}{\alpha\lambda b + 1}D < \frac{\alpha\lambda b H}{\alpha\lambda b + 1} \le H.$$

3. Main results

Before presenting our main theorem we set $\rho := \frac{1}{\alpha\sigma\eta}$ and let $\theta := \frac{p(0)}{\alpha H + \int_0^1 q(s)ds}$ where σ and H are the constants defined in Lemma 2.2.

Theorem 3.1. *Assume that f, p, q and g satisfy (H1)-(H4). If*

(H5) *There exist $u > 0$ and $v > 0$ such that $f(u) \ge \rho u$ and $f(v) < \theta v$,*

then the boundary-value problem (1.1)-(1.3) admits at least one positive solution.

Proof. Our main purpose is to make the appropriate arrangements so that Theorem 1.1 to be applicable. Define the set
$$\mathbb{K} := \left\{x \in C_0^1(I) : x \ge 0,\ x' \ge 0,\ x \text{ is concave and } \int_\eta^1 \Phi(x)(s)dg(s) \ge \sigma\|x\|\right\},$$
which is a cone in $C_0^1(I)$.

First we claim that the operator A maps $\mathbb{K}$ into $\mathbb{K}$. To this end take a point $x \in \mathbb{K}$. Then observe that it holds $Ax \ge 0, (Ax)' \ge 0$ and $(Ax)'' \le 0$. Moreover, we observe that
$$\begin{aligned}
\int_\eta^1 \Phi(Ax)(s)dg(s) \ge& \alpha \int_\eta^1 \Phi(P)(s)dg(s) \int_\eta^1 \Phi(f(x))(s)dg(s) \\
=& \alpha D \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta)f(x(\theta))d\theta dg(s) \\
=& \frac{\sigma(\alpha\lambda b + 1)}{\lambda b p(0)} \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta)f(x(\theta))d\theta dg(s) \\
=& \frac{\sigma}{p(0)}\left(\alpha + \frac{1}{\lambda b}\right) \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta)f(x(\theta))d\theta dg(s) \\
=& \sigma[\frac{\alpha}{p(0)} \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta)f(x(\theta))d\theta dg(s) \\
&+ \frac{1}{p(0)}\frac{1}{\lambda b} \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta)f(x(\theta))d\theta dg(s)].
\end{aligned}$$

Now we use Lemma 2.1 and get

$$\begin{aligned}\int_\eta^1 \Phi(Ax)(s)dg(s) \geq &\sigma[\frac{\alpha}{p(0)}\int_\eta^1 \frac{1}{p(s)}\int_s^1 q(\theta)f(x(\theta))d\theta dg(s)\\ &+\frac{1}{p(0)}\int_0^1 q(\theta)f(x(\theta))d\theta]\\ =&\sigma(Ax)'(0)\\ =&\sigma\|(Ax)\|.\end{aligned}$$

This proves our first claim.

Now consider an arbitrary $x \in \mathbb{K}$. The fact that the function x is concave implies that

$$\eta x(1) \leq x(\eta) \leq x(r) \leq x(1) \leq \|x\|, \text{ for every } r \in [\eta, 1].$$

So,

$$\begin{aligned}\sigma\|x\| &\leq \int_\eta^1 \Phi(x)(s)dg(s)\\ &= \int_\eta^1 \frac{1}{p(s)}\int_s^1 q(\theta)x(\theta)d\theta dg(s)\\ &\leq x(1)\int_\eta^1 \frac{1}{p(s)}\int_s^1 q(\theta)d\theta dg(s)\\ &= x(1)\int_\eta^1 \Phi(1)(s)dg(s)\\ &= x(1)H.\end{aligned}$$

Thus we have $x(1) \geq \frac{\sigma\|x\|}{H}$, which implies that

$$x(r) \geq \frac{\eta\sigma}{H}\|x\|, \quad r \in [\eta, 1].$$

Hence, for every $r \in [\eta, 1]$ we have

$$\frac{\eta\sigma}{H}\|x\| \leq x(r) \leq \|x\|,$$

where, notice that, by Lemma 2.2, $\frac{\eta\sigma}{H} \leq 1$. Then, by assumption (H5), there exists $u > 0$ such that $f(u) \geq \rho u$.

Set

$$M := \frac{H}{\eta\sigma}u$$

and fix a function $x \in \mathbb{K}$ with $\|x\| = M$. Then

$$\frac{\eta\sigma}{H}M \leq x(r) \leq M, \text{ for every } r \in [\eta, 1]$$

and therefore

$$\begin{aligned}(Ax)'(1) &\geq \alpha\int_\eta^1 \frac{1}{p(s)}\int_s^1 q(\theta)f(x(\theta))d\theta dg(s)\\ &\geq \alpha f(x(\eta))\int_\eta^1 \Phi(1)(s)dg(s) = \alpha H f(x(\eta))\\ &\geq \alpha H f(\frac{\eta\sigma M}{H}) = \alpha H f(u) \geq \alpha H\rho u\\ &= \alpha\rho\eta\sigma M \geq M = \|x\|.\end{aligned}$$

Thus we proved that, if $\|x\| = M$, then $\|Ax\| \geq \|x\|$.

Now, again, from assumption (H5), it follows that there exists $v > 0$ such that $0 \leq f(v) < \theta v$. Fix any function $x \in \mathbb{K}$ with $\|x\| = v$. Then $0 \leq x(r) \leq v$, $r \in I$. Therefore

$$\begin{aligned}
\|Ax\| = (Ax)'(0) &= \frac{\alpha}{p(0)} \int_\eta^1 \Phi(f(x))(s)dg(s) + \frac{1}{p(0)} \int_0^1 q(s)f(x(s))ds \\
&= \frac{\alpha}{p(0)} \int_\eta^1 \frac{1}{p(s)} \int_0^1 q(r)f(x(r))drdg(s) + \frac{1}{p(0)} \int_0^1 q(s)f(x(s))ds \\
&\leq f(v) \left[\frac{\alpha H}{p(0)} + \frac{1}{p(0)} \int_0^1 q(s)ds\right] \\
&\leq \theta v \left[\frac{\alpha H}{p(0)} + \frac{1}{p(0)} \int_0^1 q(s)ds\right] \\
&= v = \|x\|.
\end{aligned}$$

So we proved that, if $\|x\| = v$, then $\|Ax\| \leq \|x\|$.

Finally, we set $\Omega_1 := \{x \in C_0^1(I) : \|x\| < r_1\}$ and $\Omega_2 := \{x \in C_0^1(I) : \|x\| < r_2\}$, where $r_1 = min\{M, v\}$ and $r_2 = max\{M, v\}$. Without loss of generality we can assume that $M \neq v$ and hence $r_1 < r_2$. Then taking into account the fact that A is a completely continuous operator, by Theorem 1.1, the result follows. $\square$

Next we show that some information on the lower and upper limits of the quantity $f(u)/u$ at the points 0 and $+\infty$, are enough to guarrantee existence of a positive solution of the problem (1.1)-(1.3).

Corollary 3.2. *Consider the functions f, p, q and g satisfying the assumptions (H1)-(H4). Moreover assume that*

(H6) $\limsup_{x\to+\infty} \frac{f(x)}{x} = +\infty$ *and* $\liminf_{x\to 0+} \frac{f(x)}{x} = 0$.

or

(H7) $\limsup_{x\to 0+} \frac{f(x)}{x} = +\infty$ *and* $\liminf_{x\to+\infty} \frac{f(x)}{x} = 0$.

Then the boundary-value problem (1.1)-(1.3) admits at least one positive solution.

Proof. It is easy to see that each of assumptions (H6), (H7) imply the validity of (H5). Hence the result follows from Theorem 3.1.

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