Electronic Journal of Differential Equations, Vol. **2001**(2001), No. 15, pp. 1–28.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu ftp ejde.math.unt.edu (login: ftp)

Asymptotic behavior of regularizable minimizers of a Ginzburg-Landau functional in higher dimensions *

Yutian Lei

Abstract

We study the asymptotic behavior of the regularizable minimizers of a Ginzburg-Landau type functional. We also dicuss the location of the zeroes of the minimizers.

1 Introduction

Let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded and simply connected domain with smooth boundary ∂G. Let g be a smooth map from ∂G into S^{n-1} satisfying $d = \deg(g, \partial G) \neq 0$. Consider the Ginzburg-Landau-type functional

$$E_\varepsilon(u, G) = \frac{1}{p}\int_G |\nabla u|^p + \frac{1}{4\varepsilon^p}\int_G (1 - |u|^2)^2, \quad (p > 1)$$

with a small parameter $\varepsilon > 0$. It is known that this functional achieves its minimum on

$$W_p = \{v \in W^{1,p}(G, \mathbb{R}^n) : v|_{\partial G} = g\}$$

at a function u_ε. We are concerned with the asymptotic behavior of u_ε and the location of the zeroes of u_ε as $\varepsilon \to 0$.

The functional $E_\varepsilon(u, G)$ was introduced in the study of the Ginzburg-Landau vortices by F. Bethue, H. Brezis and F. Helein [1] in the case $p = n = 2$. Similar models are also used in many other theories of phase transition. The minimizer u_ε of $E_\varepsilon(u, G)$ represents a complex order parameter. The zeroes of u_ε and the module $|u_\varepsilon|$ both have physics senses, for example, in superconductivity $|u_\varepsilon|^2$ is proportional to the density of supercoducting electrons, and the zeroes of u_ε are the vortices, which were introduced in the type-II superconductors.

In the case $1 < p < n$, it is easily seen that $W^{1,p}_g(G, S^{n-1}) \neq \emptyset$. It is not difficult to prove that the existence of solution u_p for the minimization problem

$$\min\{\int_G |\nabla u|^p : u \in W^{1,p}_g(G, S^{n-1})\}$$

Mathematics Subject Classifications: 35J70.
Key words: Ginzburg-Landau functional, module and zeroes of regularizable minimizers.

Submitted December 13, 2000. Published February 23, 2001.

by taking the minimizing sequence. This solution is called a map of the least p-energy with boundary value g. Using the variational methods, we can proved that the solution u_p is also p-harmonic map on G with the boundary data g, namely, it is a weak solution of the following equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = u|\nabla u|^p.$$

As $\varepsilon \to 0$, there exists a subsequence u_{ε_k} of u_ε, the minimizer of $E_\varepsilon(u,G)$, such that

$$u_{\varepsilon_k} \to u_p, \quad \text{in } W^{1,p}(G,\mathbb{R}^n).$$

In the case $p > n$, $W_g^{1,p}(G,S^{n-1}) = \emptyset$. Thus there is no map of least p-energy on G with the boundary value g. It seems to be very difficult to study the convergence for minimizers of $E_\varepsilon(u,G)$ in W_p. Some results on the asymptotic behavior of the radial minimizers of $E_\varepsilon(u,G)$ were presented in [7].

When $p = n$, this problem was introduced in [1] (the open problem 17). M. C. Hong studied the asymptotic behavior for the regularizable minimizers of $E_\varepsilon(u,G)$ in W_n [6]. He proved that there exist $\{a_1,a_2,\dots,a_J\} \subset \overline{G}$, $J \in N$ and a subsequence u_{ε_k} of the regularizable minimizers u_ε such that

$$u_{\varepsilon_k} \xrightarrow{w} u_n, \quad \text{in } W^{1,n}_{\text{loc}}(G \setminus \{a_1,a_2,\dots,a_J\},\mathbb{R}^n) \tag{1.1}$$

as $\varepsilon_k \to 0$, where u_n is an n-harmonic map.

In this paper we shall discuss the asymptotic behavior for the regularizable minimizers of $E_\varepsilon(u,G)$ on W_n in the case $p = n$. Without loss of generality, we may assume $d > 0$. Recalling a minimizer of $E_\varepsilon(u,G)$ on W_n be called the regularizable minimizer, if it is the limit of the minimizer of the regularized functional

$$E_\varepsilon^\tau(u,G) = \frac{1}{p}\int_G (|\nabla u|^2+\tau)^{p/2} + \frac{1}{4\varepsilon^p}\int_G (1-|u|^2)^2, \quad (\tau \in (0,1))$$

on W_n in $W^{1,p}$. It is not difficult to prove that the regularizable minimizer is also a minimizer of $E_\varepsilon(u,G)$. In order to find the zeroes of the minimizers, we should first locate the singularities of the n-harmonic map u_n.

Theorem 1.1 *If $a_j \in \overline{G}, j = 1,2,\dots,J$ are the singularities of n-harmonic map u_n, then $J = d$, the degree $\deg(u_n,a_j) = 1$, and $\{a_j\}_{j=1}^d \subset G$. Moreover, for every j, there exists at least one zero of the regularizable minimizer u_ε near to a_j.*

Because the module of the minimizer has the physics sense, we have also studied its asymptotic behavior.

Theorem 1.2 *Let u_ε be a regularizable minimizer of $E_\varepsilon(u,G)$, $\rho = |u_\varepsilon|$, then there exists a constant C independent of ε such that*

$$\int_G |\nabla\rho|^n \le C, \quad \textit{and} \quad \frac{1}{\varepsilon^n}\int_G (1-\rho^2) \le C(1+|\ln\varepsilon|).$$

For any given $\eta > 0$*, denote* $G_\eta = G \setminus \cup_{j=1}^d B(a_j, \eta)$*, then as* $\varepsilon \to 0$*,*

$$\frac{1}{\varepsilon^n}\int_{G_\eta}(1-\rho^2)^2 \to 0,$$

$$\rho \to 1, \quad \textit{in } C_{\rm loc}(G_\eta, R).$$

At last, we develop the conclusion of (1.1) into following

Theorem 1.3 *There exists a subsequence* u_{ε_k} *of* u_ε *such that as* $\varepsilon \to 0$*,*

$$u_{\varepsilon_k} \to u_n, \quad \textit{in } W^{1,n}_{\rm loc}(G \setminus \cup_{j=1}^d \{a_j\}, \mathbb{R}^n).$$

We shall prove Theorems 1.2 and 1.3 in §5 and §7 respectively, and the proof of Theorem 1.1 will be given in §6.

2 Basic properties of the regularizable minimizers

First we recall the minimizer of the regularized functional

$$E_\varepsilon^\tau(u, G) = \frac{1}{n}\int_G (|\nabla u|^2 + \tau)^{n/2} + \frac{1}{4\varepsilon^n}\int_G (1-|u|^2)^2, \quad \tau \in (0,1)$$

on W_n, denoted by u_ε^τ. As $\tau \to 0$, there exists a subsequence $u_\varepsilon^{\tau_k}$ of u_ε^τ such that

$$\lim_{\tau_k \to 0} u_\varepsilon^{\tau_k} = u_\varepsilon, \quad \text{in } W^{1,n}(G, \mathbb{R}^n), \tag{2.1}$$

and the limit u_ε is one minimizer of $E_\varepsilon(u, G)$ on W_n, which is named the regularizable minimizer. It is not difficult to prove that u_ε^τ solves the problem

$$-\operatorname{div}[(|\nabla u|^2 + \tau)^{(n-2)/2}\nabla u] = \frac{1}{\varepsilon^n}u(1-|u|^2), \quad \text{on } G, \tag{2.2}$$

$$u|_{\partial G} = g$$

and satisfies the maximum principle: $|u_\varepsilon^\tau| \le 1$ on $\overline{G}$. Moreover

Proposition 2.1 (Theorem 2.2 in [6]) *For any* $\delta > 0$*, there exists a constant* C *independent of* ε *such that*

$$\overline{\lim}_{\tau \to 0}|\nabla u_\varepsilon^\tau| \le C\varepsilon^{-1}, \quad \textit{on } G^{\delta\varepsilon}, \tag{2.3}$$

where $G^{\delta\varepsilon} = \{x \in G : \operatorname{dist}(x, \partial G) \ge \delta\varepsilon\}$*.*

In this section we shall present some basic properties of the regularizable minimizer u_ε. Clearly it is a weak solution of the equation

$$-\operatorname{div}(|\nabla u|^{n-2}\nabla u) = \frac{1}{\varepsilon^n}u(1-|u|^2), \quad \text{on } G, \tag{2.4}$$

and it is known that $|u_\varepsilon| \le 1$ a.e. on $\overline{G}$ [6]. We also have

Proposition 2.2 *For any* $\delta > 0$*, there exists a constant* C *independent of* ε *such that*

$$\|\nabla u_\varepsilon\|_{L^\infty(B(x,\delta\varepsilon/8,\mathbb{R}^n))} \le C\varepsilon^{-1}, \quad \textit{if } x \in G^{\delta\varepsilon}.$$

Proof. Let $y = x\varepsilon^{-1}$ in (2.4) and denote $v(y) = u(x)$, $G_\varepsilon = \{y = x\varepsilon^{-1} : x \in G\}$, $G^\delta = \{y \in G_\varepsilon : \operatorname{dist}(y, \partial G_\varepsilon) > \delta\}$. Since that u is a weak solution of (2.4), we have

$$\int_{G_\varepsilon} |\nabla v|^{n-2}\nabla v \nabla\phi = \int_{G_\varepsilon} v(1-|v|^2)\phi, \quad \phi \in W_0^{1,n}(G_\varepsilon, \mathbb{R}^n).$$

Taking $\phi = v\zeta^n, \zeta \in C_0^\infty(G_\varepsilon, R)$, we obtain

$$\int_{G_\varepsilon} |\nabla v|^n \zeta^n \leq n \int_{G_\varepsilon} |\nabla v|^{n-1}\zeta^{n-1}|\nabla\zeta||v| + \int_{G_\varepsilon} |v|^2(1-|v|^2)\zeta^n.$$

Setting $y \in G^\delta, B(y,\delta/2) \subset G_\varepsilon$, and $\zeta = 1$ in $B(y,\delta/4), \zeta = 0$ in $G_\varepsilon \setminus B(y,\delta/2), |\nabla\zeta| \leq C(\delta)$, we have

$$\int_{B(y,\delta/2)} |\nabla v|^n\zeta^n \leq C(\delta)\int_{B(y,\delta/2)} |\nabla v|^{n-1}\zeta^{n-1} + C(\delta).$$

Using Holder inequality we can derive $\int_{B(y,\delta/4)} |\nabla v|^n \leq C(\delta)$. Combining this with the theorem of [9] yields

$$\|\nabla v\|^n_{L^\infty(B(y,\delta/8))} \leq C(\delta)\int_{B(y,\delta/4)} (1+|\nabla v|)^n \leq C(\delta)$$

which implies

$$\|\nabla u\|_{L^\infty(B(x,\varepsilon\delta/8))} \leq C(\delta)\varepsilon^{-1}.$$

Proposition 2.3 (Lemma 2.1 in [6]) *There exists a constant C independent of ε such that for $\varepsilon \in (0,1)$,*

$$E_\varepsilon(u_\varepsilon, G) \leq d\frac{(n-1)^{n/2}}{n}|S^{n-1}||\ln\varepsilon| + C. \tag{2.5}$$

Proposition 2.4 *There exists a constant C independent of ε such that*

$$\frac{1}{\varepsilon^n}\int_G (1-|u_e|^2)^2 \leq C. \tag{2.6}$$

Proof. By (3.6) in [6],

$$\int_G |\nabla u_\varepsilon|^n \geq d(n-1)^{n/2}|S^{n-1}||\ln\varepsilon| - C.$$

Applying Proposition 2.3 we may obtain (2.6).

3 A class of bad balls

Fix $\rho > 0$. For the regularizable minimizer u_ε, from Theorem 2.2 in [6] we know

$$|u_\varepsilon| \geq \frac{1}{2}, \quad \text{on } G \setminus G^{\rho\varepsilon}, \tag{3.1}$$

where $G^{\rho\varepsilon} = \{x \in G : \operatorname{dist}(x, \partial G) \geq \rho\varepsilon\}$. Thus there exists no zero of u_ε on $G \setminus G^{\rho\varepsilon}$.

Proposition 3.1 *Let u_ε be a regularizable minimizer of $E_\varepsilon(u, G)$, There exist positive constants λ, μ which are independent of $\varepsilon \in (0,1)$ such that if*

$$\frac{1}{\varepsilon^n} \int_{G^{\rho\varepsilon} \cap B^{2l\varepsilon}} (1 - |u_\varepsilon|^2)^2 \leq \mu, \tag{3.2}$$

where $B^{2l\varepsilon}$ is some ball of radius $2l\varepsilon$ with $l \geq \lambda$, then

$$|u_\varepsilon| \geq \frac{1}{2}, \quad \forall x \in G^{\rho\varepsilon} \cap B^{l\varepsilon}. \tag{3.3}$$

Proof. First it is known that there exists a constant $\beta > 0$ such that for any $x \in G^{\rho\varepsilon}$ and $0 < r \leq 1$,

$$|G^{\rho\varepsilon} \cap B(x, r)| \geq \beta r^n.$$

Next we take

$$\lambda = \min\Big(\frac{1}{4C}, \frac{1}{8}\rho\Big), \quad \mu = \frac{\beta\lambda^n}{16}$$

where C is the constant in Proposition 2.2.

Suppose that there is a point $x_0 \in G^{\rho\varepsilon} \cap B^{l\varepsilon}$ such that $|u_\varepsilon(x_0)| < 1/2$, then applying Proposition 2.2 we have

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C\varepsilon^{-1}|x - x_0| = \frac{1}{4}, \quad x \in B(x_0, \lambda\varepsilon) \cap G^{\rho\varepsilon}.$$

Hence

$$(1 - |u_\varepsilon(x)|^2)^2 > \frac{1}{16}, \quad \forall x \in B(x_0, \lambda\varepsilon) \cap G^{\rho\varepsilon},$$

$$\int_{B(x_0,\lambda\varepsilon)\cap G^{\rho\varepsilon}} (1 - |u_\varepsilon|^2)^2 > \frac{1}{16}|G^{\rho\varepsilon} \cap B(x_0, \lambda\varepsilon)| \geq \beta\frac{1}{16}(\lambda\varepsilon)^n = \mu\varepsilon^n. \tag{3.4}$$

Since $x_0 \in B^{l\varepsilon} \cap G^{\rho\varepsilon}$, we have $(B(x_0, \lambda\varepsilon) \cap G^{\rho\varepsilon}) \subset (B^{2l\varepsilon} \cap G^{\rho\varepsilon})$, thus (3.4) implies

$$\int_{B^{2l\varepsilon}\cap G^{\rho\varepsilon}} (1 - |u_\varepsilon|^2)^2 > \mu\varepsilon^n$$

which contradicts (3.2) and thus the proposition is proved.

To find the zeroes of the regularizable minimizer u_ε based on Proposition 3.1, we may take (3.2) as the ruler to distinguish the ball of radius $\lambda\varepsilon$ which contain the zeroes.

Let λ, μ be constants in Proposition 3.1. If

$$\frac{1}{\varepsilon^n}\int_{G^{\rho\varepsilon}\cap B(x^\varepsilon,2\lambda\varepsilon)}(1-|u_\varepsilon|^2)^2 \le \mu,$$

then $B(x^\varepsilon, \lambda\varepsilon)$ is called good ball. Otherwise $B(x^\varepsilon, \lambda\varepsilon)$ is called bad ball. From Proposition 3.1 we are led to

$$|u_\varepsilon| \ge \frac{1}{2}, \quad \text{on } G^{\rho\varepsilon} \setminus \cup_{x^\varepsilon\in\Lambda} B(x^\varepsilon, \lambda\varepsilon), \tag{3.5}$$

where Λ is the set of the centres of all bad balls. (3.5) and (3.1) imply that the zeroes of u_ε are contained in these bad balls.

Now suppose that $\{B(x_i^\varepsilon, \lambda\varepsilon), i \in I\}$ is a family of balls satisfying

(i) $x_i^\varepsilon \in G^{\rho\varepsilon}, i \in I$

(ii) $G^{\rho\varepsilon} \subset \cup_{i\in I} B(x_i^\varepsilon, \lambda\varepsilon)$

(iii)
$$B(x_i^\varepsilon, \lambda\varepsilon/4) \cap B(x_j^\varepsilon, \lambda\varepsilon/4) = \emptyset, i \ne j\,. \tag{3.6}$$

Let $J_\varepsilon = \{i \in I : B(x_i^\varepsilon, \lambda\varepsilon) \text{ is a bad ball}\}$.

Proposition 3.2 *There exists a positive integer N which is independent of ε such that the number of bad balls* card $J_\varepsilon \le N$.

Proof. Since (3.6) implies that every point in $G^{\rho\varepsilon}$ can be covered by finite, say m (independent of ε) balls, from (2.6) and the definition of bad balls,we have

$$\begin{aligned}
\mu\varepsilon^n \operatorname{card} J_\varepsilon &\le \sum_{i\in J_\varepsilon}\int_{B(x_i^\varepsilon,2\lambda\varepsilon)\cap G^{\rho\varepsilon}}(1-|u_\varepsilon|^2)^2\\
&\le m\int_{\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,2\lambda\varepsilon)\cap G^{\rho\varepsilon}}(1-|u_\varepsilon|^2)^2\\
&\le m\int_G(1-|u_\varepsilon|^2)^2 \le mC\varepsilon^n
\end{aligned}$$

and hence card $J_\varepsilon \le \frac{mC}{\mu} \le N$.

Similar to the argument of Theorem IV.1 in [1], we have

Proposition 3.3 *There exist a subset $J \subset J_\varepsilon$ and a constant $h \ge \lambda$ such that*

$$\begin{gathered}
\cup_{i\in J_\varepsilon} B(x_i^\varepsilon, \lambda\varepsilon) \subset \cup_{i\in J} B(x_j^\varepsilon, h\varepsilon),\\
|x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon, \quad i,j \in J, \quad i \ne j.
\end{gathered} \tag{3.7}$$

Proof. If there are two points x_1, x_2 such that (3.7) is not true with $h = \lambda$, we take $h_1 = 9\lambda$ and $J_1 = J_\varepsilon \setminus \{1\}$. In this case, if (3.7) holds we are done. Otherwise we continue to choose a pair points x_3, x_4 which does not satisfy (3.7) and take $h_2 = 9h_1$ and $J_2 = J_\varepsilon \setminus \{1,3\}$. After at most N steps we may conclude this proposition.

Applying Proposition 3.3 we may modify the family of bad balls such that the new one, denoted by $\{B(x_i^\varepsilon, h\varepsilon) : i \in J\}$, satisfies

$$\begin{gathered} \cup_{i\in J_\varepsilon} B(x_i^\varepsilon, \lambda\varepsilon) \subset \cup_{i\in J} B(x_i^\varepsilon, h\varepsilon), \\ \lambda \leq h; \quad \operatorname{card} J \leq \operatorname{card} J_\varepsilon, \\ |x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon, i, j \in J, i \neq j. \end{gathered} \tag{3.8}$$

The last condition implies that every two balls in the new family do not intersect.

As $\varepsilon \to 0$, there exist a subsequence $x_i^{\varepsilon_k}$ of x_i^ε and $a_i \in \overline{G}$ such that

$$x_i^{\varepsilon_k} \to a_i, \quad i = 1, 2, \dots, N_1 = \operatorname{card} J.$$

Perhaps there may be at least two subsequences converge to the same point, we denote by

$$a_1, a_2, \dots, a_{N_2}, \quad N_2 \leq N_1$$

the collection of distinct points in $\{a_i\}_1^{N_1}$.

To prove $a_j \overline{\in} \partial G$, it is convenient to enlarge a little G. Assume $G' \subset \mathbb{R}^n$ is a bounded, simply connected domain with smooth boundary such that $\overline{G} \subset G'$, and take a smooth map $\bar{g} : (G' \setminus G) \to S^{n-1}$ such that $\bar{g} = g$ on ∂G. We extend the definition domain of every element in $\{u : G \to \mathbb{R}^n : u|_{\partial G} = g\}$ to G' such that $u = \overline{g}$ on $G' \setminus G$. In particular, the regularizable minimizer u_ε can be defined on G'.

Fix a small constant $\sigma > 0$ such that

$$\begin{gathered} \overline{B(a_j, \sigma)} \subset G', \quad j = 1, 2, \dots, N_2; \\ 4\sigma < |a_j - a_i|, \quad i \neq j; \quad 4\sigma < \operatorname{dist}(G, \partial G'). \end{gathered}$$

Writing $\Lambda_j = \{i \in J : x_i^{\varepsilon_k} \to a_j\}, j = 1, 2, \dots, N_2$, we have

$$\begin{gathered} \cup_{i\in\Lambda_j} \overline{B(x_i^{\varepsilon_k}, h\varepsilon_k)} \subset B(a_j, \sigma), \quad j = 1, 2, \dots, N_2 \\ \cup_{j\in J} B(x_j^{\varepsilon_k}, h\varepsilon_k) \subset \cup_{j=1}^{N_2} B(a_j, \sigma/4) \\ B(x_i^{\varepsilon_k}, h\varepsilon_k) \cap B(x_j^{\varepsilon_k}, h\varepsilon_k) = \emptyset, \quad i, j \in J, i \neq j \end{gathered}$$

as long as ε_k is small enough. Let u_ε is the regularizable minimizer of $E_\varepsilon(u, G)$ and denote $d_i^k = deg(u_{\varepsilon_k}, \partial B(x_i^{\varepsilon_k}, h\varepsilon_k)), l_j^k = deg(u_{\varepsilon_k}, \partial B(a_j, \sigma))$, thus

$$l_j^k = \sum_{i\in\Lambda_j} d_i^k, \quad d = \sum_{j=1}^{N_2} l_j^k. \tag{3.9}$$

To prove that the degrees d_i^k and l_j^k are independent of ε_k, we recall a proposition stated in [6] (Lemma 3.3) or [2] (Theorem 8.2).

Proposition 3.4 *Let $\phi : S^{n-1} \to S^{n-1}$ be a C^0-map with $\deg\phi = d$. Then*

$$\int_{S^{n-1}} |\nabla_\tau \phi|^{n-1} dx \geq |d|(n-1)^{(n-1)/2}|S^{n-1}|.$$

Proposition 3.5 *There exists a constant C which is independent of ε_k such that*

$$|d_i^k| \leq C, i \in J; \quad |l_j^k| \leq C, j = 1, 2, \ldots, N_2.$$

Proof. Since $u = u_\varepsilon$ is a weak solution of (2.4), applying the theory of the local regularity in [9], we know $u \in C(\partial B(x_i^{\varepsilon_k}, h\varepsilon_k))$. Since (3.5) implies $|u| \geq 1/2$ on $\partial B(x_i^{\varepsilon_k}, h\varepsilon_k)$, thus $\phi = \frac{u}{|u|} \in C(\partial B(x_i^{\varepsilon_k}, h\varepsilon_k), S^{n-1})$. From Proposition 3.4, we have

$$|d_i^k| \leq |S^{n-1}|^{-1}(n-1)^{(1-n)/2} \int_{\partial B(x_i^{\varepsilon_k}, h\varepsilon_k)} |(\frac{u}{|u|})_\tau|^{n-1}.$$

Since $|u| \geq \frac{1}{2}$ on $G' \setminus G^{\rho\varepsilon}$, there is no zero of u_ε in it. Thus

$$\deg(u_{\varepsilon_k}, \partial B(x_i^{\varepsilon_k}, h\varepsilon_k)) = \deg(u_{\varepsilon_k}, \partial(B(x_i^{\varepsilon_k}, h\varepsilon_k) \cap G^{\rho\varepsilon_k}))$$

and

$$|d_i^k| \leq |S^{n-1}|^{-1}(n-1)^{(1-n)/2} \int_{\partial[B(x_i^{\varepsilon_k}, h\varepsilon_k) \cap G^{\rho\varepsilon}]} |(\frac{u}{|u|})_\tau|^{n-1}. \tag{3.10}$$

Substituting (2.3) and the fact $|u_{\varepsilon_k}| \geq \frac{1}{2}$ on $\partial[B(x_i^{\varepsilon_k}, h\varepsilon_k) \cap G^{\rho\varepsilon}]$ into (3.10), we obtain

$$|d_i^k| \leq C\varepsilon_k^{1-n}|S^{n-1}|^{-1}(n-1)^{(1-n)/2}(h\varepsilon_k)^{n-1} \leq C,$$

where C is a constant which is independent of ε_k. Combining this with (3.9) we can complete the proof of the proposition.

Proposition 3.5 implies that there exist a number k_j which is independent of ε_k and a subsequence of l_j^k denoted itself such that

$$l_j^k \to k_j, \quad as \quad k \to \infty.$$

Since $l_j^k, k_j \in N, \{l_j^k\}$ must be constant sequence for any fixed j, namely $l_j^k = k_j$. The same reason shows d_i^k can be writen as d_i which is also a number independent of ε_k later.

4 An estimate for the lower bound

Write $\Omega' = G' \setminus \cup_{j=1}^{N_2} B(a_j, \sigma)$. Fixing $j \in \{1, 2, \ldots, N_2\}$ and taking $i_0 \in \Lambda_j$, we have $x_{i_0} \to a_j$ as $\varepsilon \to 0$. Thus

$$\cup_{i\in\Lambda_j} \overline{B(x_i^\varepsilon, h\varepsilon)} \subset B(x_{i_0}, \sigma/4) \subset B(a_j, \sigma) \tag{4.1}$$

holds with ε small enough.

Denote $\Omega_j = B(a_j, \sigma) \setminus \cup_{i\in\Lambda_j} B(x_i^\varepsilon, h\varepsilon), \Omega_{j\sigma} = B(x_{i_0}, \sigma/4) \setminus \cup_{i\in\Lambda_j} B(x_i^\varepsilon, h\varepsilon)$. To estimate the lower bound of $\|\nabla u_\varepsilon\|_{L^n(\Omega_j)}$, the following proposition is necessary that was given by Theorem 3.9 in [6].

Proposition 4.1 *Let $A_{s,t}(x_i) = (B(x_i, s) \setminus B(x_i, t)) \cap G$ with $\varepsilon \le t < s \le R$. Assume that $u \in W_g^{1,n}(G, \mathbb{R}^n)$ and $\frac{1}{2} \le |u| \le 1$ on $A_{s,t}(x_i)$. If there is a constant C such that*

$$\frac{1}{\varepsilon^n} \int_{A_{s,t}(x_i)} (1 - |u|^2)^2 \le C.$$

Then for $\varepsilon < \varepsilon_0$ there holds

$$\int_{A_{s,t}(x_i)} |\nabla u|^n \ge |d_i|^{n/(n-1)} (n-1)^{n/2} |S^{n-1}| \ln \frac{s}{t} - C,$$

where C is a constant which is independent of ε and d_i is the degree of u on each $\partial(B(x_i, r) \cap G), t \le r \le s$.

Proposition 4.2 *Assume $Card \Lambda_j = N$. Then*

$$\int_{\Omega_j} |\nabla u_\varepsilon|^n \ge \int_{\Omega_{j,\sigma}} |\nabla u_\varepsilon|^n \ge (n-1)^{n/2} |S^{n-1}| |k_j| \ln \frac{\sigma}{\varepsilon} - C \tag{4.2}$$

where C is a constant which is independent of ε.

Proof. We give the proof following that in [6] (see Theorem 3.10), and the idea comes from [8]. Suppose $x_1, x_2, \dots, x_N$ converge to a_j, and $d_{i,R} (i = 1, 2, \dots, N)$ is the degree of u_ε around $\partial B(x_i, R)$. Let R_ε^σ denote the set of all numbers $R \in [\varepsilon, \sigma]$ such that $\partial B(x_i, R) \cap B(x_j, \varepsilon) = \emptyset$ for all $i \ne j$ and such that for some collection $J_R \subset \{1, 2, \dots, N\}$, satisfying $J_R \subset J_{R'}$ if $R' \le R$, the family $\{B(x_i, R)\}_{i \in J_R}$ is disjoint and

$$\cup_{i=1}^N B(x_i, \varepsilon) \subset \cup_{i \in J_{R'}} B(x_i, R') \subset \cup_{i \in J_R} B(x_i, R), \quad R' \le R.$$

Note that R_ε^σ is the union of closed intervals $[R_0^l, \mathbb{R}^l], 1 \le l \le L$, whose right endpoints correspond to a number $R = \mathbb{R}^l$ such that $\partial B(x_i, R) \cap \overline{B(x_j, R)} \ne \emptyset$ for some pair $i \ne j \in J_R$ and whose left endpoints correspond to a number R_0^l such that $\overline{B(x_i, \mathbb{R}^{l-1})} \setminus \cup_{j \in J_0} B(x_j, R_0^l) \ne \emptyset$ for $i \overline{\in} J_{R_0^l}$. $J_R = J^l$ is a constant for $R \in [R_0^l, \mathbb{R}^l]$ and $J^{l+1} \subset J^l, J^{l+1} \ne J^l$. Thus $L \le N$. Moreover, there exists a constant $M = M(h) > 0$ such that

$$R_0^l \le M\varepsilon, \quad \mathbb{R}^L \ge \sigma/M, \quad R_0^{l+1} \le M R^l \tag{4.3}$$

for all $l = 1, 2, \dots, L-1$. Finally, observe that for all $R \in R_\varepsilon^\sigma$ and $J \in J_R$,

$$|k_j| = |\sum_{i \in J_R} d_{i,R}| \le \sum_{i \in J_R} |d_{i,R}|^{n/(n-1)}. \tag{4.4}$$

Applying (4.3)(4.4) and proposition 4.1 we have

$$\int_{\Omega_{j,\sigma}} |\nabla u_\varepsilon|^n \quad \ge \quad \sum_{l=1}^L \sum_{i \in J^l} |\int_{A_{\mathbb{R}^l, R_0^l}(x_i)} \nabla u_\varepsilon|^n$$

$$
\begin{aligned}
&\geq \sum_{l=1}^{L}\sum_{i\in J^l}|S^{n-1}|(n-1)^{n/2}|d_{i,\mathbb{R}^l}|\ln(\mathbb{R}^l/R_0^l)-C\\
&\geq |S^{n-1}|(n-1)^{n/2}|k_j|\sum_l(\ln\mathbb{R}^l-\ln R_0^l)-C\\
&\geq (n-1)^{n/2}|S^{n-1}||k_j|\ln\frac{\sigma}{\varepsilon}-C.
\end{aligned}
$$

This and (4.1) imply that (4.2) holds.

Remark In fact the following results

$$\int_{\Omega_j}|\nabla\frac{u_\varepsilon}{|u_\varepsilon|}|^n\geq(n-1)^{n/2}|S^{n-1}||k_j|^{n/(n-1)}\ln\frac{\sigma}{\varepsilon},$$

and

$$\int_{\Omega_j}(1-|u_\varepsilon|^n)|\nabla\frac{u_\varepsilon}{|u_\varepsilon|}|^n\leq C$$

had been presented in the proof of Theorem 3.9 in [6], where C which is independent of ε. Noticing

$$\int_{\Omega_j}|u_\varepsilon|^n|\nabla\frac{u_\varepsilon}{|u_\varepsilon|}|^n=\int_{\Omega_j}|\nabla\frac{u_\varepsilon}{|u_\varepsilon|}|^n-\int_{\Omega_j}(1-|u_\varepsilon|^n)|\nabla\frac{u_\varepsilon}{|u_\varepsilon|}|^n,$$

we have

$$\int_{\Omega_j}|u_\varepsilon|^n|\nabla\frac{u_\varepsilon}{|u_\varepsilon|}|^n\geq(n-1)^{n/2}|k_j|^{n/(n-1)}|S^{n-1}|\ln\frac{\sigma}{\varepsilon}-C.$$

Theorem 4.3 *There exists a constant C which is independent of $\varepsilon,\sigma\in(0,1)$ such that*

$$\int_{\cup_{j=1}^{N_2}\Omega_j}|\nabla u_\varepsilon|^n\geq(n-1)^{n/2}|S^{n-1}|d\ln\frac{\sigma}{\varepsilon}-C,\tag{4.5}$$

$$\frac{1}{n}\int_{G_\sigma}|\nabla u_\varepsilon|^n+\frac{1}{4\varepsilon^n}\int_G(1-|u_\varepsilon|^2)^2\leq\frac{1}{n}(n-1)^{n/2}|S^{n-1}|d\ln\frac{1}{\sigma}+C\tag{4.6}$$

where $G_\sigma=G\setminus\cup_{j=1}^{N_2}B(a_j,\sigma)$.

Proof. From (4.2) and Proposition 2.3 we have

$$(n-1)^{n/2}|S^{n-1}|(\sum_{j=1}^{N_2}|k_j|)\ln\frac{\sigma}{\varepsilon}\leq(n-1)^{n/2}|S^{n-1}|d\ln\frac{1}{\varepsilon}+C$$

or $(\sum_{j=1}^{N_2}|k_j|-d)\ln\frac{1}{\varepsilon}\leq C$. It is seen as ε small enough

$$\sum_{j=1}^{N_2}|k_j|\leq d=\sum_{j=1}^{N_2}k_j$$

which implies

$$k_j \geq 0. \tag{4.7}$$

This and (3.9) imply

$$\sum_{j=1}^{N_2} |k_j| = \sum_{j=1}^{N_2} k_j = d. \tag{4.8}$$

Substituting (4.8) into (4.2) yields (4.5), and (4.6) may be concluded from (4.5) and Proposition 2.3.

From (4.6) and the fact $|u_\varepsilon| \leq 1$ a.e. on G, we may conclude that there exists a subsequence u_{ε_k} of u_ε such that

$$u_{\varepsilon_k} \xrightarrow{w} u_*, \quad W^{1,n}(G_\sigma, \mathbb{R}^n) \tag{4.9}$$

as $\varepsilon_k \to 0$. Compare (4.9) with (1.1) we known $u_* = u_n$ on G_σ, and

$$\{a_j\}_{j=1}^{N_2} = \{a_j\}_{j=1}^{J}. \tag{4.10}$$

These points were called the singularities of u_n.

To show these singularities $a_j \overline{\in} \partial G$, the following conclussion is necessary.

Proposition 4.4 *Assume $a \in \partial G$ and $\sigma \in (0, R)$ with a small constant R. If*

$$u \in W^{1,n}(A_{R,\sigma}(a), S^{n-1}) \cap C^0, \quad u = \overline{g}$$

on $(G' \setminus G) \cap B(a, R)$ and $\deg(u, \partial B(a, R)) = 1$, then there exists a constant C which is independent of σ such that

$$\int_{A_{R,\sigma}(a)} |\nabla u|^n \geq 2^{\frac{1}{n}} (n-1)^{n/2} |S^{n-1}| \ln \frac{1}{\sigma} - C\,. \tag{4.11}$$

Proof. Similar to the proof of Lemma VI.1 in [1], we may write G as the half space

$$\{(x_1, x_2, \dots, x_n) : x_n > 0\}$$

locally and a as 0 by a conformal change.

Denote $S_t = \partial B(0, t), t \in (\sigma, R)$. Noticing that $\overline{g}$ is smooth on $G' \setminus G$, we have

$$\sup_{\overline{G'} \setminus G} |\overline{g}_\tau| \leq C_1.$$

Taking t sufficiently small such that

$$t \leq (n-1)^{1/2} \frac{(2^{n-1} - 1)^{1/(n-1)}}{2C_1},$$

then

$$\int_{S_t^-} |\bar{g}_\tau|^{n-1} \leq |S_t^-| C_1^{n-1} \leq |S^{n-1}| t^{n-1} C_1^{n-1} \leq (n-1)^{(n-1)/2} |S^{n-1}| (1 - 2^{1-n}) \tag{4.12}$$

with $R < 1$ small enough, where $S_t^- = S_t \cap \{x_n < 0\}$. On the other hand we can be led to

$$(n-1)^{(n-1)/2}|S^{n-1}| \le \int_{S_t} |u_\tau|^{n-1} = \int_{S_t^+} |u_\tau|^{n-1} + \int_{S_t^-} |\bar{g}_\tau|^{n-1}$$

from Proposition 3.4. Here $S_t^+ = S_t \setminus S_t^-$. Combining this with (4.12) yields

$$\begin{aligned} \int_{S_t^+} |u_\tau|^n &\ge |S_t^+|^{-1/(n-1)}(\int_{S_t^+} |u_\tau|^{n-1})^{n/(n-1)} && (4.1)\\ &\ge 2^{\frac{1}{n}}|S^{n-1}|(n-1)^{n/2}t^{-1}. && (4.2) \end{aligned}$$

Integrating this over (σ, R), we obtain

$$\int_{A_{R,\sigma}} |\nabla u|^n \ge 2^{\frac{1}{n}}|S^{n-1}|(n-1)^{n/2} \ln \frac{R}{\sigma}$$

which implies (4.11). To prove $k_j = 1$ for any j, we suppose $R > 2\sigma$ is a small constant such that

$$\overline{B(a_j, R)} \subset G'; \quad B(a_j, R) \cap B(a_i, R) = \emptyset, i \ne j. \tag{4.13}$$

Denote $\Pi = \{v \in W^{1,n}(\Omega', S^{n-1}) \cap C^0 : \deg(v, \partial B(a_j, r)) = k_j, r \in (\sigma, R), j = 1, 2, \dots, N_2\}$.

Proposition 4.5 *For any $v \in \Pi$, if $k_j \ge 0, j = 1, 2, \dots, N_2$, then there exists a constant $C = C(R)$ which is independent of σ such that*

$$\int_{\Omega'} |\nabla v|^n \ge (n-1)^{n/2}|S^{n-1}|(\sum_{j=1}^{N_2} k_j^{\frac{n}{n-1}}) \ln \frac{1}{\sigma} - C. \tag{4.14}$$

Proof. Write $A_{R,\sigma}(a_j) = B(a_j, R) \setminus B(a_j, \sigma)$, thus $\cup_{j=1}^{N_2} A_{R,\sigma}(a_j) \subset \Omega'$. From Proposition 3.4 we have

$$\begin{aligned} k_j = |k_j| &\le (n-1)^{(1-n)/2}|S^{n-1}|^{-1} \int_{S^{n-1}} |v_\tau|^{n-1}\\ &\le (n-1)^{(1-n)/2}|S^{n-1}|^{(n-1)/n}(\int_{S^{n-1}} |v_\tau|^n)^{(n-1)/n} \end{aligned}$$

namely

$$\int_{S^{n-1}} |v_\tau|^n \ge (n-1)^{n/2}|S^{n-1}|k_j^{n/(n-1)}.$$

On the other hand, we may obtain

$$\int_{\Omega'} |\nabla v|^n \ \ge \ \sum_{j=1}^{N_2} \int_{A_{R,\sigma}(a_j)} |\nabla v|^n$$

$$
\begin{aligned}
&\geq \sum_{j=1}^{N_2} \int_\sigma^R \int_{S^{n-1}} r^{-n} |\nabla_\tau v|^n r^{n-1} d\zeta dr \\
&\geq (n-1)^{n/2} |S^{n-1}| \sum_{j=1}^{N_2} k_j^{n/(n-1)} \int_\sigma^R r^{-1} dr \\
&= (n-1)^{n/2} |S^{n-1}| (\sum_{j=1}^{N_2} k_j^{n/(n-1)}) \ln \frac{R}{\sigma}
\end{aligned}
$$

which implies (4.14).

5 The proof of Theorem 1.2

Let $u_\varepsilon$be a regularizable minimizer of $E_\varepsilon(u,G)$. Proposition 2.4 has given one estimate of convergence rate of $|u_\varepsilon|$. Moreover, we also have

Theorem 5.1 *There exists a constant C which is independent of $\varepsilon \in (0,1)$ such that*

$$
\frac{1}{\varepsilon^n} \int_G (1 - |u_\varepsilon|^2) \leq C(1 + \ln \frac{1}{\varepsilon}). \tag{5.1}
$$

Proof. The minimizer $u = u_\varepsilon^\tau$ of the regularized functional $E_\varepsilon^\tau(u,G)$ solves (2.2). Taking the inner product of the both sides of (2.2) with u and integrating over G we have

$$
\begin{aligned}
\frac{1}{\varepsilon^n} \int_G |u|^2 (1 - |u|^2) &= -\int_G div(v^{(n-2)/2} \nabla u) u \\
&= \int_G v^{(n-2)/2} |\nabla u|^2 - \int_{\partial G} v^{(n-2)/2} u u_n \qquad (5.2) \\
&\leq \int_G v^{(n-2)/2} |\nabla u|^2 + C \int_{\partial G} v^{n/2} + C
\end{aligned}
$$

where n denotes the unit outward normal to ∂G and u_n the derivative with respect to n.

To estimate $\int_{\partial G} v^{n/2}$, we choose a smooth vector field ν such that $\nu|_{\partial G} = n$. Multiplying (2.2) by $(\nu \cdot \nabla u)$ and integrating over G, we obtain

$$
\begin{aligned}
\frac{1}{\varepsilon^n} \int_G u(1 - |u|^2)(\nu \cdot \nabla u) &= -\int_G div(v^{(n-2)/2} \nabla u)(\nu \cdot \nabla u) \\
&= \int_G v^{(n-2)/2} \nabla u \cdot (\nu \cdot \nabla u) - \int_{\partial G} v^{(n-2)/2} |u_n|^2.
\end{aligned}
$$

Combining this with

$$
\begin{aligned}
\frac{1}{\varepsilon^n} \int_G u(1 - |u|^2)(\nu \cdot \nabla u) &= \frac{1}{2\varepsilon^n} \int_G (1 - |u|^2)(\nu \cdot \nabla(|u|^2)) \\
&= -\frac{1}{4\varepsilon^n} \int_G (1 - |u|^2)^2 \operatorname{div} \nu
\end{aligned}
$$

and

$$\begin{aligned}
&\int_G v^{(n-2)/2}\nabla u\cdot\nabla(\nu\cdot\nabla u)\\
&= \int_G v^{(n-2)/2}|\nabla u|^2\operatorname{div}\nu+\frac{1}{n}\int_G \nu\cdot\nabla(v^{n/2})\\
&= \int_G v^{(n-2)/2}|\nabla u|^2\operatorname{div}\nu+\frac{1}{n}\int_{\partial G} v^{n/2}-\frac{1}{n}\int_G v^{n/2}\operatorname{div}\nu
\end{aligned}$$

we obtain

$$\int_{\partial G} v^{(n-2)/2}|u_n|^2 \le \frac{C}{4\varepsilon^n}\int_G (1-|u|^2)^2 + C\int_G v^{n/2}+\frac{1}{n}\int_{\partial G} v^{n/2}.$$

Thus

$$\begin{aligned}
\int_{\partial G} v^{n/2} &= \int_{\partial G} v^{(n-2)/2}(|u_n|^2+|g_t|^2+\tau)\\
&\le C\int_{\partial G} v^{(n-2)/2}+\frac{1}{n}\int_{\partial G} v^{n/2}+CE_\varepsilon^\tau(u_\varepsilon^\tau,G).
\end{aligned}$$

Substituting this into (5.2) yields

$$\frac{1}{\varepsilon^n}\int_G |u|^2(1-|u|^2)\le CE_\varepsilon^\tau(u_\varepsilon^\tau,G).$$

Let $\tau\to 0$, applying (2.1) and Proposition 2.3 we have

$$\frac{1}{\varepsilon^n}\int_G |u_\varepsilon|^2(1-|u_\varepsilon|^2)\le CE_\varepsilon(u_\varepsilon,G)\le C(1+|\ln\varepsilon|)$$

which and (2.6) imply (5.1).

Theorem 5.2 *Denote $\rho=|u_\varepsilon|$. There exists a constant C which is independent of $\varepsilon\in(0,1)$ such that*

$$\|\nabla\rho\|_{L^n(G)}\le C. \tag{5.3}$$

Proof. Denote $u=u_\varepsilon$. From the Remark in §4 we know

$$\int_{\Omega_j}|u|^n|\nabla\frac{u}{|u|}|^n dx\ge (n-1)^{n/2}|k_j|^{\frac{n}{n-1}}|S^{n-1}|\ln\frac{\sigma}{\varepsilon}-C.$$

Thus we may modify (4.5) as

$$\int_{\cup_{j=1}^{N_2}\Omega_j}\rho^n|\nabla\frac{u}{|u|}|^n\ge (n-1)^{n/2}|S^{n-1}|d\ln\frac{\sigma}{\varepsilon}-C.$$

Combining this with

$$\int_{\cup_{j=1}^{N_2}\Omega_j}|\nabla u|^n\ge\int_{\cup_{j=1}^{N_2}\Omega_j}\rho^n|\nabla\frac{u}{|u|}|^n+\int_{\cup_{j=1}^{N_2}\Omega_j}|\nabla\rho|^n-C$$

and Proposition 2.3, we derive

$$\int_{\cup_{j=1}^{N_2}\Omega_j} |\nabla\rho|^n \le C. \tag{5.4}$$

On the other hand, from (2.1) and Proposition 2.1 we are led to

$$\int_{G^{\rho\varepsilon}\cap B(x_i,h\varepsilon)} |\nabla u_\varepsilon|^n = \lim_{\tau_k\to 0}\int_{G^{\rho\varepsilon}\cap B(x_i,h\varepsilon)} |\nabla u_\varepsilon{}^{\tau_k}|^n \le C(\lambda\varepsilon)^n(C/\varepsilon)^n \le C,$$

for $i\in\Lambda_j$. Summarizing for i and using (5.4) we can obtain (5.3).

Theorem 5.3 *For the $\sigma>0$ in Theorem 4.4, then as $\varepsilon\to 0$,*

$$\frac{1}{\varepsilon^n}\int_{G_{3\sigma}}(1-\rho^2)^2 \to 0, \tag{5.5}$$

where $G_{3\sigma} = G\setminus \cup_{j=1}^{N_2} B(a_j,3\sigma)$.

Proof. The regularizable minimizer u_ε satisfies

$$\int_{G_\sigma} |\nabla u|^{n-2}\nabla u\nabla\phi = \frac{1}{\varepsilon^n}\int_{G_\sigma} u\phi(1-|u|^2), \tag{5.6}$$

where $\phi\in W_0^{1,n}(G_\sigma,\mathbb{R}^n)$ since u_ε is a weak solution of (2.4). Denoting $u = u_\varepsilon^\tau = \rho w, \rho = |u|, w = \frac{u}{|u|}$ in G_σ and taking $\phi = \rho w\zeta, \zeta\in W_0^{1,n}(G_\sigma,\mathbb{R}^n)$, we have

$$\int_{G_\sigma} |\nabla u|^{n-2}(w\nabla\rho+\rho\nabla w)(\rho\zeta\nabla w+\rho w\nabla\zeta+w\zeta\nabla\rho) = \frac{1}{\varepsilon^n}\int_{G_\sigma}\rho^2\zeta(1-\rho^2). \tag{5.7}$$

Substituting $2w\nabla w = \nabla(|w|^2) = 0$ into (5.7), we obtain

$$\int_{G_\sigma} |\nabla u|^{n-2}(\rho\nabla\rho\nabla\zeta + |\nabla u|^2\zeta) = \frac{1}{\varepsilon^n}\int_{G_\sigma}\rho^2\zeta(1-\rho^2). \tag{5.8}$$

Set $S = \{x\in G_\sigma : \rho(x) > 1-\varepsilon^\beta\}$ for some fixed $\beta\in(0,n/2)$ and $\overline{\rho} = \max(\rho, 1-\varepsilon^\beta)$, thus $\rho=\overline{\rho}$ on S. In (5.8) taking $\zeta = (1-\overline{\rho})\psi$, where $\psi\in C^\infty(G_\sigma,R), \psi = 0$ on $G_\sigma\setminus G_{2\sigma}, 0<\psi<1$ on $G_{2\sigma}\setminus G_{3\sigma}, \psi=1$ on $G_{3\sigma}$, we have

$$\begin{aligned} &\int_{G_\sigma} |\nabla u|^{n-2}\rho\nabla\rho\cdot\nabla\bar{\rho}\psi + \frac{1}{\varepsilon^n}\int_{G_\sigma} l^2(1-\rho^2)(1-\bar{\rho})\psi \\ =\; &\int_{G_\sigma}|\nabla u|^{n-2}\rho\nabla\rho\nabla\psi(1-\bar{\rho}) + \int_{G_\sigma}|\nabla u|^n\psi(1-\overline{\rho}) \end{aligned} \tag{5.9}$$

Noticing $1/2\le l\le 1$ in G_σ and applying (4.6) we obtain

$$\frac{1}{\varepsilon^n}\int_{G_{3\sigma}}(1-\overline{\rho})(1-\rho^2) + \int_{S\cap G_{3\sigma}}|\nabla u|^{n-2}|\nabla\rho|^2 \le C\varepsilon^\beta. \tag{5.10}$$

On the other hand, (2.6) implies

$$\varepsilon^{2\beta}|G_\sigma \setminus S| \le \int_{G_\sigma\setminus S}(1-l^2)^2 \le C\varepsilon^n,$$

namely $|G_\sigma \setminus S| \le C\varepsilon^{n-2\beta}$. Then there exists a small constant $\varepsilon_0 > 0$ such that

$$G_{3\sigma} \subset S \cup E$$

as $\varepsilon \in (0,\varepsilon_0)$ where E is a set, the measure of which converges to zero. Thus

$$\lim_{\varepsilon\to 0}\int_{G_{3\sigma}}(1-\rho^2)(1-\overline{\rho}) = \lim_{\varepsilon\to 0}\int_{G_{3\sigma}}(1+\rho)(1-\rho)^2.$$

By (5.10),

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\int_{G_{3\sigma}}(1+\rho)^2(1-\rho)^2 \\ \le\ &\lim_{\varepsilon\to 0}\frac{2}{\varepsilon^n}\int_{G_{3\sigma}}(1-\overline{\rho})(1-\rho^2) = 0 \end{aligned}$$

This is our conclusion.

Theorem 5.4 *Assume $B(x,2\sigma) \subset G_\sigma$ satisfies*

$$\frac{1}{\varepsilon^n}\int_{B(x,\sigma)}(1-|u_\varepsilon|^2)^2 \to 0, \ \textit{as } \varepsilon \to 0, \tag{5.11}$$

then $|u_\varepsilon| \to 1$ in $C(B(x,\sigma),R)$.

Proof. Since $B(x,2\sigma) \subset G_\sigma$, there exists ε_0 sufficiently small so that $B(x,\sigma) \subset G^{2\delta\varepsilon_0}$. We always assume $\varepsilon < \varepsilon_0$. For $x_0 \in B(x,\sigma)$, set $\alpha = |u_\varepsilon(x_0)|$. Proposition 2.2 implies

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| < C\varepsilon^{-1}\tau\varepsilon, \quad \text{if } x \in B(x_0,\tau\varepsilon),$$

where $\tau = (1-\alpha)(NC)^{-1}, C$ is the constant in Proposition 2.2 and N is a large number such that $\tau < \delta$. Thus $B(x_0,\tau\varepsilon) \subset B(x,\sigma)$ and

$$|u_\varepsilon(x)| \le \alpha + C\tau, \quad \text{if } x \in B(x_0,\tau\varepsilon),$$

$$\int_{B(x_0,\tau\varepsilon)}(1-|u_\varepsilon(x)|^2)^2 \ge (1-1/N)^2(1-\alpha)^{n+2}\pi\varepsilon^n(NC)^{-n}.$$

Combining this with (5.11) we obtain $(1-\alpha)^{n+2} = o(1)$ as $\varepsilon \to 0$. Thus it is not difficult to complete the proof of Theorem.

6 The proof of Theorem 1.1

It is known that the singularities of u_n are in $\overline{G}$ from the discussion in §3. Since $\deg(g, \partial G) > 0$, we can see that the zeroes of u_ε are also in G . Moreover, the zeroes are contained in finite bad balls, i.e. $B(x_i^\varepsilon, h\varepsilon), i \in J$. As $\varepsilon \to 0, B(x_i^\varepsilon, h\varepsilon) \to a_j, i \in \Lambda_j$. This implies that the zeroes of u_ε distribute near these singularities of u_n as $\varepsilon \to 0$. Thus it is necessary to describe these singularities $\{a_j\}, j = 1, 2, \ldots, N_2$.

Proposition 6.1 $k_j = \deg(u_n, a_j)$.

Proof. Denote $\Omega' = G' \setminus \cup_{j=1}^{N_2} B(a_j, \sigma)$. Combining (4.6) and

$$\int_{G'\setminus G} |\nabla u_\varepsilon|^n = \int_{G'\setminus G} |\nabla \bar{g}|^n \leq C,$$

we have

$$\int_{\Omega'} |\nabla u_\varepsilon|^n \leq C + (n-1)^{n/2} |S^{n-1}| |d| \ln \sigma|, \tag{6.1}$$

where C is a constant which is independent of ε. For R in (4.13), from (6.1) we have

$$\int_{A_{R,\sigma}(a_j)} |\nabla u_\varepsilon|^n \leq C.$$

Then we know that there exists a constant $r \in (\sigma, R)$ such that

$$\int_{\partial B(a_j, r)} |\nabla u_\varepsilon|^n \leq C(r)$$

by using integral mean value theorem. Thus there exists a subsequence u_{ε_k} of u_ε such that

$$u_{\varepsilon_k} \to u_n, \quad \text{in } C(\partial B(a_j, r))$$

as $\varepsilon_k \to 0$, which implies

$$k_j = \deg(u_\varepsilon, \partial B(a_j, \sigma)) = \deg(u_n, a_j).$$

Proposition 6.2 $k_j = 0$ *or* $k_j = 1$.

Proof. From the regularity results on n-harmonic maps (see [3][5] or [9]), we know $u_n \in C^0(G_\sigma, \mathbb{R}^n)$. Set

$$w = \begin{cases} \bar{g} & \text{on } G' \setminus G, \\ u_n & \text{on } G_\sigma, \end{cases}$$

then $w \in \Pi$. Using Proposition 4.5 and (4.7) we have

$$\int_{\Omega'} |\nabla w|^n \geq (n-1)^{n/2} |S^{n-1}| (\sum_{j=1}^{N_2} k_j^{\frac{n}{n-1}}) \ln \frac{1}{\sigma} - C(R). \tag{6.2}$$

On the other hand, (6.1) and (4.9) imply

$$u_{\varepsilon_k} \xrightarrow{w} w, \quad \text{in } W^{1,n}(\Omega', \mathbb{R}^n).$$

Noting this and the weak lower semicontinuity of $\int_{\Omega'} |\nabla u|^n$, applying (6.1) we have

$$\int_{\Omega'} |\nabla w|^n \leq \underline{\lim}_{\varepsilon_k \to 0} \int_{\Omega'} |\nabla u_{\varepsilon_k}|^n \leq (n-1)^{n/2} |S^{n-1}| d \ln \frac{1}{\sigma} + C. \tag{6.3}$$

Combining this with (6.2), we obtain

$$(\sum_{j=1}^{N_2} k_j^{\frac{n}{n-1}} - d) \ln \frac{1}{\sigma} \leq C \quad \text{or} \quad \sum_{j=1}^{N_2} k_j^{\frac{n}{n-1}} \leq d = \sum_{j=1}^{N_2} k_j$$

for σ small enough. Thus $(k_j^{1/(n-1)} - 1) k_j \leq 0$ which implies that the Proposition holds.

Proposition 6.3 $k_j > 0$, $j = 1, 2, \ldots, N_2$.

Proof. Suppose $k_1 = 0$ and $k_2, k_3, \ldots, k_{N_2} > 0$. Similar to the proof of Theorem 4.3 we have

$$\int_{\cup_{j=2}^{N_2} \Omega_j} |\nabla u_\varepsilon|^n \geq (n-1)^{n/2} |S^{n-1}| d \ln \frac{\sigma}{\varepsilon} - C.$$

By this we can rewrite (4.6) as

$$\int_{G \setminus \cup_{j=2}^{N_2} B(a_j, \sigma)} |\nabla u_\varepsilon|^n + \frac{1}{4\varepsilon^n} \int_G (1 - |u_\varepsilon|^2)^2 \leq C(\sigma).$$

Thus similar to the proof of Theorem 5.3 we may modify (5.5) as

$$\frac{1}{\varepsilon^n} \int_{G \setminus \cup_{j=2}^{N_2} B(a_j, 3\sigma)} (1 - |u_\varepsilon|^2)^2 \to 0 \tag{6.4}$$

as $\varepsilon \to 0$. Noticing

$$G \cap B(a_1, \sigma) \subset G \cap B(a_1, R) \subset G \setminus \cup_{j=2}^{N_2} B(a_j, R) \subset G \setminus \cup_{j=2}^{N_2} B(a_j, 3\sigma)$$

we have

$$\frac{1}{\varepsilon^n} \int_{G \cap B(a_1, \sigma)} (1 - |u_\varepsilon|^2)^2 \to 0. \tag{6.5}$$

On the other hand, the definition of a_1 implies that there exists at least one bad ball $B(x_0^\varepsilon, h\varepsilon)$ such that

$$G \cap B(x_0^\varepsilon, h\varepsilon) \subset G \cap B(a_1, \sigma).$$

Applying the definition of bad ball we obtain

$$\frac{1}{\varepsilon^n} \int_{G \cap B(a_1, \sigma)} (1 - |u_\varepsilon|^2)^2 \geq \frac{1}{\varepsilon^n} \int_{G \cap B(x_0^\varepsilon, h\varepsilon)} (1 - |u_\varepsilon|^2)^2 \geq \mu > 0$$

which is contrary to (6.5). This contradiction shows $k_1 > 0$.

Remark We may conclude $k_j = 1, j = 1, 2, \dots, N_2$ from Proposition 6.2 and Proposition 6.3. Noticing $d = \sum_{j=1}^{N_2} k_j$, we obtain

$$N_2 = d, \quad 1 = k_j = \sum_{i\in\Lambda_j} d_i.$$

Thus on one hand, although the number of the singularities of $n-$ harmonic maps is indefinite (see Theorem A and Theorem C in [3]), we can say that for this $n-$ harmonic map u_n, the limit of the regularizable minimizer u_{ε_k} in $W^{1,n}$ as $k \to \infty$, the number of its singularities is just the degree d by applying (4.10). On the other hand, there exists at least one $i_0 \in \Lambda_j$ such that $d_{i_0} \neq 0$. Then we know that there exists at least one zero of u_ε in $B(x_{i_0}^\varepsilon, h\varepsilon)$ by using Kronecker's theorem.

Theorem 6.4 $a_j \in G, \quad j = 1, 2, \dots, d.$

Proof. Suppose $a_1 \in \partial G$, $a_2, a_3, \dots, a_d \in G$. Set

$$\Omega_\sigma = (G' \setminus B(a_1, R)) - \cup_{j=2}^d B(a_j, \sigma), \quad w = \begin{cases} u_n & \text{on } G_\sigma, \\ \bar{g} & \text{on } G' \setminus G. \end{cases}$$

Using Proposition 4.5 on Ω_σ we have

$$\int_{\Omega_\sigma} |\nabla w|^n \geq (n-1)^{n/2}|S^{n-1}|(d-1)\ln\frac{1}{\sigma} - C(R). \tag{6.6}$$

Taking $u = w, a = a_1$ in Proposition 4.4 we have

$$\int_{A_{R,\sigma}(a_1)} |\nabla w|^n \geq 2^{\frac{1}{n}}(n-1)^{n/2}|S^{n-1}|\ln\frac{1}{\sigma} - C.$$

Combining this with (6.6) yields

$$\int_{\Omega'} |\nabla w|^n \geq (d + 2^{\frac{1}{n}} - 1)(n-1)^{n/2}|S^{n-1}|\ln\frac{1}{\sigma} - C.$$

Compare this to (6.3) we obtain

$$(d + 2^{\frac{1}{n}} - 1 - d)\ln\frac{1}{\sigma} \leq C$$

where C is a constant which is independent of σ. It is impossible as σ small enough, so $a_1 \in G$.

7 The proof of Theorem 1.3

Theorem 7.1 *Let u_ε be the regularizable minimizer of $E_\varepsilon(u, G)$. Then there exists a subsequence u_{ε_k} of u_ε such that*

$$u_{\varepsilon_k} \to u_n, \quad \textit{in } W^{1,n}_{\rm loc}(G \setminus \cup_{j=1}^d \{a_j\}, \mathbb{R}^n).$$

Proof. Step 1: Suppose the ball $B(x_0, 2\sigma) \subset G \setminus \cup_{j=1}^d \{a_j\}$, where the constant σ may be sufficiently small but independent of ε. Since (4.6) implies

$$E_\varepsilon(u_\varepsilon, B(x_0, 2\sigma) \setminus B(x_0, \sigma)) \le C,$$

we know there is a constant $r \in (\sigma, 2\sigma)$ such that

$$\int_{\partial B(x_0,r)} |\nabla u_\varepsilon|^n + \frac{1}{\varepsilon^n} \int_{\partial B(x_0,r)} (1 - |u_\varepsilon|^2)^2 \le C(r), \tag{7.1}$$

by applying the integral mean value theorem. Thus, there exists a subsequence u_{ε_k} of u_ε such that

$$u_{\varepsilon_k} \to u_n, \quad \text{in } C(\partial B(x_0, r), \mathbb{R}^n),$$

which leads to

$$\frac{u_{\varepsilon_k}}{|u_{\varepsilon_k}|} \to u_n, \quad \text{in } C(\partial B(x_0, r), \mathbb{R}^n). \tag{7.2}$$

Step 2: Denote $\rho = |u_\varepsilon|$ on $B = B(x_0, r)$. It is not difficult to prove that the minimizer w of the problem

$$\min\{\int_B |\nabla u|^n : u \in W^{1,n}_{\frac{u_\varepsilon}{|u_\varepsilon|}}(B, S^{n-1})\} \tag{7.3}$$

exists. Noting u_ε be a minimizer of $E_\varepsilon(u, G)$, we have

$$E_\varepsilon(u_\varepsilon, B) \le \frac{1}{n} \int_B |\nabla(\rho w)|^n + \frac{1}{4\varepsilon^n} \int_B (1 - \rho^2)^2.$$

Obviously (4.6) and $|u_\varepsilon| \ge 1/2$ on B imply

$$\frac{1}{2^n} \int_B |\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^n \le \int_B |\nabla u_\varepsilon|^n \le C,$$

thus

$$\int_B |\nabla w|^n \le \int_B |\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^n \le C. \tag{7.4}$$

Applying this we may claim that

$$\int_B |\nabla u_\varepsilon|^n \le C\varepsilon^\lambda + \int_B |\nabla w|^n, \tag{7.5}$$

for some $\lambda > 0$. Its proof can be seen in §8.

Step 3: Let w^τ is a solution of

$$\min\{\int_B (|\nabla w|^2 + \tau)^{n/2} : w \in W^{1,n}_{\frac{u_\varepsilon}{|u_\varepsilon|}}(B, S^{n-1})\}, \quad \tau \in (0, 1). \tag{7.6}$$

It is easy to see that w^τ solves

$$-\operatorname{div}(v_\varepsilon^{(n-2)/2} \nabla w) = w|\nabla w|^2 v_\varepsilon^{(n-2)/2}, \quad v_\varepsilon = |\nabla w|^2 + \tau. \tag{7.7}$$

as $\tau \to 0$. Noticing $\frac{u_\varepsilon}{|u_\varepsilon|} \in W^{1,n}_{\frac{u_\varepsilon}{|u_\varepsilon|}}(B, S^{n-1})$ we have

$$\begin{aligned} \int_B |\nabla w^\tau|^n &\le \int_B (|\nabla w^\tau|^2 + \tau)^{n/2} \\ &\le \int_B (|\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^2 + \tau)^{n/2} \le \int_B (|\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^2 + 1)^{n/2} \le C \end{aligned} \tag{7.8}$$

by using (7.4), where C is a constant which is independent of ε, τ. Then there exist $w^* \in W^{1,n}_{\frac{u_\varepsilon}{|u_\varepsilon|}}(B, S^{n-1})$ and a subsequence of w^τ such that

$$w^\tau \xrightarrow{w} w^*, \quad \text{in } W^{1,n}(B, \mathbb{R}^n). \tag{7.9}$$

Noting the weak lower semicontinuity of $\int_B |\nabla w|^n$, we have

$$\begin{aligned} \int_B |\nabla w^*|^n &\le \underline{\lim}_{\tau\to 0} \int_B |\nabla w^\tau|^n \\ &\le \overline{\lim}_{\tau\to 0} \int_B |\nabla w^\tau|^n \le \overline{\lim}_{\tau\to 0} \int_B (|\nabla w^\tau|^2 + \tau)^{n/2}. \end{aligned} \tag{7.10}$$

The fact that w^τ solves (7.6) implies

$$\overline{\lim}_{\tau\to 0} \int_B (|\nabla w^\tau|^2 + \tau)^{n/2} \le \lim_{\tau\to 0} \int_B (|\nabla w_*|^2 + \tau)^{n/2} = \int_B |\nabla w_*|^n,$$

where w_* is a solution of (7.3). This and (7.10) lead to

$$\int_B |\nabla w^*|^n \le \underline{\lim}_{\tau\to 0} \int_B |\nabla w^\tau|^n \le \overline{\lim}_{\tau\to 0} \int_B |\nabla w^\tau|^n \le \int_B |\nabla w_*|^n. \tag{7.11}$$

Since $w^* \in W^{1,n}_{\frac{u_\varepsilon}{|u_\varepsilon|}}(B, S^{n-1})$, we know w^* also solves (7.3), namely

$$\int_B |\nabla w_*|^n = \int_B |\nabla w^*|^n.$$

Combining this with (7.11) yields

$$\lim_{\tau\to 0} \int_B |\nabla w^\tau|^n = \int_B |\nabla w^*|^n,$$

which and (7.9) imply

$$\nabla w^\tau \to \nabla w^*, \quad \text{in } L^n(B, \mathbb{R}^n). \tag{7.12}$$

Step 4: Similar to the discussion of Step 3, we may derive the following conclusion: Let u^τ be a solution of

$$\min\{\int_B (|\nabla u|^2 + \tau)^{n/2} : u \in W^{1,n}_{u_n}(B, S^{n-1})\}, \quad \tau \in (0,1). \tag{7.13}$$

Then u^τ satisfies

$$\int_B |\nabla u^\tau|^n \le C, \tag{7.14}$$

where C is which is independent of τ, and u^τ solves

$$-\operatorname{div}(v^{(n-2)/2}\nabla u) = u|\nabla u|^2 v^{(n-2)/2}, \quad v = |\nabla u|^2 + \tau. \tag{7.15}$$

As $\tau \to 0$, there exists a subsequence of u^τ denoted itself such that

$$\nabla u^\tau \to \nabla u^*, \quad \text{in } L^n(B, \mathbb{R}^n), \tag{7.16}$$

where u^* is a minimizer of $\int_B |\nabla u|^n$ in $W^{1,n}_{u_n}(B, S^{n-1})$. It is well-known that u^* is a map of the least n-energy, and also an n-harmonic map.

Fix $R > 2\sigma$ such that $B(x_0, R) \subset G \setminus \cup_{j=1}^d \{a_j\}$. Applying the regularity results on the map of the least n-energy (for example, Theorem 3.1 in [5]), we have

$$\sup_{B(x_0,r)} |\nabla u^*|^n \le \sup_{\overline{B(x_0,R)}} |\nabla u^*|^n := C_0. \tag{7.17}$$

It is obvious that C_0 is a constant which is independent of r.

Step 5: From (7.7) subtracts (7.15). Then

$$-\operatorname{div}(v_\varepsilon^{(n-2)/2}\nabla w - v^{(n-2)/2}\nabla u) = w|\nabla w|^2 v_\varepsilon^{(n-2)/2} - u|\nabla u|^2 v^{(n-2)/2}. \tag{7.18}$$

Multiplying both sides of (7.18) by $w - u$ and integrating over B we obtain

$$\begin{aligned}
&-\int_{\partial B} (v_\varepsilon^{(n-2)/2} w_\nu - v^{(n-2)/2} u_\nu)(w-u) \\
&+\int_B (v_\varepsilon^{(n-2)/2}\nabla w - v^{(n-2)/2}\nabla u)\nabla(w-u) \\
&= \int_B (w|\nabla w|^2 v_\varepsilon^{(n-2)/2} - u|\nabla u|^2 v^{(n-2)/2})(w-u),
\end{aligned}$$

where ν denotes the unit outside-norm vector of ∂B. Thus

$$\begin{aligned}
&|\int_B (v_\varepsilon^{(n-2)/2}\nabla w - v^{(n-2)/2}\nabla u)\nabla(w-u)| \\
&\le |\int_{\partial B} (v_\varepsilon^{(n-2)/2} w_\nu - v^{(n-2)/2} u_\nu)(w-u)| \qquad (7.19)\\
&\quad +|\int_B (w|\nabla u|^2 v^{(n-2)/2} - u|\nabla u|^2 v^{(n-2)/2})(w-u)| \\
&\quad +|\int_B (w|\nabla w|^2 v_\varepsilon^{(n-2)/2} - w|\nabla u|^2 v^{(n-2)/2})(w-u)| \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

First we give an estimate for I_1. Let $w = w^\tau$ is a solution of (7.6). Integrating both sides of (7.7) over B, we have

$$-\int_{\partial B} v_\varepsilon^{(n-2)/2} w_\nu = \int_B w|\nabla w|^2 v_\varepsilon^{(n-2)/2},$$

which and (7.8) imply

$$|\int_{\partial B} v_\varepsilon^{(n-2)/2} w_\nu| \le \int_B v_\varepsilon^{n/2} \le C. \tag{7.20}$$

An analogous discussion shows that for the solution $u = u^\tau$ of (7.13) which equips with (7.14), we may also obtain

$$|\int_{\partial B} v^{(n-2)/2} u_\nu| \le \int_B |\nabla u|^n \le C. \tag{7.21}$$

Applying (7.20)(7.21) we derive

$$\begin{aligned} I_1 &\le \sup_{\partial B} |w-u| (|\int_{\partial B} v_\varepsilon^{(n-2)/2} w_\nu| + |\int_{\partial B} v^{(n-2)/2} u_\nu|) \\ &\le C \sup_{\partial B} |w-u| = C \sup_{\partial B} |\frac{u_\varepsilon}{|u_\varepsilon|} - u_n|, \end{aligned} \tag{7.22}$$

where C is independent of ε, τ. For the estimate of I_3, we have

$$\begin{aligned} I_3 &\le \int_B |u-w| ||\nabla u|^2 v^{(n-2)/2} - |\nabla w|^2 v_\varepsilon^{(n-2)/2}| \\ &\le 2 \int_B ||\nabla u|^2 v^{(n-2)/2} - |\nabla w|^2 v_\varepsilon^{(n-2)/2}|. \end{aligned} \tag{7.23}$$

For estimating I_2, we multiply both sides of (7.15) by $(u-w)$ and integrate over B, then

$$\begin{aligned} &-\int_{\partial B} v^{(n-2)/2} u_\nu (u-w) + \int_B v^{(n-2)/2} \nabla u \nabla (u-w) \\ &= \int_B |\nabla u|^2 v^{(n-2)/2} u(u-w) = \int_B |\nabla u|^2 v^{(n-2)/2} (1-uw). \end{aligned}$$

Thus, we have

$$\begin{aligned} I_2 &\le \int_B |\nabla u|^2 v^{(n-2)/2} |u-w|^2 = 2\int_B |\nabla u|^2 v^{(n-2)/2} (1-uw) \\ &\le 2|\int_{\partial B} v^{(n-2)/2} u_\nu (u-w)| + 2|\int_B v^{(n-2)/2} \nabla u \nabla (u-w)|. \end{aligned}$$

Noting (7.21) we may derive

$$I_2 \le C \sup_{\partial B} |\frac{u_\varepsilon}{|u_\varepsilon|} - u_n| + 2|\int_B v^{(n-2)/2} \nabla u \nabla (u-w)|. \tag{7.24}$$

Step 6: Substituting (7.22)-(7.24) into (7.19) yields

$$\begin{aligned} &|\int_B (v_\varepsilon^{(n-2)/2} \nabla w - v^{(n-2)/2} \nabla u) \nabla (w-u)| \\ &\le C \sup_{\partial B} |\frac{u_\varepsilon}{|u_\varepsilon|} - u_n| + 2|\int_B v^{(n-2)/2} \nabla u \nabla (u-w)| \\ &\quad +2 \int_B |v_\varepsilon^{(n-2)/2} |\nabla w|^2 - v^{(n-2)/2} |\nabla u|^2|. \end{aligned}$$

Letting $\tau \to 0$ and applying (7.12)(7.16) we obtain

$$\begin{aligned}
&|\int_B (|\nabla w^*|^{(n-2)/2}\nabla w^* - |\nabla u^*|^{(n-2)/2}\nabla u^*)\nabla(w^* - u^*)| \\
\leq\ & C\sup_{\partial B}|\frac{u_\varepsilon}{|u_\varepsilon|} - u_n| + 2|\int_B |\nabla u^*|^{n-1}\nabla(u^* - w^*)| + 2\int_B ||\nabla w^*|^n - |\nabla u^*|^n|.
\end{aligned}$$

Using Lemma 1.2 in [4], we have

$$2^{n-1}\int_B |\nabla w^* - \nabla u^*|^n \leq |\int_B (|\nabla w^*|^{(n-2)/2}\nabla w^* - |\nabla u^*|^{(n-2)/2}\nabla u^*)\nabla(w^* - u^*)|.$$

Thus

$$(2^{n-1} - 2)\int_B |\nabla w^* - \nabla u^*|^n \leq C\sup_{\partial B}|\frac{u_\varepsilon}{|u_\varepsilon|} - u_n| + 2|\int_B |\nabla u^*|^{n-1}\nabla(u^* - w^*)|.$$

Denote $\psi(\varepsilon) = \int_B |\nabla w^* - \nabla u^*|^n$ and let $\varepsilon \to 0$, then

$$(2^{n-1} - 2)\psi(\varepsilon) \leq o(1) + 2(C_0|B|)^{(n-1)/n}(\psi(\varepsilon))^{1/n} \tag{7.25}$$

holds by using (7.2), where C_0 is the constant in (7.17).

We claim that for some small constant $\sigma > 0$, the following holds:

$$\psi(\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0. \tag{7.26}$$

Suppose (7.26) is not true, then there exists $\tau > 0$, for any $\varepsilon_0 > 0$, such that as $\varepsilon < \varepsilon_0$ we have $\psi(\varepsilon) \geq 2\tau > \tau$ or

$$(\psi(\varepsilon))^{(n-1)/n} > \tau^{(n-1)/n}, \quad \forall \varepsilon < \varepsilon_0. \tag{7.27}$$

Taking σ small enough so that

$$2(C_0|B(x_0, r)|)^{(n-1)/n} = (2^{n-2} - 1)\tau^{(n-1)/n},$$

we obtain from (7.25)

$$\begin{aligned}
&(\psi(\varepsilon))^{1/n}[(\psi(\varepsilon))^{(n-1)/n} - \frac{2(C_0|B|)^{(n-1)/n}}{2^{n-1} - 2}] \qquad (7.28)\\
=\ & (\psi(\varepsilon))^{1/n}[(\psi(\varepsilon))^{(n-1)/n} - \frac{1}{2}\tau^{(n-1)/n}] = o(1).
\end{aligned}$$

Substituting (7.27) into (7.28) we derive $(\psi(\varepsilon))^{1/n} = o(1)$, which is contrary to (7.27).

Step 7: Noting the weak lower semicontinuity of the functional $\int_B |\nabla u|^n$, from (4.9) we are led to

$$\int_B |\nabla u_n|^n \leq \underline{\lim}_{\varepsilon_k \to 0}\int_B |\nabla u_{\varepsilon_k}|^n.$$

Combining this with (7.5) and (7.26) we obtain

$$\begin{aligned}\int_B |\nabla u_n|^n \le \underline{\lim}_{\varepsilon_k\to 0}\int_B |\nabla u_{\varepsilon_k}|^n &\le \overline{\lim}_{\varepsilon_k\to 0}\int_B |\nabla u_{\varepsilon_k}|^n\\ &\le \lim_{\varepsilon_k\to 0}\int_B |\nabla w^*|^n = \int_B |\nabla u^*|^n.\end{aligned}$$

Recalling the definition of u^* in Step 4, and noticing $u_n \in W^{1,n}_{u_n}(B,S^{n-1})$, we know that u_n is also a minimizer of $\int_B |\nabla u|^n$ and

$$\lim_{\varepsilon_k\to 0}\int_B |\nabla u_{\varepsilon_k}|^n = \int_B |\nabla u_n|^n = \int_B |\nabla u^*|^n, \tag{7.29}$$

which and (4.9) imply

$$\nabla u_{\varepsilon_k} \to \nabla u_n, \quad \text{ in } L^n(B,\mathbb{R}^n).$$

Combining this with the fact

$$u_{\varepsilon_k} \to u_n, \quad \text{ in } L^n(B,\mathbb{R}^n),$$

which can be deduced from (4.6), we derive

$$u_{\varepsilon_k} \to u_n, \quad \text{ in } W^{1,n}(B,\mathbb{R}^n).$$

Then it is not difficult to complete the proof of this theorem.

8 The proof of (7.5)

To prove (7.5), we will introduce a comparison function first. Consider the functional

$$E(\rho,B) = \frac{1}{n}\int_B (|\nabla\rho|^2+1)^{n/2} + \frac{1}{2\varepsilon^n}\int_B (1-\rho)^2.$$

It is easy to prove that the minimizer ρ_1 of $E(\rho,B)$ on $W^{1,n}_{|u_\varepsilon|}(B,R^+)$ exists and satisfies

$$-div(v^{(n-2)/2}\nabla\rho) = \frac{1}{\varepsilon^n}(1-\rho) \quad on \quad B, \tag{8.2}$$

$$\rho|_{\partial B} = |u_\varepsilon|, \tag{8.3}$$

where $v = |\nabla\rho|^2+1$. Since $1/2 \le |u_\varepsilon| \le 1$ on B, it follows from the maximum principle that

$$1/2 \le |u_\varepsilon| \le \rho_1 \le 1 \tag{8.4}$$

on $\overline{B}$.

Applying (4.6) we see easily that

$$E(\rho_1,B) \le E(|u_\varepsilon|,B) \le CE_\varepsilon(u_\varepsilon,B) \le C. \tag{8.5}$$

Multiplying (8.2) by $(\nu\cdot\nabla\rho)$, where $\rho=\rho_1$, and integrating over B, we obtain

$$-\int_{\partial B} v^{(n-2)/2}(\nu\cdot\nabla\rho)^2+\int_B v^{(n-2)/2}\nabla\rho\cdot\nabla(\nu\cdot\nabla\rho)=\frac{1}{\varepsilon^n}\int_B(1-\rho)(\nu\cdot\nabla\rho), \tag{8.6}$$

where ν denotes the unit outside norm vector on ∂B. Using (8.5) we have

$$\begin{aligned}&|\textstyle\int_B v^{(n-2)/2}\nabla\rho\nabla(\nu\cdot\nabla\rho)|\le C\int_B v^{(n-2)/2}|\nabla\rho|^2+\frac12|\int_B v^{(n-2)/2}\nu\cdot\nabla v|\\ &\le C+\textstyle\frac1n|\int_B\nu\cdot\nabla(v^{n/2})|\le C+\frac1n\int_B|div(\nu v^{n/2})-v^{n/2}div\nu|\\ &C+\textstyle\frac1n\int_{\partial B}v^{n/2}.\end{aligned}\tag{8.7}$$

Combining (8.3)(7.1) and (8.5) we also have

$$\begin{aligned}&|\textstyle\frac{1}{\varepsilon^n}\int_B(1-\rho)(\nu\cdot\nabla\rho)|\le\frac{1}{2\varepsilon^n}|\int_B(1-\rho)^2div\nu-\int_{\partial B}(1-\rho)^2|\\ &\le\textstyle\frac{1}{2\varepsilon^n}\int_B(1-\rho)^2|div\nu|+\frac{1}{2\varepsilon^n}\int_{\partial B}(1-\rho)^2\le C.\end{aligned}$$

Substituting this and (8.7) into (8.6) yields

$$|\int_{\partial B}v^{(n-2)/2}(\nu\cdot\nabla\rho)^2|\le C+\frac1n\int_{\partial B}v^{n/2}. \tag{8.8}$$

Applying (8.3)(7.1) and (8.8), we obtain for any $\delta\in(0,1)$,

$$\begin{aligned}&\textstyle\int_{\partial B}v^{n/2}=\int_{\partial B}v^{(n-2)/2}[1+(\tau\cdot\nabla\rho)^2+(\nu\cdot\nabla\rho)^2]\\ &=\textstyle\int_{\partial B}v^{(n-2)/2}[1+(\tau\cdot\nabla|u_\varepsilon|)^2+(\nu\cdot\nabla\rho)^2]\\ &\le\textstyle\int_{\partial B}v^{(n-2)/2}+\int_{\partial B}v^{(n-2)/2}(\nu\cdot\nabla\rho)^2\\ &\quad+\textstyle(\int_{\partial B}v^{n-2})^{(n-2)/n}(\int_{\partial B}(\tau\cdot\nabla|u_\varepsilon|)^n)^{2/n}\\ &\le C(\delta)+(\textstyle\frac1n+2\delta)\int_{\partial B}v^{n/2},\end{aligned}$$

where τ denotes the unit tangent vector on ∂B. Hence it follows by choosing $\delta>0$ so small that

$$\int_{\partial B}v^{n/2}\le C. \tag{8.9}$$

Now we multiply both sides of (8.2) by $(1-\rho)$ and integrate over B. Then

$$\int_B v^{(n-2)/2}|\nabla\rho|^2+\frac{1}{\varepsilon^n}\int_B(1-\rho)^2=-\int_{\partial B}v^{(n-2)/2}(\nu\cdot\nabla\rho)(1-\rho).$$

From this, using (7.1)(8.3)(8.4) and (8.9) we obtain

$$\begin{aligned}&E(\rho_1,B)\le C|(\nu\cdot\nabla\rho)(1-\rho)|\\ &\le C|\textstyle\int_{\partial B}v^{n/2}|^{(n-1)/n}|\int_{\partial B}(1-\rho)^2|^{1/n}\\ &\le C|\textstyle\int_{\partial B}(1-|u_\varepsilon|)^2|^{1/n}\le C\varepsilon\end{aligned}\tag{8.10}$$

Since u_ε is a minimizer of $E_\varepsilon(u,B)$, we have

$$\begin{aligned} &E_\varepsilon(u_\varepsilon,B)\le E_\varepsilon(\rho_1 w,B)\\ &=\tfrac{1}{n}\textstyle\int_B(|\nabla\rho_1|^2+\rho_1^2|\nabla w|^2)^{n/2}+\tfrac{1}{4\varepsilon^n}\textstyle\int_B(1-\rho_1^2)^2,\end{aligned}\tag{8.11}$$

where w is a solution of (7.3). On on hand,

$$\begin{aligned} &\textstyle\int_B(|\nabla\rho_1|^2+\rho_1^2|\nabla w|^2)^{n/2}dx-\textstyle\int_B(\rho_1^2|\nabla w|^2)^{n/2}dx\\ &=\tfrac{n}{2}\textstyle\int_B\int_0^1[(|\nabla\rho_1|^2+\rho_1^2|\nabla w|^2)^{(n-2)/2}s+(\rho_1^2|\nabla w|^2)^{(n-2)/2}(1-s)]ds|\nabla\rho_1|^2dx\\ &\le C\textstyle\int_B(|\nabla\rho_1|^n+|\nabla\rho_1|^2|\nabla w|^{(n-2)/2})dx.\end{aligned}\tag{8.12}$$

On the other hand, by using (8.10) and (7.4) we have

$$\int_B|\nabla\rho_1|^2|\nabla w|^{(n-2)/2}\le(\int_B|\nabla\rho_1|^{4n/(n+2)})^{(n+2)/2n}(\int_B|\nabla w|^n)^{(n-2)/2n}\le C\varepsilon^\lambda.\tag{8.13}$$

Combining (8.11)-(8.13), we can derive

$$E_\varepsilon(u_\varepsilon,B)\le\frac{1}{n}\int_B\rho_1^n|\nabla w|^n+C\varepsilon^\lambda,$$

where λ is a constant only depending on n. Thus (7.5) can be seen by (8.4).

References

[1] F. Bethuel, H. Brezis, F. Helein: *Ginzburg-Landau Vortices,* Birkhauser. 1994.

[2] H. Brezis, J. Coron, E. Leib: *Harmonic maps with defects,* Comm. Math. Phys. ,**107** (1996).649-705.

[3] B. Chen, R.Hardt: *Prescribing singularities for p-harmonic maps,* Ind. Univ. Math. J. ,**44** (1995).575-602.

[4] Y. Chen, M. Hong, N. Hungerbuhler: *Heat flow of p-harmonic maps with values into spheres,* Math. Z., **215** (1994), 25-35.

[5] R. Hardt, F. Lin: *Mappings minimizing the L^p norm of the gradient,* Comm. Pure Appl. Math., **40** (1987), 555-588.

[6] M. Hong: *Asymptotic behavior for minimizers of a Ginzburg-Landau type functional in higher dimensions associated with n-harmonic maps,* Adv. in Diff. Eqns., **1** (1996), 611-634.

[7] Y. Lei, Z. Wu, H. Yuan: *Radial minimizers of a Ginzburg-Landau type functional*, Electron. J. Diff Eqns., **1999** (1999), No. 30, 1-21.

[8] M. Struwe: *On the asymptotic behaviour of minimizers of the Ginzburg-Landau model in 2 dimensions,* Diff. and Inter. Eqns., **7** (1994), 1613-1624.

[9] P. Tolksdorf: *Everywhere regularity for some quasilinear systems with a lack of ellipticity,* Anna. Math. Pura. Appl., **134** (1983), No. 30, 241-266.

Yutian Lei
Mathematics Department, SuZhou University
1 Shizi street
Suzhou, Jiangsu, 215006
P. R. China
e-mail: lythxl@163.com