Electronic Journal of Differential Equations, Vol. **2000**(2000), No. 76, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu ftp ejde.math.unt.edu (login: ftp)

Transformation to Liénard form *

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Abstract

We show that certain two-dimensional differential systems can be transformed to a system of Liénard type. This enables known criteria for the existence of a centre for Liénard systems to be exploited, so extending the range of techniques which are available for proving that conditions which are known to be necessary for a centre are also sufficient.

1 Introduction

In the investigation of two-dimensional dynamical systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

the twin questions of the conditions under which a critical point is a centre and the number of limit cycles are often encountered. Recall that a *limit cycle* is an isolated closed orbit, while a critical point is a *centre* if all orbits in its neighbourhood are closed.

Much of the published work refers to specific classes of systems. These may be polynomial systems of a fixed degree (cubic systems, for example - that is, systems in which P and Q are cubic polynomials) or they may be of a specific form, for example

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x). \tag{2}$$

Systems of this form are said to be of Liénard type. They are well understood, and there is a very extensive literature on them, not least because they often arise in applications.

Systems of the form (1) in which P and Q are polynomials are of particular interest, and several authors have transformed such systems to Liénard form in order to exploit the many known results on the existence of limit cycles and their number. For example, Coppel [5,6] and Cherkas and Zhilevich [2] transformed various quadratic systems in this way to prove that there is at most one periodic orbit encircling the origin. Kooij [9] obtained the same conclusion for some cubic systems in a similar fashion, though in this case a sequence of

* *Mathematics Subject Classifications:* 34C07.
Key words: Ordinary differential equations, polynomial systems, Liénard systems.

Submitted October 31, 2000. Published December 22, 2000.

transformations was required. Other instances are found in [7] and [18], where the results are relevant to models of ecological competition. These are just a few of the examples in the literature and are indicative of the variety of instances where the technique has proved useful. Our interest here is not so much in the number of limit cycles, but rather in the use of transformation to a Liénard form to prove that a critical point is a centre.

Experience has shown that in the search for centre conditions, necessity and sufficiency should be treated separately; the literature is littered with incomplete sets of conditions which are claimed to be both necessary and sufficient. Proof of necessity often involves extensive use of computer algebra, and it is the availability of such systems which has led to many of the recent advances. The eventual requirement is to eliminate variables from the focal values, which are polynomials in the coefficients arising in P and Q. In many cases these are polynomials of very high degree with coefficients which are very large integers. As pointed out in [20] the computations are on occasion beyond the scope of the available elimination algorithms. A particular instance is the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3. \end{aligned} \tag{3}$$

This system was first investigated by Kukles and has been the subject of a number of papers, including [3,8,10,11 and 21].

A variety of methods have been developed for proving the sufficiency of centre conditions, and in practice several are used in combination. The simplest criterion for a centre is when the system is Hamiltonian:

$$P_x + Q_y = 0 \tag{4}$$

in a neighbourhood of the origin. The other classical condition is that of symmetry. The origin is a centre if the system is invariant under the transformation $(x, y, t) \longmapsto (-x, y, -t)$ or $(x, y, t) \longmapsto (x, -y, -t)$; the system is symmetric in the y-axis in the first case and in the x-axis in the second. Clearly the same conclusion holds if the system is symmetric in any line through the origin. This is most easily tested when the system is written in complex form:

$$i\dot{z} = -z + \sum A_{k,j} z_k \bar{z}^j.$$

The origin is a centre if there exists θ such that $A_{kj}e^{i(k-j-1)\theta}$ is real for all k, j (see [13]).

Condition (4) can be generalised. The origin is a centre if there is an integrating factor (sometimes called a Dulac function) B such that $(BP)_x + (BQ)_y = 0$ in a neighbourhood of the origin. Geometrically this happens when there is a transformation of time, depending on the space variables x and y, to a Hamiltonian system. The task is to find an integrating factor and a systematic way of doing so is described in [14]. A function C is said to be *invariant* if there exists a polynomial L such that $\dot{C} = CL$. The idea is to find integrating factors which are products of the form $C_1^{k_1} C_2^{k_2} \ldots$, where each C_i is an invariant polynomial or exponential of a polynomial. The original idea goes back to Darboux in the late nineteenth century and systems with such an integrating factor are said to be *Darboux integrable*. Though it would be attractive to develop a fully automatic method to find such integrating factors it was found that a modicum of user intervention is desirable and so the process described in [15] is semi-automatic in the sense that a number of procedures are developed which the user calls as required.

However, not all centre conditions can be proved to be sufficient by means of the above approaches. A method was developed by Cherkas, and described in [4], in which the polynomial system is transformed to Liénard form. As noted above, because of the independent interest in Liénard systems, there is a very large literature, and known results can then be used. In this paper we extend the scope of the transformation described in [4] to cover a wider class of polynomial systems, including the Kukles system (3). The transformed system need not be, and is not usually, polynomial. This approach extends the range of techniques which are available for proving that centre conditions which are known to be necessary are also in fact sufficient.

The centre conditions for a Liénard system on which this method depends are in fact derived, albeit at one remove, from the symmetry of a related system. Systems which can be so transformed are sometimes said to display *generalised symmetry.* Centres can be classified depending on whether they arise from symmetry in a line (time-reversible systems) or are Darboux integrable or have generalised symmetry.

System (2) is derived from the second order equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{5}$$

with $f(x) = F'(x)$. It is usual to suppose that $g(x)\mathrm{sgn}\ x > 0$ for x small and $x \neq 0$. Sometimes the corresponding system in the phase plane is used:

$$\dot{x} = y, \quad \dot{y} = -f(x)y - g(x). \tag{6}$$

The relevant results on Liénard systems are summarised in the following, where $F(x) = \int_0^x f(\xi)d\xi$ and $G(x) = \int_0^x g(\xi)d\xi$.

Lemma 1 *(i) The origin is a centre for (6) if and only if there is an analytic function Φ with $\Phi(0) = 0$ such that $G(x) = \Phi(F(x))$.*

(ii) The origin is a centre for (6) if and only if there is a function $z(x)$ with $z'(0) < 0$ such that

$$F(z) = F(x) \text{ and } G(z) = G(x).$$

Proofs of these results are given in [4], for instance.

In some cases, it is possible to transform a polynomial system to a generalised Liénard system:

$$\dot{x} = h(y) - F(x), \quad \dot{y} = -g(x)$$

where $h(0) = 0$ and $h'(0) \neq 0$. Lemma 1 holds for this system.

2 The Transformation

In this paper we consider polynomial systems of the general form

$$\dot{x} = \Sigma p_k(x)y^k, \quad \dot{y} = \Sigma q_k(x)y^k \tag{7}$$

where $p_k(x)$ and $q_k(x)$, $(k = 0, ...)$ are polynomials. We could in exactly the same way consider systems which are polynomial in x with coefficients which are polynomials in y. The Kukles system (3) is a very specific instance of this form.

Cherkas considered systems of this form with $p_k = 0$ for $k \geq 2$ and $q_k = 0$ for $k \geq 3$; his approach was discussed in [4], and is summarized as follows.

Lemma 2 *Suppose that $p_1(x) \neq 0$ for $-\alpha < x < \beta$, where $\alpha, \beta > 0$. For $-\alpha < x < \beta$ the system*

$$\dot{x} = p_0(x) + p_1(x)y, \quad \dot{y} = q_0(x) + q_1(x)y + q_2(x)y^2$$

can be transformed to a system of the form (6) with

$$f = -(p_0' - p_0p_1' + q_1 - 2p_0q_2p_1^{-1})\phi\,,$$
$$g = (p_0q_1 - q_0p_1 - p_0^2q_2p_1^{-1})\phi^2,$$

where $\phi(x) = (p_1(x))^{-1} \exp\left(-\int_0^x q_2(t)/p_1(t)dt\right)$.

The transformation can be achieved by a change of independent variable. Let $dt/d\tau = \phi(x)$; routine calculation then leads to the desired result. This transformation was used in both [4] and [12] to confirm the sufficiency of the centre conditions previously shown to be necessary. In [4] we used part (ii) of Lemma 1, while in [12] we used part (i).

However, Lemma 2 does not cover systems such as

$$\dot{x} = p_0 + p_1y, \quad \dot{y} = q_0 + q_1y + q_2y^2 + q_3y^3 \tag{8}$$

(where, of course, the p_k and q_k are functions of x alone). Cherkas [1] was able to transform such systems to Liénard form when $p_0 = 0$ but required a particular solution in order to be able to do so; the transformation he used was

$$y = Z(x)Y(Y+1)^{-1}$$

where Z is a particular solution of

$$\frac{dy}{dx} = \frac{q_0 + q_1y + q_2y^2 + q_3y^3}{p_1y}.$$

The transformation of such systems to Liénard form is also discussed by Sadovskii [17].

The purpose here is to present a different approach to systems of the form (8). We use a transformation of the form

$$y = \frac{a(x)Y}{1 + b(x)Y} \tag{9}$$

where a and b are differentiable functions to be chosen, with $a(0) \neq 0$. The transformation is invertible in a neighbourhood of the origin. We consider systems

$$\begin{aligned} \dot{x} &= p_1(x)y, \\ \dot{y} &= q_0(x) + q_1(x)y + q_2(x)y^2 + q_3(x)y^3, \end{aligned} \tag{10}$$

where $q_0(0) = 0$ (so the origin is a critical point). Since such systems generalise the form (3), they are sometimes said to be of *Kukles type.* System (8) can easily be transformed to (10), so our consideration of the latter is without loss of generality.

A routine calculation leads to

$$\begin{aligned} \dot{x} &= P_1(x)Y, \\ \dot{Y} &= Q_0(x) + Q_1(x)Y + Q_2(x)Y^2 + Q_3(x)Y^3, \end{aligned} \tag{11}$$

where

$$\begin{gathered} P_1 = ap_1, \quad Q_0 = a^{-1}q_0,\ Q_1 = q_1 + 3a^{-1}bq_0, \\ Q_2 = aq_2 + 2bq_1 + 3b^2a^{-1}q_0 - a'p_1 \end{gathered}$$

and

$$Q_3 = a^2q_3 + abq_2 + b^2q_1 + b^3a^{-1}q_0 + (ab' - a'b)p_1.$$

If a, b are chosen so that $Q_3 = 0$, then (11) can be further transformed to a Liénard system in accordance with Lemma 2. This requires $u = b/a$ to satisfy the differential equation

$$u'p_1 = -(q_0u^3 + q_1u^2 + q_2u + q_3). \tag{12}$$

We therefore lose no generality by taking $a(x) \equiv 1$. The difficulty is that it may not be possible to solve equation (12) explicitly - and an explicit solution would be required to be able to use results such as those given in Lemma 1. In some situations, of course, an explicit solution can be found (a particular instance is when $q_1 = q_3 = 0$).

Lemma 3 *Suppose that b/a satisfies equation (12). Then the system (10) can be transformed to Liénard form by means of (9).*

System (11) is itself of Liénard form if both Q_2 and Q_3 are zero. The functions a and b are then determined by a pair of differential equations. Again, an explicit solution may not be possible.

The remaining possibility is to choose a and b so that $Q_1 = Q_2 = 0$. The transformed system is then of the form

$$\begin{aligned} \dot{x} &= u, \\ \dot{u} &= -g(x) - f(x)u^3, \end{aligned} \tag{13}$$

and the question arises whether results analogous to those given in Lemma 1 apply. The functions a and b are now given by the relations

$$\begin{aligned} lb &= -\tfrac{1}{3} q_0^{-1} q_1 a, \\ a' &= a p_1^{-1} \left(q_2 - \tfrac{1}{3} q_1^2 q_0^{-1} \right). \end{aligned} \tag{14}$$

Thus

$$a(x) = \exp\left[\int_0^x (3q_0 q_2 - q_1^2) q_0^{-1} p_1^{-1} \right]. \tag{15}$$

We suppose that

(i) $p_1(0) \neq 0$,

(ii) $(q_1(x))^2 (q_0(x))^{-1}$ tends to a finite limit as $x \to 0$.

Both these are very natural conditions: (i) states that the y-axis is not invariant, which must be the case if there are closed orbits surrounding the origin, and (ii) is certainly satisfied by (3), for example.

Theorem 4 *System (10) can be transformed to (13), with a, b given by (15) and (14) respectively, where*

$$g = -q_0 p_1^{-1} a^{-2}$$

and

$$f = \tfrac{1}{3} a \left(\frac{q_1}{q_0} \right)' - p_1^{-1} a \left(q_3 - \tfrac{1}{3} q_1 q_2 q_0^{-1} + \tfrac{2}{27} q_1^3 q_0^{-2} \right).$$

The origin is certainly a centre for (11) when the system is symmetric in the x-axis. This requires $Q_1 = Q_3 = 0$, which exactly corresponds to $f(x) = 0$. The following result is a straightforward observation.

Theorem 5 *Suppose that the origin is a critical point of (10) of focus type. Given p_1, q_0, q_1, q_2, the origin is a centre for (10) if*

$$q_3 = \tfrac{1}{3} p_1 \left(\frac{q_1}{q_0} \right)' - \tfrac{1}{3} q_1 q_2 q_0^{-1} + \tfrac{2}{27} q_1^3 q_0^{-2}.$$

It might be expected that a transformation

$$y = \frac{a(x)Y + c(x)}{1 + b(x)Y}$$

would lead to a more general result. This is not the case. The function $c(x)$ does not appear in Q_3, and an additional term arises in the equation for $\dot{x}$:

$$\dot{x} = P_0(x) + P_1(x)Y.$$

Then c is determined by the need for $P_0(x) = 0$, and no generalisation is achieved.

We now turn to systems of the form (13); we consider

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(x) - f(x)\phi(y), \end{aligned} \tag{16}$$

where $y\phi(y) > 0$. There is no analogue in this case of the form (2).

We suppose that f and g are C^1, $g(0) = 0$ and $g(x)\text{sgn } x > 0$ for $x \neq 0$. Let $G(x) = \int_0^x g(\xi)d\xi$, and define

$$u = \sqrt{2G(x)} \text{ sgn } g(x).$$

The transformation $x \longmapsto u$ has an inverse; let this be $x = \xi(u)$. Let $k(x) = f(x)/g(x)$, and define $k^*(u) = uk(\xi(u))$. In terms of y and u, system (16) is

$$\begin{aligned} \dot{u} &= u^{-1}g(\xi(u))y, \\ \dot{y} &= -g(\xi(u)) - f(\xi(u))\phi(y). \end{aligned} \tag{17}$$

Now $u^{-1}g(\xi(u)) \to 1$ as $u \to 0$. The orbits of (17) are the same as those of

$$\begin{aligned} \dot{u} &= y, \\ \dot{y} &= -u - k^*(u)\phi(y). \end{aligned} \tag{18}$$

If k^* is an odd function then (18) is unchanged under the transformation $(u, y, t) \longmapsto (-u, y, -t)$; it follows that the origin is a centre.

Conversely, compare (18) with the system

$$\dot{u} = y, \quad \dot{y} = -u - K(u)\phi(y), \tag{19}$$

where $K(u) = \frac{1}{2}(k^*(u) - k^*(-u))$. The origin is a centre for (19), again by symmetry in the y-axis. But system (18) is rotated with respect to (19) in a neighbourhood of the origin. This follows from the fact that the vector product of the two vector fields is $\frac{1}{2}y\phi(y)(k^*(u) + k^*(-u))$, which is of one sign in a neighbourhood of the origin. Hence, if the origin is a centre, k^* is odd. It follows, as in [4], that there is a unique function $z(x)$ satisfying $z'(0) < 0$ and $z(0) = 0$, such that

$$G(z) = G(x), \quad k(z) = k(x).$$

Theorem 6 *Suppose that f, g, ϕ are C^1 functions, with $xg(x) > 0$ (x small, $x \neq 0$) and $y\phi(y) > 0$ (y small, $y \neq 0$). Then the origin is a centre for system (16) if and only if there is a unique function $z(x)$ with $z(0) = 0$, $z'(0) < 0$, such that*

$$G(z) = G(x) \text{ and } k(z) = k(x),$$

where G and k are as defined above.

It might be thought that the ideas described above could be used in relation to systems more general than (10). The simplest example would be

$$\dot{x} = p_1 y + p_2 y^2, \quad \dot{y} = q_0 + q_1 y + q_2 y^2 + q_3 y^3 + q_4 y^4,$$

where the p_k and q_k are again functions of x. The transformation (9) gives

$$\begin{aligned} \dot{x} &= P_1(x)Y + P_2(x)Y^2, \\ \dot{Y} &= Q_0(x) + Q_1(x)Y + Q_2(x)Y^2 + Q_3(x)Y^3 + Q_4(x)Y^4, \end{aligned}$$

where

$$\begin{aligned} P_1 &= a^2 p_1, \quad P_2 = a^2(ap_2 + bp_1), \\ Q_0 &= q_0, \quad Q_1 = aq_1 + 4bq_0, \\ Q_2 &= a^2 q_2 + 3abq_1 + 6b^2 q_0 - aa'p_1, \\ Q_3 &= a^3 q_3 + 2a^2 bq_2 + 3ab^2 q_1 + 4b^3 q_0 - a(ap_2 + bp_1)a' + a(ab' - a'b)p_1, \\ Q_4 &= a^4 q_4 + a^3 bq_3 + a^2 b^2 q_2 + ab^3 q_1 + b^4 q_0 + a(ab' - a'b)(ap_2 + bp_1). \end{aligned}$$

If we can choose a, b so that $P_2 = Q_4 = 0$, then we can use Theorem 4 to obtain the desired form. We can ensure that $P_2 = 0$ by choosing $b = -ap_2 p_1^{-1}$. However, then

$$Q_4 = a^4(q_4 - q_3 p_2 p_1^{-1} + q_2 p_2^2 p_1^{-2} - q_1 p_2^3 p_1^{-3} + q_0 p_4^4 p_1^{-4}).$$

We have $Q_4 = 0$ only if

$$p_1^4 q_4 - p_1^3 p_2 q_3 + p_1^2 p_2^2 q_2 - p_1 p_2^3 q_1 - p_4^4 q_0 = 0,$$

a relation which is not usually satisfied. As noted previously no further benefit is derived from using (16) instead of (9).

The approach which we have described is designed to extend the range of the techniques available for proving that a critical point is a centre. We conclude by giving a simple illustration of the use of the ideas which we have described. Ordinary differential equations are used extensively to model biological population dynamics (see [14] for example); they are appropriate when spatial detail is less significant than changes in populations with time. The systems used often consist of polynomial equations, and their analysis is helped by considering just two taxonomic categories (see [19]). In [16] a model in which intraprophic predation is taken into account is described and analysed. In nondimensional form the differential system that arises is

$$\acute{x} = xR(x, y), \quad \dot{y} = yS(x, y),$$

where

$$\begin{aligned} R(x, y) &= \xi(-\kappa\eta x + y)(1 + \eta x + y)^{-1} - \delta, \\ S(x, y) &= 1 - \varepsilon y - \xi x(1 + \eta x + y)^{-1}, \end{aligned}$$

and $\xi, \kappa, \eta, \delta$ and ε are parameters.

After transforming the origin to a critical point in the first quadrant and rescaling time the system is of the form

$$\begin{aligned} \dot{x} &= p_0(x) + yp_1(x), \\ \dot{y} &= q_0(x) + yq_1(x) + y^2 q_2(x) + y^3 q_3(x). \end{aligned}$$

For simplicity, we choose κ so that $p_0 = 0$. The functions p_1 and $q_i (i = 0, ..., 3)$ are all linear:

$$\begin{aligned} &p_1(x) = Cx + k_5, \quad q_0(x) = k_4 x, \quad q_1(x) = k_2 + k_3 x, \\ &q_2(x) = k_0 - k_1 x, \quad q_3(x) = -\varepsilon, \end{aligned}$$

where C and the k_i are functions of the parameters in R and S. The origin is a critical point of focus type if $k_2^2 + 4k_4k_5 < 0$. We can then use Theorem 5 to deduce that it is a centre when

$$\begin{aligned} k_2(2k_2^2 - 9k_4k_5) &= 0 \\ k_2(2k_2k_3 - 3k_0k_4 - 3Ck_4) &= 0 \\ 2k_2k_3^2 - 3k_4(k_0k_3 - k_1k_2) + 9\varepsilon k_4^2 &= 0 \\ k_3(2k_3^2 + 9k_1k_4) &= 0. \end{aligned}$$

From the first of these equations, $k_2 = 0$ or $2k_2^2 - 9k_4k_5 = 0$. The latter is inconsistent with the requirement that $k_2^2 + 4k_4k_5 < 0$. Here we must have $k_2 = 0$. We conclude that the system has a centre if

$$k_2 = 0, \ 2k_3^2 + 9k_1k_4 = 0, \ 3\varepsilon k_4 = k_0k_3.$$

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