Electronic Journal of Differential Equations, Vol. **2001**(2001), No.04, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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A generalization of the Landesman-Lazer condition *

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Abstract

In this paper we prove the existence of solutions to the semi-linear problem

$$\begin{gathered} u''(x) + m^2 u(x) + g(x, u(x)) = f(x) \\ u(0) = u(\pi) = 0 \end{gathered}$$

at resonance. We assume a Landesman-Lazer type condition and use a variational method based on the Saddle Point Theorem.

1 Introduction

Let us consider the nonlinear boundary-value problem

$$\begin{gathered} u''(x) + m^2 u(x) + g(u(x)) = f(x)\,, \quad x \in (0,\pi)\,, \\ u(0) = u(\pi) = 0\,, \end{gathered} \tag{1.1}$$

at resonance. Here $m \in \mathbb{N}$, $f \in C(0,\pi)$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\lim_{s\to\infty} g(s) = g_+ \quad \text{and} \quad \lim_{s\to-\infty} g(s) = g_- \tag{1.2}$$

exist and are finite numbers. Fučík [3, Th.6.4] proved that (1.1) has at least one solution provided that

$$\begin{aligned} &\int_0^\pi \left[g_-(\sin mx)^+ - g_+(\sin mx)^-\right] dx \\ &< \int_0^\pi f(x)\sin mx\, dx < \int_0^\pi \left[g_+(\sin mx)^+ - g_-(\sin mx)^-\right] dx\,. \end{aligned} \tag{1.3}$$

Intensive study of problem (1.1) started with the paper [5] by Landesman and Lazer in 1970. Their result [5] has been generalized in various directions. For a survey of results and exhaustive list of the bibliography up to 1979, we refer

* *Mathematics Subject Classifications:* 35J70, 35P30, 47H15.
Key words: resonance, eigenvalue, Landesman-Lazer condition.

Submitted September 8, 2000. Published January 3, 2001.
Partially supported by the Grant Agency of Czech Republic, No. 201/00/0376

the reader to Fučík [3]. A list of works published since 1980, can be found in Drábek [1]. Chun-Lei Tang [4] defined the function $F(s) = \frac{2G(s)}{s} - g(s)$ and the constants $F_+ = \liminf_{s\to+\infty} F(s)$, $F_- = \limsup_{s\to-\infty} F(s)$ to prove that for $m = 1$, Problem (1.1) is solvable under the condition

$$\begin{aligned} \int_0^\pi \left[F_-(\sin x)^+ - F_+(\sin x)^-\right] dx &< \int_0^\pi f(x)\sin x\, dx \qquad (1.4)\\ &< \int_0^\pi [F_+(\sin x)^+ - F_-(\sin x)^-]\, dx\,. \end{aligned}$$

In this paper, we generalize the Landesman-Lazer type conditions (1.3) and (1.4) to prove solvability of (1.1) when the nonlinearity g that satisfies:

- $g(s)$ is a continuous and odd function
- For $s \ge \delta > 0$, $g(s) = \epsilon + K\sin s$ with $K > \epsilon > 0$ (see Theorem 2).

Note that for this linearity, there is no function f satisfying conditions (1.3) or (1.4). We will use a variational method based on the modification of the Saddle Point Theorem introduced by Rabinowitz [7].

2 Preliminaries

It is known that the spectrum of the linear problem

$$\begin{gathered} u''(x) + \lambda u(x) = 0\,, \quad x \in (0,\pi)\,, \\ x(0) = x(\pi) = 0 \end{gathered} \qquad (2.1)$$

is the set $C = \{\lambda : \lambda = m^2,\ m \in N\}$.

Notation: We shall use the classical spaces $C(0,\pi)$, $L^p(0,\pi)$ of continuous and measurable real-valued functions whose p-th power of the absolute value is Lebesgue integrable, respectively. H is the Sobolev space of absolutely continuous functions $u : (0,\pi) \to \mathbb{R}$ such that $u' \in L^2(0,\pi)$ and $u(0) = u(\pi) = 0$. We denote by the symbols $\|\cdot\|$, and $\|\cdot\|_2$ the norm in H, and in $L^2(0,\pi)$, respectively. Let H^- be the subspace of H spanned by all eigenfunctions corresponding to the eigenvalues $1, 4, \dots, m^2$ and let H^+ be the subspace of H spanned by all eigenfunctions corresponding to the eigenvalues greater or equal to $(m+1)^2$. Then $H = H^- \oplus H^+$, $\dim(H^-) < \infty$ and $\dim(H^+) = \infty$.

Let $I : H \to \mathbb{R}$ be a functional such that $I \in C^1(H,\mathbb{R})$ (continuously differentiable). We say that u is a critical point of I, if

$$I'(u)v = 0 \quad \text{for all} \quad v \in H\,.$$

We say that I satisfies Palais-Smale condition (PS) if every sequence (u_n) for which $I(u_n)$ is bounded in H and $I'(u_n) \to 0$ (as $n \to \infty$) possesses a convergent subsequence.

Now we can formulate a variation of the Saddle Point Theorem, due to Lupo and Micheletti [6].

Theorem 2.1 *Let $I \in C^1(H, \mathbb{R})$ and*

(a) $\inf_{\|u\| \to \infty} I(u) = -\infty$ *for* $u \in H^-$

(b) $\lim_{\|u\| \to \infty} I(u) = +\infty$ *for* $u \in H^+$ *and I is bounded on bounded sets in* H^+

(c) I satisfies Palais-Smale Condition (PS).

Then functional I has a critical point in H.

In the original Saddle Point Theorem [7], instead of a) and b) above, the author assumes the following two conditions

($\tilde{a}$) There exists a bounded neighborhood D of 0 in H^- and a constant α such that $I/\partial D \leq \alpha$,

($\tilde{b}$) There is a constant $\beta > \alpha$ such that $I/H^+ \geq \beta$.

It is obvious, that conditions ($\tilde{a}$), ($\tilde{b}$) follow from conditions (a), (b).

We investigate the boundary-value problem

$$u''(x) + m^2 u(x) + g(x, u(x)) = f(x)\,, \quad x \in (0, \pi)\,, \qquad (2.2)$$
$$u(0) = u(\pi) = 0,$$

where $f \in L^1(0, \pi)$, $m \in \mathbb{N}$ and $g : (0, \pi) \times \mathbb{R} \to \mathbb{R}$ is Caratheodory type function, i.e. $g(\cdot, s)$ is measurable for all $s \in \mathbb{R}$ and $g(x, \cdot)$ is continuous for a.e. $x \in (0, \pi)$.

By a solution of (2.2) we mean a function $u \in C^1(0, \pi)$ such that u' is absolutely continuous, u satisfies the boundary conditions and the equations (2.2) holds a.e. in $(0, \pi)$.

We study (2.2) by using of varitional methods. More precisely, we look for critical points of the functional $J : H \to \mathbb{R}$, which is defined by

$$J(u) = \frac{1}{2} \int_0^\pi \left[(u')^2 - m^2 u^2\right] dx - \int_0^\pi \left[G(x, u) - fu\right] dx\,, \qquad (2.3)$$

where

$$G(x, s) = \int_0^s g(x, t)\, dt\,.$$

Every critical point $u \in H$ of the functional J satisfies

$$\int_0^\pi \left[u'v' - m^2 uv\right] dx - \int_0^\pi \left[g(x, u)v - fv\right] dx = 0 \quad \text{for all} \quad v \in H\,.$$

Then u is also a weak solution of (2.2) and vice versa. The usual regularity argument for ODE yields immediately (see Fučík [3]) that any weak solution of (2.2) is also the solution in the sense mentioned above.

We will suppose that g satisfies the growth restriction

$$|g(x, s)| \leq c|s| + p(x) \quad \text{for a.e. } x \in (0, \pi) \quad \text{and for all } s \in \mathbb{R}\,, \qquad (2.4)$$

with $p \in L^1(0,\pi)$ and $c > 0$. Moreover,

$$\lim_{|s|\to\infty} \frac{g(x,s)}{s} = 0 \quad \text{uniformly for a.e. } x \in (0,\pi)\,. \tag{2.5}$$

We define

$$G_+(x) = \liminf_{s\to+\infty} \frac{G(x,s)}{s}\,, \quad G_-(x) = \limsup_{s\to-\infty} \frac{G(x,s)}{s}\,.$$

Assume that the following potential Landesman-Lazer type condition holds:

$$\begin{aligned} &\int_0^\pi \big[G_-(x)(\sin mx)^+ - G_+(x)(\sin mx)^-\big]\,dx \\ &< \int_0^\pi f(x)\sin mx\,dx < \int_0^\pi \big[G_+(x)(\sin mx)^+ - G_-(x)(\sin mx)^-\big]\,dx\,. \end{aligned} \tag{2.6}$$

Set

$$F(x,s) = \begin{cases} \dfrac{2G(x,s)}{s} - g(x,s) & s \neq 0\,,\\ g(x,0) & s = 0\,, \end{cases}$$

$$g_+(x) = \liminf_{s\to+\infty} g(x,s)\,, \quad g_-(x) = \limsup_{s\to-\infty} g(x,s),$$

$$F_+(x) = \liminf_{s\to+\infty} F(x,s)\,, \quad F_-(x) = \limsup_{s\to-\infty} F(x,s)\,.$$

We generalize conditions (1.3) and (1.4) by assuming that f and g satisfy one of the following set of inequalities: Either

$$\begin{aligned} &\int_0^\pi \big[g_-(x)(\sin mx)^+ - g_+(x)(\sin mx)^-\big]\,dx \\ &< \int_0^\pi f(x)\sin mx\,dx < \int_0^\pi \big[g_+(x)(\sin mx)^+ - g_-(x)(\sin mx)^-\big]\,dx\,, \end{aligned} \tag{2.7}$$

or

$$\begin{aligned} &\int_0^\pi \big[F_-(x)(\sin mx)^+ - F_+(x)(\sin mx)^-\big]\,dx \\ &< \int_0^\pi f(x)\sin mx\,dx < \int_0^\pi \big[F_+(x)(\sin mx)^+ - F_-(x)(\sin mx)^-\big]\,dx\,. \end{aligned} \tag{2.8}$$

We shall prove that for any g,

$$g_+(x) \le G_+(x)\,, \quad F_+(x) \le G_+(x)\,, \quad G_-(x) \le g_-(x)\,, \quad G_-(x) \le F_-(x)\,. \tag{2.9}$$

Therefore, the potential Landesman-Lazer type condition (2.6) is more general than the conditions (2.7) and (2.8).

Let us prove the first two inequalities in (2.9). The proof of the other two is similar. It follows from the definition of the function $g_+(x)$ that $\forall\varepsilon > 0$, $\exists R > 0$

such that $\forall s > R$, $g(x,s) \geq g_+(x) - \varepsilon$ for $x \in (0,\pi)$. Then for $s > R$ and $x \in (0,\pi)$ we have

$$\frac{G(x,s)}{s} \geq \frac{1}{s}\int_0^R g(x,t)\,dt + (g_+(x)-\varepsilon)\frac{s-R}{s}.$$

Hence, the inequality $g_+(x) \leq G_+(x)$ follows.

We use the argument from C.-L. Tang [4] and prove that $F_+(x) \leq G_+(x)$. It follows from (2.8) that $F_+(x) > -\infty$ for a.e. $x \in (0,\pi)$. For these x and arbitrary $\varepsilon > 0$, we set

$$F_\varepsilon(x) = \begin{cases} F_+(x) - \varepsilon & \text{if } F_+(x) \in \mathbb{R}\,, \\ 1/\varepsilon & \text{if } F_+(x) = \infty\,. \end{cases}$$

Then for any $\varepsilon > 0$ there exists $K(x) > 0$ such that

$$F(x,s) \geq F_\varepsilon(x) \quad \text{for all} \quad s \geq K(x)\,.$$

Since

$$\frac{\partial}{\partial\tau}\Big(-\frac{G(x,\tau)}{\tau^2}\Big) = -\frac{g(x,\tau)\cdot\tau^2 - 2G(x,\tau)\cdot\tau}{\tau^4} = \frac{F(x,\tau)}{\tau^2} \geq \frac{F_\varepsilon(x)}{\tau^2}$$

for any $s > t > K(x)$, we have

$$\int_t^s \frac{\partial}{\partial\tau}\Big(-\frac{G(x,\tau)}{\tau^2}\Big)\,d\tau \geq \int_t^s \frac{F_\varepsilon(x)}{\tau^2}\,d\tau\,;$$

i.e.

$$\frac{G(x,t)}{t^2} - \frac{G(x,s)}{s^2} \geq F_\varepsilon(x)\big(t^{-1} - s^{-1}\big)\,.$$

Assumption (2.5) implies that $G(x,s)/s^2 \to 0$. Since $\varepsilon > 0$ was arbitrary, passing to the limit as $s \to +\infty$ in the last inequality, we obtain $G(x,t)/t^2 \geq F_\varepsilon(x)/t$ and so $G_+(x) \geq F_+(x)$.

Example. We suppose that the nonlinearities $g_i(x,s) = g_i(s)$ $(i = 1,2,3,4)$ are continuous, odd and for all $s \geq \epsilon > 0$: $g_1(s) = 1$, $g_2(s) = |\sin s|$, $g_3(s) = \frac{4}{\pi} - |\sin s|$, $g_4(s) = 1 + \sin s$. We set

$$\begin{aligned} M_6 &= \{f \in L^1(0,\pi) \text{ such that } f \text{ satisfies (2.6)}\}, \\ M_7 &= \{f \in L^1(0,\pi) \text{ such that } f \text{ satisfies (2.7)}\}, \\ M_8 &= \{f \in L^1(0,\pi) \text{ such that } f \text{ satisfies (2.8)}\}. \end{aligned}$$

Then for g_1, one has $M_6 = M_7 = M_8 \neq \emptyset$, $g_+ = F_+ = G_+ = 1$;
for g_2, one has $M_7 = \emptyset$, $\emptyset \neq M_8 \subset\subset M_6$, $g_+ = 0 < F_+ = \frac{4}{\pi} - 1 < G_+ = \frac{2}{\pi}$;
for g_3, one has $M_8 = \emptyset$, $\emptyset \neq M_7 \subset\subset M_6$, $F_+ = 0 < g_+ = \frac{4}{\pi} - 1 < G_+ = \frac{2}{\pi}$;
for g_4, one has $M_7 = M_8 = \emptyset$, $\emptyset \neq M_6$, $g_+ = F_+ = 0 < G_+ = 1$.

3 Main result

Theorem 3.1 *Under the assumptions (2.4), (2.5), and (2.6), Problem (2.2) has at least one solution in H.*

Proof Firstly, we note that by (2.5),

$$\lim_{\|u\|\to\infty} \int_0^\pi \frac{G(x,u)-fu}{\|u\|^2}\,dx = 0\,. \tag{3.1}$$

We shall prove that the functional J defined by (2.3) satisfies the assumptions in Theorem 2.1 (Saddle Point Theorem).

For Assumption (a), we argue by contradiction. Suppose that

$$\inf_{\|u\|\to\infty} J(u) = -\infty \quad \text{for} \quad u \in H^-$$

is not true. Then there is a sequence $(u_n) \subset H^-$ such that $\|u_n\| \to \infty$ and a constant c_- satisfying

$$\liminf_{n\to\infty} J(u_n) \ge c_-. \tag{3.2}$$

From the definition of J and from (3.2) it follows that

$$\liminf_{n\to\infty}\left[\frac{1}{2}\int_0^\pi \frac{(u_n')^2 - m^2u_n^2}{\|u_n\|^2}\,dx - \int_0^\pi \frac{G(x,u_n)-fu_n}{\|u_n\|^2}\,dx\right] \ge 0. \tag{3.3}$$

For $u \in H^-$ we have

$$\int_0^\pi [(u')^2 - m^2u^2]\,dx = \|u\|^2 - m^2\|u\|_2^2 \le 0 \tag{3.4}$$

and the equality in (3.4) holds only for $u = k\sin mx$, $k \in \mathbb{R}$. Set $v_n = u_n/\|u_n\|$. Since $\dim H^- < \infty$ there is $v_0 \in H^-$ such that $v_n \to v_0$ strongly in H^- (also strongly in $L^2(0,\pi)$). Then (3.1), (3.3), and (3.4) yield

$$v_0 = k\sin mx\,,$$

where $k = \frac{1}{m}\sqrt{\frac{2}{\pi}}$ or $k = -\frac{1}{m}\sqrt{\frac{2}{\pi}}$, ($\|v_0\| = 1$). Let $k = \frac{1}{m}\sqrt{\frac{2}{\pi}}$. We divide (3.2) by $\|u_n\|$ to obtain

$$\liminf_{n\to\infty}\left[\frac{1}{2}\int_0^\pi \frac{(u_n')^2 - m^2u_n^2}{\|u_n\|}\,dx - \int_0^\pi \frac{G(x,u_n)-fu_n}{\|u_n\|}\,dx\right] \ge 0\,. \tag{3.5}$$

Because $u_n \in H^-$ the first integral in (3.5) is less than or equal to zero and we have

$$\liminf_{n\to\infty}\left[\int_0^\pi -\frac{G(x,u_n)}{u_n}v_n\,dx\right] + k\int_0^\pi f\sin mx\,dx \ge 0\,. \tag{3.6}$$

We know that $v_n \to k\sin mx$, $k>0$ in H^-. Because of the compact imbedding $H^- \subset C(0,\pi)$, we have $v_n \to k\sin mx$ in $C(0,\pi)$ and we get

$$\lim_{n\to\infty} u_n(x) = \begin{cases} +\infty & \text{for } x\in(0,\pi) \text{ such that } \sin mx > 0\,,\\ -\infty & \text{for } x\in(0,\pi) \text{ such that } \sin mx < 0\,. \end{cases}$$

We note that from (2.6) it follows that there exist constants R, r and functions $A_+(x), A_-(x) \in L^1(0,\pi)$ such that $A_+(x) \le G_+(x)$, $G_-(x) \le A_-(x)$ for a.e. $x\in(0,\pi)$ and for all $s\ge R$, $s\le r$, respectively. We obtain from Fatou's lemma and (3.6)

$$\int_0^\pi f(x)\sin mx\,dx \ge \int_0^\pi \big[G_+(x)(\sin mx)^+ - G_-(x)(\sin mx)^-\big]\,dx\,, \tag{3.7}$$

a contradiction to (2.6). We proceed for the case $k=-\frac{1}{m}\sqrt{\frac{2}{\pi}}$. Then Assumption a) of Theorem 1 is verified.

For Assumption (b), we prove that

$$\lim_{\|u\|\to\infty} J(u) = \infty \quad \text{for all } u\in H^+$$

and that J is bounded on bounded sets. Because of the compact imbedding of H into $C(0,\pi)$ ($\|u\|_{C(0,\pi)} \le c_1\|u\|$), and of H into $\mathrm{L}^2(0,\pi)$ ($\|u\|_2 \le c_2\|u\|$), and the assumption (2.4) one has

$$\begin{aligned}\int_0^\pi \Big[G(x,u(x)) - f(x)u(x)\Big]\,dx &\le \int_0^\pi \Big[c\frac{u^2(x)}{2} + p(x)|u(x)| - f(x)u(x)\Big]\,dx\\ &\le c_2\frac{c}{2}\|u\|^2 + (\|p\|_1 + \|f\|_1)c_1\|u\|\,. \end{aligned} \tag{3.8}$$

Hence J is bounded on bounded subsets of H.

Since $u\in H^+$, we have

$$\|u\|^2 \ge (m+1)^2\|u\|_2^2\,. \tag{3.9}$$

The definition of J, (3.1), and (3.9) yield

$$\lim_{\|u\|\to\infty}\frac{J(u)}{\|u\|^2} \ge \lim_{\|u\|\to\infty}\frac{1}{2}(2m+1)\frac{\|u\|_2^2}{\|u\|^2}\,. \tag{3.10}$$

If $\|u\|_2^2/\|u\|^2 \to 0$ then it follows from the definition of J and (3.1) that

$$\lim_{\|u\|\to\infty}\frac{J(u)}{\|u\|^2} = \frac{1}{2}\,. \tag{3.11}$$

Then (3.10) and (3.11) imply $\lim_{\|u\|\to\infty} J(u) = \infty$; therefore, Assumption b) of Theorem 1 is satisfied.

For Assumption (c), we show that J satisfies the Palais-Smale condition. First, we suppose that the sequence (u_n) is unbounded and there exists a constant c_3 such that

$$\Big|\frac{1}{2}\int_0^\pi \big[(u_n')^2 - m^2u_n^2\big]\,dx - \int_0^\pi \big[G(x,u_n) - fu_n\big]\,dx\Big| \le c_3 \tag{3.12}$$

and

$$\lim_{n\to\infty}\|J'(u_n)\| = 0\,. \tag{3.13}$$

Let (w_k) be an arbitrary sequence bounded in H. It follows from (3.13) and the Schwarz inequality that

$$\begin{aligned}
&\big|\lim_{\substack{n\to\infty\\k\to\infty}}\int_0^\pi [u_n'w_k' - m^2u_nw_k]\,dx - \int_0^\pi [g(x,u_n)w_k - fw_k]\,dx\big| \\
&= \ \big|\lim_{\substack{n\to\infty\\k\to\infty}} J'(u_n)w_k\big| \\
&\le \ \lim_{\substack{n\to\infty\\k\to\infty}} \|J'(u_n)\|\cdot\|w_k\| = 0\,.
\end{aligned} \tag{3.14}$$

By (2.4) and (2.5) we obtain

$$\lim_{\substack{n\to\infty\\k\to\infty}}\int_0^\pi \Big[\frac{g(x,u_n)}{\|u_n\|}w_k - \frac{f}{\|u_n\|}w_k\Big]\,dx = 0\,. \tag{3.15}$$

Put $v_n = u_n/\|u_n\|$. Due to compact imbedding $H \subset L^2(0,\pi)$ there is $v_0 \in H$ such that (up to subsequence) $v_n \rightharpoonup v_0$ weakly in H, $v_n \to v_0$ strongly in $L^2(0,\pi)$. One has from (3.14), (3.15)

$$\lim_{\substack{n\to\infty\\k\to\infty}}\int_0^\pi \Big[\frac{u_n'}{\|u_n\|}w_k' - m^2\frac{u_n}{\|u_n\|}w_k\Big]\,dx = 0 \tag{3.16}$$

and also

$$\lim_{\substack{n\to\infty\\m\to\infty\\k\to\infty}}\int_0^\pi \big[(v_n - v_m)'w_k' - m^2(v_n - v_m)w_k\big]\,dx = 0\,. \tag{3.17}$$

We set $k = n$ and $w_k = v_n$, $k = m$ and $w_k = v_m$ in (3.17) and subtract these equalities we get

$$\lim_{\substack{n\to\infty\\m\to\infty}}\Big[\|v_n - v_m\|^2 - m^2\|v_n - v_m\|_2^2\Big] = 0\,. \tag{3.18}$$

Since $v_n \to v_0$ strongly in $L^2(0,\pi)$ then $\|v_n - v_m\|_2 \to 0$. Since (3.18) holds then v_n is a Cauchy sequence in H and $v_n \to v_0$ strongly in H. Hence it follows from (3.16) and from the usual regularity argument for ordinary differential equations (see Fučík [3]) that either $v_0 = \frac{1}{m}\sqrt{\frac{2}{\pi}}\sin mx$ or $v_0 = -\frac{1}{m}\sqrt{\frac{2}{\pi}}\sin mx$ ($\|v_0\| = 1$). Suppose that $v_0 = \frac{1}{m}\sqrt{\frac{2}{\pi}}\sin mx$. Setting $w_k = \sin mx$ in (3.14), we get

$$\lim_{n\to\infty}\int_0^\pi \big[-g(x,u_n(x))\sin mx + f(x)\sin mx\big]\,dx = 0\,. \tag{3.19}$$

Let $\overline{H}$ be the subspace of H spanned by the eigenfunctions $\sin x$, $\sin 2x$, $\dots$, $\sin(m-1)x$. Then we write $v_n = \overline{v}_n + a_n \sin mx + \tilde{v}_n$, where $\overline{v}_n \in \overline{H}$, $\tilde{v}_n \in H^+$ and $a_n \in \mathbb{R}$ (likewise $u_n = \overline{u}_n + \|u_n\| a_n \sin mx + \tilde{u}_n$). If we set $k = n$ and $w_k = -\overline{v}_n + a_n \sin mx + \tilde{v}_n$ in (3.14), we get

$$\begin{aligned}&\lim_{n\to\infty} \Big\{ \int_0^\pi \big[u_n'(-\overline{v}_n + a_n \sin mx + \tilde{v}_n)' - m^2 u_n(-\overline{v}_n + a_n \sin mx + \tilde{v}_n)\big]\,dx \\ &- \int_0^\pi \big[g(x,u_n)(-\overline{v}_n + a_n \sin mx + \tilde{v}_n) - f(-\overline{v}_n + a_n \sin mx + \tilde{v}_n)\big]\,dx\Big\} \\ &= 0\,. \end{aligned} \tag{3.20}$$

It follows from (3.19) and (3.20) that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{\|u_n\|}\Big\{ &\int_0^\pi \big[-u_n'\overline{u}_n' + m^2 u_n \overline{u}_n + u_n'\tilde{u}_n' - m^2 u_n \tilde{u}_n\big]\,dx \\ &- \int_0^\pi \Big[\frac{g(x,u_n)}{u_n} u_n(-\overline{u}_n + \tilde{u}_n) - f(-\overline{u}_n + \tilde{u}_n)\Big]\,dx\Big\} \quad = \quad 0\,. \end{aligned} \tag{3.21}$$

For $\overline{u}_n \in \overline{H}$ we have $\|\overline{u}_n\|^2 \le (m-1)^2 \|\overline{u}_n\|_2^2$, and for $\tilde{u}_n \in H^+$ we have $\|\tilde{u}_n\|^2 \ge (m+1)^2 \|\tilde{u}_n\|_2^2$. It follows from the orthogonality $\int_0^\pi \overline{u}_n' \tilde{u}_n'\,dx = 0$ that the first integral in (3.21) satisfies

$$\begin{aligned} &\int_0^\pi \big[-u_n'\overline{u}_n' + m^2 u_n \overline{u}_n + u_n'\tilde{u}_n' - m^2 u_n \tilde{u}_n\big]\,dx \\ &= -\|\overline{u}_n\|^2 + m^2\|\overline{u}_n\|_2^2 + \|\tilde{u}_n\|^2 - m^2 \|\tilde{u}_n\|_2^2 \\ &\ge -\|\overline{u}_n\|^2 + \frac{m^2}{(m-1)^2}\|\overline{u}_n\|^2 + \|\tilde{u}_n\|^2 - \frac{m^2}{(m+1)^2}\|\tilde{u}_n\|^2 \\ &= \frac{2m-1}{(m-1)^2}\|\overline{u}_n\|^2 + \frac{2m+1}{(m+1)^2}\|\tilde{u}_n\|^2\,. \end{aligned} \tag{3.22}$$

It follows from (2.5) and (2.4) that $\forall \varepsilon > 0$, $\exists R > 0$ such that for a.e. $x \in (0,\pi)$ and all $|s| > R$,

$$\Big|\frac{g(x,s)}{s}\Big| < \varepsilon\,.$$

Also for a.e. $x \in (0,\pi)$, and all $|s| \le R$,

$$|g(x,s)| \le cR + p(x)\,.$$

It follows from the imbedding $H \subset L^2(0,\pi)$ ($\|u\|_2 \le \|u\|$) that the second integral in the equations (3.21) satisfies

$$\begin{aligned} &\int_0^\pi \Big[\frac{g(x,u_n)}{u_n} u_n(-\overline{u}_n + \tilde{u}_n) - f(-\overline{u}_n + \tilde{u}_n)\Big]\,dx \\ &\le \quad \varepsilon(\|\overline{u}_n\|^2 + \|\tilde{u}_n\|^2) + (cR + \|p\|_1 + \|f\|_1)(\|\overline{u}\|_{C(0,\pi)} + \|\tilde{u}\|_{C(0,\pi)})\,. \end{aligned} \tag{3.23}$$

It follows from (3.21), (3.22), (3.23), and the imbedding $H\subset C(0,\pi)$ ($\|u\|_{C(0,\pi)}\le c_1\|u\|$) that there are constants $\varrho_1>0$, $\varrho_2>0$ such that

$$0\ge\lim_{n\to\infty}\frac{1}{\|u_n\|}\big[\varrho_1\|\overline{u}_n\|^2+\varrho_2\|\tilde{u}_n\|^2-(cR+\|p\|_1+\|f\|_1)c_1(\|\overline{u}_n\|+\|\tilde{u}_n\|)\big]\,.$$

Let $u_n^\perp=\overline{u}_n+\tilde{u}_n$. Then it holds $\|u_n^\perp\|^2=\|\overline{u}_n\|^2+\|\tilde{u}_n\|^2$ and $\|\overline{u}_n\|+\|\tilde{u}_n\|\le\sqrt{2}\|u_n^\perp\|$. Since there are constants $\varrho>0$ and $\beta>0$ such that

$$0\ge\lim_{n\to\infty}\frac{1}{\|u_n\|}\big(\varrho\|u_n^\perp\|^2-\beta\|u_n^\perp\|\big)\,.$$

Therefore,

$$0=\lim_{n\to\infty}\frac{\|u_n^\perp\|^2}{\|u_n\|}\,. \tag{3.24}$$

Now we divide (3.12) by $\|u_n\|$. We get

$$\lim_{n\to\infty}\Big\{\frac12\int_0^\pi\Big[\frac{u_n'}{\|u_n\|}u_n'-m^2\frac{u_n}{\|u_n\|}u_n\Big]\,dx-\int_0^\pi\frac{G(u_n)-fu_n}{\|u_n\|}\,dx\Big\}=0\,. \tag{3.25}$$

We obtain from (3.24) the following equality

$$\begin{aligned}&\lim_{n\to\infty}\frac12\int_0^\pi\Big[\frac{u_n'}{\|u_n\|}u_n'-m^2\frac{u_n}{\|u_n\|}u_n\Big]\,dx\\ &=\lim_{n\to\infty}\frac12\int_0^\pi\Big[\frac{((u_n^\perp)')^2}{\|u_n\|}-m^2\frac{(u_n^\perp)^2}{\|u_n\|}\Big]\,dx=0\,.\end{aligned} \tag{3.26}$$

We know that $v_n\to k\sin mx$, $k>0$ in H. Due to compact imbedding $H\subset C(0,\pi)$ we have $v_n\to k\sin mx$ in $C(0,\pi)$ and we get

$$\lim_{n\to\infty}u_n(x)=\begin{cases}+\infty & \text{for } x\in(0,\pi) \text{ such that } \sin mx>0\,,\\ -\infty & \text{for } x\in(0,\pi) \text{ such that } \sin mx<0\,.\end{cases}$$

Using Fatou's lemma, (3.25), and (3.26) we conclude

$$\int_0^\pi f(x)\sin mx\,dx\ge\int_0^\pi\big[G_+(x)(\sin mx)^+-G_-(x)(\sin mx)^-\big]\,dx\,. \tag{3.27}$$

This is a contradiction to (2.6). This implies that the sequence $\{\|u_n\|\}$ is bounded. Then there exists $u_0\in H$ such that $\|u_n\|\rightharpoonup u_0$ in H, $\|u_n\|\to u_0$ in $L^2(0,\pi)$, $C(0,\pi)$ (taking a subsequence if it is necessary). It follows from the equality (3.14) that

$$\begin{aligned}\lim_{\substack{n\to\infty\\ m\to\infty\\ k\to\infty}}\Big\{&\int_0^\pi\big[(u_n-u_m)'w_k'-m^2(u_n-u_m)w_k\big]\,dx\\ &-\int_0^\pi\big[g(x,u_n)-g(x,u_m)\big]w_k\,dx\Big\}=0\,.\end{aligned} \tag{3.28}$$

The strong convergence $u_n \to u_0$ in $C(0,\pi)$ and the assumption (2.4) imply

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \int_0^\pi \big[g(x,u_n) - g(x,u_m)\big]\big(u_n - u_m\big)\,dx = 0\,. \tag{3.29}$$

If we set $w_k = u_n$, $w_k = u_m$ in (3.29) and subtract these equalities, then

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \int_0^\pi [(u_n' - u_m')^2 - m^2(u_n - u_m)^2]\,dx = 0\,. \tag{3.30}$$

Hence the strong convergence $u_n \to u_0$ in $L^2(0,\pi)$ and (3.30) imply the strong convergence $u_n \to u_0$ in H. This shows that J satisfies Palais-Smale condition and the proof of Theorem 2 is complete.

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