Asymptotic behavior of regularizable minimizers of a Ginzburg-Landau functional in higher dimensions *

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Abstract

We study the asymptotic behavior of the regularizable minimizers of a Ginzburg-Landau type functional. We also discuss the location of the zeroes of the minimizers.

1 Introduction

Let $G \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded and simply connected domain with smooth boundary $\partial G$. Let $g$ be a smooth map from $\partial G$ into $S^{n-1}$ satisfying $d = \text{deg}(g, \partial G) \neq 0$. Consider the Ginzburg-Landau-type functional

$$E_{\varepsilon}(u,G) = \frac{1}{p} \int_G |\nabla u|^p + \frac{1}{4\varepsilon^p} \int_G (1 - |u|^2)^2, \quad (p > 1)$$

with a small parameter $\varepsilon > 0$. It is known that this functional achieves its minimum on

$$W_p = \{v \in W^{1,p}(G, \mathbb{R}^n) : v|_{\partial G} = g\}$$

at a function $u_\varepsilon$. We are concerned with the asymptotic behavior of $u_\varepsilon$ and the location of the zeroes of $u_\varepsilon$ as $\varepsilon \to 0$.

The functional $E_{\varepsilon}(u,G)$ was introduced in the study of the Ginzburg-Landau vortices by F. Bethue, H. Brezis and F. Helein [1] in the case $p = n = 2$. Similar models are also used in many other theories of phase transition. The minimizer $u_\varepsilon$ of $E_{\varepsilon}(u,G)$ represents a complex order parameter. The zeroes of $u_\varepsilon$ and the module $|u_\varepsilon|$ both have physics senses, for example, in superconductivity $|u_\varepsilon|^2$ is proportional to the density of superconducting electrons, and the zeroes of $u_\varepsilon$ are the vortices, which were introduced in the type-II superconductors.

In the case $1 < p < n$, it is easily seen that $W^{1,p}_g(G, S^{n-1}) \neq \emptyset$. It is not difficult to prove that the existence of solution $u_p$ for the minimization problem

$$\min\{\int_G |\nabla u|^p : u \in W^{1,p}_g(G, S^{n-1})\}$$

*Mathematics Subject Classifications: 35J70.
Key words: Ginzburg-Landau functional, module and zeroes of regularizable minimizers.

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Asymptotic behavior of regularizable minimizers

by taking the minimizing sequence. This solution is called a map of the least
\( p \)-energy with boundary value \( g \). Using the variational methods, we can proved
that the solution \( u_p \) is also \( p \)-harmonic map on \( G \) with the boundary data \( g \),
namely, it is a weak solution of the following equation

\[- \text{div}(|\nabla u|^{p-2}\nabla u) = u|\nabla u|^p.\]

As \( \varepsilon \to 0 \), there exists a subsequence \( u_{\varepsilon_k} \) of \( u_\varepsilon \), the minimizer of \( E_\varepsilon(u,G) \), such that

\[ u_{\varepsilon_k} \rightharpoonup u_p, \quad \text{in } W^{1,p}(G,\mathbb{R}^n). \]

In the case \( p > n \), \( W^{1,p}(G,S^{n-1}) = \emptyset \). Thus there is no map of least
\( p \)-energy on \( G \) with the boundary value \( g \). It seems to be very difficult to
study the convergence for minimizers of \( E_\varepsilon(u,G) \) in \( W^p \). Some results on the
asymptotic behavior of the radial minimizers of \( E_\varepsilon(u,G) \) were presented in [7].

When \( p = n \), this problem was introduced in [1] (the open problem 17). M.
C. Hong studied the asymptotic behavior for the regularizable minimizers of
\( E_\varepsilon(u,G) \) in \( W^n \) [6]. He proved that there exist \( \{a_1, a_2, \ldots, a_J\} \subset G, J \in \mathbb{N} \)
and a subsequence \( u_{\varepsilon_k} \) of the regularizable minimizers \( u_\varepsilon \) such that

\[ u_{\varepsilon_k} \overset{w}{\rightharpoonup} u_n, \quad \text{in } W^{1,n}\text{loc}(G \setminus \{a_1, a_2, \ldots, a_J\},\mathbb{R}^n) \quad (1.1) \]
as \( \varepsilon_k \to 0 \), where \( u_n \) is an \( n \)-harmonic map.

In this paper we shall discuss the asymptotic behavior for the regularizable
minimizers of \( E_\varepsilon(u,G) \) on \( W_n \) in the case \( p = n \). Without loss of generality,
we may assume \( d > 0 \). Recalling a minimizer of \( E_\varepsilon(u,G) \) on \( W_n \) be called the
regularizable minimizer, if it is the limit of the minimizer of the regularized
functional

\[ E_\varepsilon^\tau(u,G) = \frac{1}{p} \int_G (|\nabla u|^2 + \tau)^{p/2} + \frac{1}{4\varepsilon^p} \int_G (1 - |u|^2)^2, \quad (\tau \in (0,1)) \]
on \( W_n \) in \( W^{1,p} \). It is not difficult to prove that the regularizable minimizer is
also a minimizer of \( E_\varepsilon(u,G) \). In order to find the zeroes of the minimizers, we
should first locate the singularities of the \( n \)-harmonic map \( u_n \).

**Theorem 1.1** If \( a_j \in G, j = 1,2,\ldots,J \) are the singularities of \( n \)-harmonic
map \( u_n \), then \( J = d \), the degree \( \text{deg}(u_n,a_j) = 1 \), and \( \{a_j\}^d_{j=1} \subset G \). Moreover,
for every \( j \), there exists at least one zero of the regularizable minimizer \( u_\varepsilon \) near
to \( a_j \).

Because the module of the minimizer has the physics sense, we have also
studied its asymptotic behavior.

**Theorem 1.2** Let \( u_\varepsilon \) be a regularizable minimizer of \( E_\varepsilon(u,G) \), \( \rho = |u_\varepsilon| \), then there exists a constant \( C \) independent of \( \varepsilon \) such that

\[ \int_G |\nabla \rho|^n \leq C, \text{ and } \frac{1}{\varepsilon^n} \int_G (1 - \rho^2) \leq C(1 + |\ln \varepsilon|). \]
For any given \( \eta > 0 \), denote \( G_\eta = G \setminus \cup_{j=1}^d B(a_j, \eta) \), then as \( \varepsilon \to 0 \),

\[
\frac{1}{\varepsilon^n} \int_{G_\eta} (1 - \rho^2)^2 \to 0,
\]

\( \rho \to 1, \quad \text{in } C_{\text{loc}}(G_\eta, \mathbb{R}) \).

At last, we develop the conclusion of (1.1) into following

\begin{align*}
\text{Theorem 1.3} & \quad \text{There exists a subsequence } u_{\varepsilon k} \text{ of } u_{\varepsilon} \text{ such that as } \varepsilon \to 0, \\
& \quad u_{\varepsilon k} \to u_n, \quad \text{in } W^{1,n}_{\text{loc}}(G \setminus \cup_{j=1}^d \{a_j\}, \mathbb{R}^n).
\end{align*}

We shall prove Theorems 1.2 and 1.3 in \( \S 5 \) and \( \S 7 \) respectively, and the proof of Theorem 1.1 will be given in \( \S 6 \).

2 Basic properties of the regularizable minimizers

First we recall the minimizer of the regularized functional

\[
E_\varepsilon(u, G) = \frac{1}{n} \int_G (|\nabla u|^2 + \tau/n)^{n/2} + \frac{1}{4\varepsilon^n} \int_G (1 - |u|^2)^2, \quad \tau \in (0, 1)
\]
on \( W_n \), denoted by \( u^\varepsilon \). As \( \tau \to 0 \), there exists a subsequence \( u_{\varepsilon k}^\tau \) of \( u^\varepsilon \) such that

\[
\lim_{\tau_k \to 0} u_{\varepsilon k}^\tau = u_\varepsilon, \quad \text{in } W^{1,n}(G, \mathbb{R}^n),
\]

and the limit \( u_\varepsilon \) is one minimizer of \( E_\varepsilon(u, G) \) on \( W_n \), which is named the regularizable minimizer. It is not difficult to prove that \( u_\varepsilon \) solves the problem

\[
- \text{div}(|\nabla u|^2 + \tau)^{(n-2)/2} \nabla u = \frac{1}{\varepsilon^n} u(1 - |u|^2), \quad \text{on } G,
\]

\[
u_{\partial G} = g
\]

and satisfies the maximum principle: \( |u_\varepsilon^\tau| \leq 1 \) a.e. on \( G \). Moreover

\begin{prop} (Theorem 2.2 in [6]) \end{prop}

For any \( \delta > 0 \), there exists a constant \( C \) independent of \( \varepsilon \) such that

\[
\lim_{\tau_k \to 0} |\nabla u_{\varepsilon k}^\tau| \leq C\varepsilon^{-1}, \quad \text{on } G^{\delta \varepsilon},
\]

where \( G^{\delta \varepsilon} = \{ x \in G : \text{dist}(x, \partial G) \geq \delta \varepsilon \} \).

In this section we shall present some basic properties of the regularizable minimizer \( u_\varepsilon \). Clearly it is a weak solution of the equation

\[
- \text{div}(|\nabla u|^{n-2} \nabla u) = \frac{1}{\varepsilon^n} u(1 - |u|^2), \quad \text{on } G,
\]

and it is known that \( |u_\varepsilon| \leq 1 \) a.e. on \( G \) [6]. We also have

\begin{prop} \end{prop}

For any \( \delta > 0 \), there exists a constant \( C \) independent of \( \varepsilon \) such that

\[
\|\nabla u_\varepsilon\|_{L^\infty(B(x, \delta \varepsilon/8, \mathbb{R}^n)} \leq C\varepsilon^{-1}, \quad \text{if } x \in G^{\delta \varepsilon}.
\]
Proof. Let \( y = x\varepsilon^{-1} \) in (2.4) and denote \( v(y) = u(x), G_\varepsilon = \{ y = x\varepsilon^{-1} : x \in G \}, \ G^\delta = \{ y \in G_\varepsilon : \text{dist}(y, \partial G_\varepsilon) > \delta \}. \) Since that \( u \) is a weak solution of (2.4), we have

\[
\int_{G_\varepsilon} |\nabla v|^{n-2} \nabla v \nabla \phi = \int_{G_\varepsilon} v(1 - |v|^2) \phi, \quad \phi \in W^{1,n}_0(G_\varepsilon, \mathbb{R}^n).
\]

Taking \( \phi = v \zeta^n, \zeta \in C^\infty_0(G_\varepsilon, \mathbb{R}), \) we obtain

\[
\int_{G_\varepsilon} |\nabla v|^n \zeta^n \leq n \int_{G_\varepsilon} |\nabla v|^{n-1} |\nabla \zeta| |v| + \int_{G_\varepsilon} |v|^2 (1 - |v|^2) \zeta^n.
\]

Setting \( y \in G_\delta, B(y, \delta/2) \subset G_\varepsilon, \) and \( \zeta = 1 \) in \( B(y, \delta/4), \zeta = 0 \) in \( G_\varepsilon \setminus B(y, \delta/2), |\nabla \zeta| \leq C(\delta), \) we have

\[
\int_{B(y, \delta/2)} |\nabla v|^n \zeta^n \leq C(\delta) \int_{B(y, \delta/2)} |\nabla v|^{n-1} \zeta^{n-1} + C(\delta).
\]

Using Holder inequality we can derive \( \int_{B(y, \delta/4)} |\nabla v|^n \leq C(\delta). \) Combining this with the theorem of [9] yields

\[
\|\nabla v\|_{L^n(B(y, \delta/8))} \leq C(\delta) \int_{B(y, \delta/4)} (1 + |\nabla v|^n) \leq C(\delta)
\]

which implies

\[
\|\nabla u\|_{L^n(B(x, \delta/8))} \leq C(\delta) \varepsilon^{-1}.
\]

**Proposition 2.3 (Lemma 2.1 in [6])** There exists a constant \( C \) independent of \( \varepsilon \) such that for \( \varepsilon \in (0, 1), \)

\[
E_\varepsilon(u_\varepsilon, G) \leq d^{(n-1)n/2} |S^{n-1}| |\ln \varepsilon| + C. \quad (2.5)
\]

**Proposition 2.4** There exists a constant \( C \) independent of \( \varepsilon \) such that

\[
\frac{1}{\varepsilon^n} \int_G (1 - |u_\varepsilon|^2)^2 \leq C. \quad (2.6)
\]

**Proof.** By (3.6) in [6],

\[
\int_G |\nabla u_\varepsilon|^n \geq d^{(n-1)n/2} |S^{n-1}| |\ln \varepsilon| - C.
\]

Applying Proposition 2.3 we may obtain (2.6).
3 A class of bad balls

Fix $\rho > 0$. For the regularizable minimizer $u_\varepsilon$, from Theorem 2.2 in [6] we know

$$|u_\varepsilon| \geq \frac{1}{2}, \quad \text{on } G \setminus G^{\rho_\varepsilon},$$

(3.1)

where $G^{\rho_\varepsilon} = \{x \in G : \text{dist}(x, \partial G) \geq \rho_\varepsilon\}$. Thus there exists no zero of $u_\varepsilon$ on $G \setminus G^{\rho_\varepsilon}$.

**Proposition 3.1** Let $u_\varepsilon$ be a regularizable minimizer of $E_\varepsilon(u, G)$. There exist positive constants $\lambda, \mu$ which are independent of $\varepsilon \in (0, 1)$ such that if

$$\frac{1}{\varepsilon^n} \int_{G^{\rho_\varepsilon} \cap B^{2l_\varepsilon}} (1 - |u_\varepsilon|^2)^2 \leq \mu,$$

(3.2)

where $B^{2l_\varepsilon}$ is some ball of radius $2l_\varepsilon$ with $l \geq \lambda$, then

$$|u_\varepsilon| \geq \frac{1}{2}, \quad \forall x \in G^{\rho_\varepsilon} \cap B^{l_\varepsilon}.$$  

(3.3)

**Proof.** First it is known that there exists a constant $\beta > 0$ such that for any $x \in G^{\rho_\varepsilon}$ and $0 < r \leq 1$,

$$|G^{\rho_\varepsilon} \cap B(x, r)| \geq \beta r^n.$$

Next we take

$$\lambda = \min\left(\frac{1}{4C}, \frac{1}{8\rho}\right), \quad \mu = \frac{\beta \lambda^n}{16},$$

where $C$ is the constant in Proposition 2.2.

Suppose that there is a point $x_0 \in G^{\rho_\varepsilon} \cap B^{l_\varepsilon}$ such that $|u_\varepsilon(x_0)| < 1/2$, then applying Proposition 2.2 we have

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C \varepsilon^{-1} |x - x_0| = \frac{1}{4}, \quad x \in B(x_0, \lambda \varepsilon) \cap G^{\rho_\varepsilon}.$$  

Hence

$$(1 - |u_\varepsilon(x)|^2)^2 > \frac{1}{16}, \quad \forall x \in B(x_0, \lambda \varepsilon) \cap G^{\rho_\varepsilon},$$

$$\int_{B(x_0, \lambda \varepsilon) \cap G^{\rho_\varepsilon}} (1 - |u_\varepsilon|^2)^2 > \frac{1}{16} |G^{\rho_\varepsilon} \cap B(x_0, \lambda \varepsilon)| \geq \beta \frac{1}{16} (\lambda \varepsilon)^n = \mu \varepsilon^n.$$  

(3.4)

Since $x_0 \in B^{l_\varepsilon} \cap G^{\rho_\varepsilon}$, we have $(B(x_0, \lambda \varepsilon) \cap G^{\rho_\varepsilon}) \subset (B^{2l_\varepsilon} \cap G^{\rho_\varepsilon})$, thus (3.4) implies

$$\int_{B^{2l_\varepsilon} \cap G^{\rho_\varepsilon}} (1 - |u_\varepsilon|^2)^2 > \mu \varepsilon^n$$

which contradicts (3.2) and thus the proposition is proved.

To find the zeroes of the regularizable minimizer $u_\varepsilon$ based on Proposition 3.1, we may take (3.2) as the ruler to distinguish the ball of radius $\lambda \varepsilon$ which contain the zeroes.
Let $\lambda, \mu$ be constants in Proposition 3.1. If
\[
\frac{1}{\varepsilon^n} \int_{G^{\varepsilon} \cap B(x^\varepsilon, 2\lambda \varepsilon)} (1 - |u^\varepsilon|^2)^2 \leq \mu,
\]
then $B(x^\varepsilon, \lambda \varepsilon)$ is called good ball. Otherwise $B(x^\varepsilon, \lambda \varepsilon)$ is called bad ball. From Proposition 3.1 we are led to
\[
|u^\varepsilon| \geq \frac{1}{2}, \quad \text{on } G^{\varepsilon} \setminus \bigcup_{x^\varepsilon \in \Lambda} B(x^\varepsilon, \lambda \varepsilon), \quad (3.5)
\]
where $\Lambda$ is the set of the centres of all bad balls. (3.5) and (3.1) imply that the zeroes of $u^\varepsilon$ are contained in these bad balls.

Now suppose that $\{B(x^\varepsilon_i, \lambda \varepsilon), i \in I\}$ is a family of balls satisfying
\begin{enumerate}[(i)]
  \item $x^\varepsilon_i \in G^{\varepsilon}, i \in I$
  \item $G^{\varepsilon} \subset \bigcup_{i \in I} B(x^\varepsilon_i, \lambda \varepsilon)$
  \item $B(x^\varepsilon_i, \lambda \varepsilon/4) \cap B(x^\varepsilon_j, \lambda \varepsilon/4) = \emptyset, i \neq j.$ \quad (3.6)
\end{enumerate}
Let $J^\varepsilon = \{i \in I : B(x^\varepsilon_i, \lambda \varepsilon) \text{ is a bad ball}\}$.

**Proposition 3.2** There exists a positive integer $N$ which is independent of $\varepsilon$ such that the number of bad balls $\text{card} \ J^\varepsilon \leq N$.

**Proof.** Since (3.6) implies that every point in $G^{\varepsilon}$ can be covered by finite, say $m$ (independent of $\varepsilon$) balls, from (2.6) and the definition of bad balls, we have
\[
\mu \varepsilon^n \text{card} \ J^\varepsilon \leq \sum_{i \in J^\varepsilon} \int_{B(x^\varepsilon_i, 2\lambda \varepsilon) \cap G^{\varepsilon}} (1 - |u^\varepsilon|^2)^2 \leq m \int_{\bigcup_{i \in J^\varepsilon} B(x^\varepsilon_i, 2\lambda \varepsilon) \cap G^{\varepsilon}} (1 - |u^\varepsilon|^2)^2 \leq mC \varepsilon^n
\]
and hence $\text{card} \ J^\varepsilon \leq \frac{mC \varepsilon^n}{\mu} \leq N$.

Similar to the argument of Theorem IV.1 in [1], we have

**Proposition 3.3** There exist a subset $J \subset J^\varepsilon$ and a constant $h \geq \lambda$ such that
\[
\bigcup_{i \in J} B(x^\varepsilon_i, \lambda \varepsilon) \subset \bigcup_{i \in J} B(x^\varepsilon_i, h \varepsilon),
\]
\[
|x^\varepsilon_i - x^\varepsilon_j| > 8h \varepsilon, \quad i, j \in J, \quad i \neq j. \quad (3.7)
\]
Proof. If there are two points \( x_1, x_2 \) such that (3.7) is not true with \( h = \lambda \), we take \( h_1 = 9\lambda \) and \( J_1 = J_\varepsilon \setminus \{1\} \). In this case, if (3.7) holds we are done. Otherwise we continue to choose a pair points \( x_3, x_4 \) which does not satisfy (3.7) and take \( h_2 = 9h_1 \) and \( J_2 = J_\varepsilon \setminus \{1, 3\} \). After at most \( N \) steps we may conclude this proposition.

Applying Proposition 3.3 we may modify the family of bad balls such that the new one, denoted by \( \{B(x_i^\varepsilon, h\varepsilon) : i \in J\} \), satisfies

\[
\bigcup_{i \in J} B(x_i^\varepsilon, \lambda\varepsilon) \subset \bigcup_{i \in J} B(x_i^\varepsilon, h\varepsilon),
\]

\[
\lambda \leq h; \quad \text{card } J \leq \text{card } J_\varepsilon,
\]

\[
|x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon, i, j \in J, i \neq j.
\]

The last condition implies that every two balls in the new family do not intersect.

As \( \varepsilon \to 0 \), there exist a subsequence \( x_i^{\varepsilon_k} \) of \( x_i^\varepsilon \) and \( a_i \in \overline{G} \) such that

\[
x_i^{\varepsilon_k} \to a_i, \quad i = 1, 2, \ldots, N_1 = \text{card } J.
\]

Perhaps there may be at least two subsequences converge to the same point, we denote by

\[
a_1, a_2, \ldots, a_{N_2}, \quad N_2 \leq N_1
\]

the collection of distinct points in \( \{a_i\}_{i=1}^{N_1} \).

To prove \( a_j \in \partial G \), it is convenient to enlarge a little \( G \). Assume \( G' \subset \mathbb{R}^n \) is a bounded, simply connected domain with smooth boundary such that \( \overline{G} \subset G' \), and take a smooth map \( \tilde{g} : (G' \setminus G) \to S^{n-1} \) such that \( \tilde{g} = g \) on \( \partial G \). We extend the definition domain of every element in \( \{u : G \to \mathbb{R}^n : u|_{\partial G} = g\} \) to \( G' \) such that \( u = \tilde{g} \) on \( G' \setminus G \). In particular, the regularizable minimizer \( u_\varepsilon \) can be defined on \( G' \).

Fix a small constant \( \sigma > 0 \) such that

\[
\overline{B(a_j, \sigma)} \subset G', \quad j = 1, 2, \ldots, N_2;
\]

\[
4\sigma < |a_j - a_i|, \quad i \neq j; \quad 4\sigma < \text{dist}(G, \partial G').
\]

Writing \( A_j = \{i \in J : x_i^{\varepsilon_k} \to a_j, j = 1, 2, \ldots, N_2 \}, \) we have

\[
\bigcup_{i \in A_j} B(x_i^{\varepsilon_k}, h\varepsilon_k) \subset B(a_j, \sigma), \quad j = 1, 2, \ldots, N_2
\]

\[
\bigcup_{j \in J} B(x_j^{\varepsilon_k}, h\varepsilon_k) \subset \bigcup_{j=1}^{N_2} B(a_j, \sigma/4)
\]

\[
B(x_j^{\varepsilon_k}, h\varepsilon_k) \cap B(x_j^{\varepsilon_k}, h\varepsilon_k) = \emptyset, \quad i, j \in J, i \neq j
\]

as long as \( \varepsilon_k \) is small enough. Let \( u_\varepsilon \) is the regularizable minimizer of \( E_\varepsilon(u, G) \) and denote \( d_k^i = \text{deg}(u_\varepsilon, \partial B(x_i^{\varepsilon_k}, h\varepsilon_k)), l_k^i = \text{deg}(u_\varepsilon, \partial B(a_j, \sigma)) \), thus

\[
l_k^i = \sum_{\varepsilon \in A_j} d_k^i, \quad d = \sum_{j=1}^{N_2} l_k^j.
\]

To prove that the degrees \( d_k^i \) and \( l_k^i \) are independent of \( \varepsilon_k \), we recall a proposition stated in [6] (Lemma 3.3) or [2] (Theorem 8.2).
Proposition 3.4 Let $\phi : S^{n-1} \to S^{n-1}$ be a $C^0$-map with $\deg \phi = d$. Then
\[
\int_{S^{n-1}} |\nabla \phi|^{n-1} \, dx \geq |d|(n-1)^{(n-1)/2}|S^{n-1}|.
\]

Proposition 3.5 There exists a constant $C$ which is independent of $\varepsilon_k$ such that
\[
|d_k| \leq C, \quad i \in J; \quad |l_j| \leq C, \quad j = 1, 2, \ldots, N_2.
\]

Proof. Since $u = u\varepsilon$ is a weak solution of (2.4), applying the theory of the local regularity in [9], we know $u \in C(\partial B(x_1^{\varepsilon_k}, h\varepsilon_k))$. Since (3.5) implies $|u| \geq 1/2$ on $\partial B(x_1^{\varepsilon_k}, h\varepsilon_k)$, thus $\phi = \frac{n}{|u|} \in C(\partial B(x_1^{\varepsilon_k}, h\varepsilon_k), S^{n-1})$. From Proposition 3.4, we have
\[
|d_k| \leq |S^{n-1}|^{-1} (n-1)^{(1-n)/2} \int_{\partial B(x_1^{\varepsilon_k}, h\varepsilon_k)} |\frac{u}{|u|}| r^{n-1}.
\]

Since $|u| \geq \frac{1}{2}$ on $G' \cap G^\varepsilon$, there is no zero of $u\varepsilon$ in it. Thus
\[
\deg(u\varepsilon_k, \partial B(x_1^{\varepsilon_k}, h\varepsilon_k)) = \deg(u\varepsilon_k, \partial(B(x_1^{\varepsilon_k}, h\varepsilon_k) \cap G^\varepsilon))
\]
and
\[
|d_k| \leq |S^{n-1}|^{-1} (n-1)^{(1-n)/2} \int_{\partial(B(x_1^{\varepsilon_k}, h\varepsilon_k) \cap G^\varepsilon)} |\frac{u}{|u|}| r^{n-1}. \tag{3.10}
\]

Substituting (2.3) and the fact $|u\varepsilon_k| \geq \frac{1}{2}$ on $\partial(B(x_1^{\varepsilon_k}, h\varepsilon_k) \cap G^\varepsilon)$ into (3.10), we obtain
\[
|d_k| \leq C \varepsilon_k^{-n} |S^{n-1}|^{-1} (n-1)^{(1-n)/2} (h\varepsilon_k)^{n-1} \leq C,
\]
where $C$ is a constant which is independent of $\varepsilon_k$. Combining this with (3.9) we can complete the proof of the proposition.

Proposition 3.5 implies that there exist a number $k_j$ which is independent of $\varepsilon_k$ and a subsequence of $l_j^k$ denoted itself such that
\[
l_j^k \to k_j, \quad \text{as} \quad k \to \infty.
\]

Since $l_j^k, k_j \in N$, $\{l_j^k\}$ must be constant sequence for any fixed $j$, namely $l_j^k = k_j$. The same reason shows $d_k$ can be written as $d_i$ which is also a number independent of $\varepsilon_k$ later.

4 An estimate for the lower bound

Write $\Omega' = G' \setminus \bigcup_{j=1}^{N_2} B(a_j, \sigma)$. Fixing $j \in \{1, 2, \ldots, N_2\}$ and taking $i_0 \in A_j$, we have $x_{i_0} \to a_j$ as $\varepsilon \to 0$. Thus
\[
\bigcup_{i \in A_j} B(x_i^\varepsilon, h\varepsilon) \subset B(x_{i_0}, \sigma/4) \subset B(a_j, \sigma) \tag{4.1}
\]
holds with $\varepsilon$ small enough.

Denote $\Omega_j = B(a_j, \sigma) \setminus \bigcup_{i \in A_j} B(x_i^\varepsilon, h\varepsilon), \Omega_j^\sigma = B(x_{i_0}, \sigma/4) \setminus \bigcup_{i \in A_j} B(x_i^\varepsilon, h\varepsilon)$. To estimate the lower bound of $\|\nabla u\varepsilon\|_{L^\infty(\Omega_j)}$ the following proposition is necessary that was given by Theorem 3.9 in [6].
Proposition 4.1  Let $A_{s,t}(x_i) = (B(x_i, s) \setminus B(x_i, t)) \cap G$ with $\varepsilon \leq t < s \leq R$. Assume that $u \in W^{1,n}_G(G, \mathbb{R}^n)$ and $\frac{1}{2} \leq |u| \leq 1$ on $A_{s,t}(x_i)$. If there is a constant $C$ such that

$$\frac{1}{\varepsilon^n} \int_{A_{s,t}(x_i)} (1 - |u|^2)^2 \leq C.$$ 

Then for $\varepsilon < \varepsilon_0$ there holds

$$\int_{A_{s,t}(x_i)} |\nabla u|^n \geq |d_i|^n(n-1)(n-1)^{n/2} |S^{n-1}| \ln \frac{s}{t} - C,$$

where $C$ is a constant which is independent of $\varepsilon$ and $d_i$ is the degree of $u$ on each $\partial(B(x_i, r) \cap G), t \leq r \leq s$.

Proposition 4.2  Assume $\text{Card} A_{j} = N$. Then

$$\int_{\Omega_j} |\nabla u_\varepsilon|^n \geq \int_{\Omega_{j,s}} |\nabla u_\varepsilon|^n \geq (n-1)n^{n/2} |S^{n-1}| |k_j| \ln \frac{\sigma}{|\varepsilon|} - C$$

(4.2)

where $C$ is a constant which is independent of $\varepsilon$.

Proof. We give the proof following that in [6] (see Theorem 3.10), and the idea comes from [8]. Suppose $x_1, x_2, \ldots, x_N$ converge to $a_j$, and $d_{i,R}(i = 1, 2, \ldots, N)$ is the degree of $u_\varepsilon$ around $\partial B(x_i, R)$. Let $R^\sigma_{\varepsilon}$ denote the set of all numbers $R \in [\varepsilon, \sigma]$ such that $\partial B(x_i, R) \cap B(x_j, \varepsilon) = \emptyset$ for all $i \neq j$ and such that for some collection $J_R \subset \{1, 2, \ldots, N\}$, satisfying $J_R \subset J_{R'}$ if $R' \leq R$, the family $\{B(x_i, R)\}_{i \in J_R}$ is disjoint and

$$\bigcup_{i=1}^{N} B(x_i, \varepsilon) \subset \bigcup_{i \in J_{R}} B(x_i, R') \subset \bigcup_{i \in J_{R}} B(x_i, R), \quad R' \leq R.$$

Note that $R^\sigma_{\varepsilon}$ is the union of closed intervals $[R^0_{\varepsilon}, R^0_l], 1 \leq l \leq L$, whose right endpoints correspond to a number $R = R^L$ such that $\partial B(x_i, R) \cap B(x_j, \varepsilon) = \emptyset$ for some pair $i \neq j \in J_R$ and whose left endpoints correspond to a number $R^0_{\varepsilon}$ such that $B(x_i, R^0_l) \setminus \cup_{i \in J_{R}} B(x_i, R^0_l) \neq \emptyset$ for $i \in J_{R_0}$. $J_R = J^l$ is a constant for $R \in [R^0_{\varepsilon}, R^0_L]$ and $J_{l+1} \subset J^l$, $J_{l+1} \neq J^l$. Thus $L \leq N$. Moreover, there exists a constant $M = M(h) > 0$ such that

$$R^0_l \leq M \varepsilon, \quad R^L \geq \sigma/M, \quad R^{l+1} \leq MR^l$$

(4.3)

for all $l = 1, 2, \ldots, L - 1$. Finally, observe that for all $R \in R^\sigma_{\varepsilon}$ and $J \in J_{R}$,

$$|k_j| = \left| \sum_{i \in J_R} d_{i,R} \right| \leq \sum_{i \in J_R} |d_{i,R}|^{n/(n-1)}.$$  

(4.4)

Applying (4.3)(4.4) and proposition 4.1 we have

$$\int_{\Omega_{j,s}} |\nabla u_\varepsilon|^n \geq \sum_{l=1}^{L} \sum_{i \in J_{R^l}} \int_{A_{R^l_{\varepsilon}, R^l_{\varepsilon}}(x_i)} |\nabla u_\varepsilon|^n$$
\[ \sum_{l=1}^{L} \sum_{i \in J} |S^{l-1}|(n-1)^{n/2}|d_{i, R_l}| \ln(R_l/R_0) - C \]
\[ \geq |S^{l-1}|(n-1)^{n/2}|k_j| \sum_{l} (\ln R_l - \ln R_0) - C \]
\[ \geq (n-1)^{n/2}|S^{l-1}||k_j| \ln \frac{\sigma}{\varepsilon} - C. \]

This and (4.1) imply that (4.2) holds.

**Remark** In fact the following results
\[ \int_{\Omega_j} |\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^n \geq (n-1)^{n/2}|S^{l-1}||k_j|^{n/(n-1)} \ln \frac{\sigma}{\varepsilon}, \]
and
\[ \int_{\Omega_j} (1 - |u_\varepsilon|^n)|\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^n \leq C \]
had been presented in the proof of Theorem 3.9 in [6], where \( C \) which is independent of \( \varepsilon \). Noticing
\[ \int_{\Omega_j} |u_\varepsilon|^n|\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^n = \int_{\Omega_j} |\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^n - \int_{\Omega_j} (1 - |u_\varepsilon|^n)|\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^n, \]
we have
\[ \int_{\Omega_j} |u_\varepsilon|^n|\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^n \geq (n-1)^{n/2}|k_j|^{n/(n-1)}|S^{l-1}| \ln \frac{\sigma}{\varepsilon} - C. \]

**Theorem 4.3** There exists a constant \( C \) which is independent of \( \varepsilon, \sigma \in (0, 1) \) such that
\[ \int_{\bigcup_{j=1}^{N_2} |u_\varepsilon|^n \geq (n-1)^{n/2}|S^{l-1}| \ln \frac{\sigma}{\varepsilon} - C, \] (4.5)
\[ \frac{1}{n} \int_{G_\varepsilon} |\nabla u_\varepsilon|^n + \frac{1}{4n} \int_{G_\varepsilon} (1 - |u_\varepsilon|^2)^2 \leq \frac{1}{n} (n-1)^{n/2}|S^{l-1}| \ln \frac{1}{\sigma} + C \] (4.6)
where \( G_\varepsilon = G \setminus \bigcup_{j=1}^{N_2} B(a_j, \sigma) \).

**Proof.** From (4.2) and Proposition 2.3 we have
\[ (n-1)^{n/2}|S^{l-1}| |\sum_{j=1}^{N_2} |k_j|| \ln \frac{\sigma}{\varepsilon} \leq (n-1)^{n/2}|S^{l-1}| \ln \frac{1}{\varepsilon} + C \]
or \((\sum_{j=1}^{N_2} |k_j| - d) \ln \frac{1}{\varepsilon} \leq C. \) It is seen as \( \varepsilon \) small enough
\[ \sum_{j=1}^{N_2} |k_j| \leq d = \sum_{j=1}^{N_2} k_j \]
which implies
\[ k_j \geq 0. \quad (4.7) \]
This and (3.9) imply
\[ \sum_{j=1}^{N_2} |k_j| = \sum_{j=1}^{N_2} k_j = d. \quad (4.8) \]
Substituting (4.8) into (4.2) yields (4.5), and (4.6) may be concluded from (4.5) and Proposition 2.3.

From (4.6) and the fact \(|u_\varepsilon| \leq 1\) a.e. on \(G\), we may conclude that there exists a subsequence \(u_{\varepsilon_k}\) of \(u_\varepsilon\) such that
\[ u_{\varepsilon_k} \to u^*, \quad W^{1,n}(G_\sigma, \mathbb{R}^n) \quad (4.9) \]
as \(\varepsilon_k \to 0\). Compare (4.9) with (1.1) we known \(u^* = u_n\) on \(G_\sigma\), and
\[ \{a_j\}_{j=1}^{N_2} = \{a_j\}_{j=1}^{J}. \quad (4.10) \]
These points were called the singularities of \(u_n\).

To show these singularities \(a_j \in \partial G\), the following conclusion is necessary.

**Proposition 4.4** Assume \(a \in \partial G\) and \(\sigma \in (0,R)\) with a small constant \(R\). If
\[ u \in W^{1,n}(A_{R,\sigma}(a), S^{n-1}) \cap C^0, \quad u = \overline{g} \]
on \((G' \setminus G) \cap B(a,R)\) and \(\deg(u, \partial B(a,R)) = 1\), then there exists a constant \(C\) which is independent of \(\sigma\) such that
\[ \int_{A_{R,\sigma}(a)} |\nabla u|^n \geq 2^n (n-1)^{n/2} |S^{n-1}| |\ln \frac{1}{\sigma} - C|. \quad (4.11) \]

**Proof.** Similar to the proof of Lemma VI.1 in [1], we may write \(G\) as the half space
\[ \{(x_1, x_2, \ldots, x_n): x_n > 0\} \]
locally and \(a\) as 0 by a conformal change.

Denote \(S_t = \partial B(0,t), t \in \sigma, R\). Noticing that \(\overline{g}\) is smooth on \(G' \setminus G\), we have
\[ \sup_{\sigma \setminus G} |\overline{g}| \leq C_1. \]
Taking \(t\) sufficiently small such that
\[ t \leq (n-1)^{1/2} \frac{(2^{n-1} - 1)^{1/(n-1)}}{2C_1}, \]
than
\[ \int_{S_t} |\overline{g}|^{n-1} \leq |S_t| \leq |S^{n-1}| |t^{n-1} C_1^{n-1} \leq (n-1)^{(n-1)/2} |S^{n-1}| (1 - 2^{1-n}) \quad (4.12) \]
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with $R < 1$ small enough, where $S_t^r = S_t \cap \{x_n < 0\}$. On the other hand we can be led to

$$(n - 1)^{(n-1)/2}|S^{n-1}| \leq \int_{S_t} |u_r|^{n-1} = \int_{S_t^+} |u_r|^{n-1} + \int_{S_t^-} |\overline{g_r}|^{n-1}$$

from Proposition 3.4. Here $S_t^+ = S_t \setminus S_t^-$. Combining this with (4.12) yields

$$\int_{S_t^+} |u_r|^n \geq |S_t^+|^{-1/(n-1)} \left( \int_{S_t^+} |u_r|^{n-1} \right)^{n/(n-1)} \geq 2^n |S^{n-1}|(n-1)^{n/2}t^{-1}. \quad (4.2)$$

Integrating this over $(\sigma, R)$, we obtain

$$\int_{A_{R, \sigma}} |\nabla v|^n \geq \frac{2^n |S^{n-1}|(n-1)^{n/2} \ln \frac{R}{\sigma}}{n(n-1)/2}$$

which implies (4.11). To prove $k_j = 1$ for any $j$, we suppose $R > 2\sigma$ is a small constant such that

$$B(a_j, R) \subset G' \quad B(a_j, R) \cap B(a_i, R) = \emptyset, i \neq j. \quad (4.13)$$

Denote $\Pi = \{v \in W^{1,n}(\Omega', S^{n-1}) : \deg(v, \partial B(a_j, r)) = k_j, r \in (\sigma, R), j = 1, 2, \ldots, N_2\}$.

**Proposition 4.5** For any $v \in \Pi$, if $k_j \geq 0$, $j = 1, 2, \ldots, N_2$, then there exists a constant $C = C(R)$ which is independent of $\sigma$ such that

$$\int_{\Omega'} |\nabla v|^n \geq \left( \frac{1}{\sigma} \sum_{j=1}^{N_2} k_j^{1/n} \right) \frac{1}{\sigma} \cdot \frac{1}{\sigma} |S^{n-1}| = C. \quad (4.14)$$

**Proof.** Write $A_{R, \sigma}(a_j) = B(a_j, R) \setminus B(a_j, \sigma)$, thus $\bigcup_{j=1}^{N_2} A_{R, \sigma}(a_j) \subset \Omega'$. From Proposition 3.4 we have

$$k_j = |k_j| \leq (n - 1)^{(1-n)/2}|S^{n-1}|^{-1} \int_{S^{n-1}} |v_r|^{n-1} \leq (n - 1)^{(1-n)/2}|S^{n-1}|^{-1/(n-1)} \left( \int_{S^{n-1}} |v_r|^{n-1} \right)^{(n-1)/n}$$

namely

$$\int_{S^{n-1}} |v_r|^{n} \geq (n - 1)^{n/2}|S^{n-1}|^{1/(n-1)}.$$

On the other hand, we may obtain

$$\int_{\Omega'} |\nabla v|^n \geq \sum_{j=1}^{N_2} \int_{A_{R, \sigma}(a_j)} |\nabla v|^n$$
\[
\geq \sum_{j=1}^{N_z} \int_{S^{n-1}} r^{-n} |\nabla \tau v|^n r^{n-1} d\zeta \, dr \\
\geq (n - 1)^{n/2} |S^{n-1}| \sum_{j=1}^{N_z} k_j^{n/(n-1)} \int_\sigma r^{-1} \, dr \\
= (n - 1)^{n/2} |S^{n-1}| \left( \sum_{j=1}^{N_z} k_j^{n/(n-1)} \right) \ln \frac{R}{\sigma}
\]

which implies (4.14).

## 5 The proof of Theorem 1.2

Let \( u_\varepsilon \) be a regularizable minimizer of \( E_\varepsilon(u, G) \). Proposition 2.4 has given one estimate of convergence rate of \( |u_\varepsilon| \). Moreover, we also have

**Theorem 5.1** There exists a constant \( C \) which is independent of \( \varepsilon \in (0, 1) \) such that

\[
\frac{1}{\varepsilon^n} \int_G (1 - |u_\varepsilon|^2) \leq C(1 + \ln \frac{1}{\varepsilon}). \tag{5.1}
\]

**Proof.** The minimizer \( u = u^\varepsilon \) of the regularized functional \( E_\varepsilon^\tau(u, G) \) solves (2.2). Taking the inner product of the both sides of (2.2) with \( u \) and integrating over \( G \) we have

\[
\frac{1}{\varepsilon^n} \int_G |u|^2 (1 - |u|^2) = - \int_G \text{div}(u^{(n-2)/2} \nabla u)u \\
= \int_G u^{(n-2)/2} |\nabla u|^2 - \int_{\partial G} v^{(n-2)/2} u u_n \tag{5.2}
\]

where \( n \) denotes the unit outward normal to \( \partial G \) and \( u_n \) the derivative with respect to \( n \).

To estimate \( \int_{\partial G} v^{n/2} \), we choose a smooth vector field \( \nu \) such that \( \nu|_{\partial G} = n \).

Multiplying (2.2) by \( (\nu \cdot \nabla u) \) and integrating over \( G \), we obtain

\[
\frac{1}{\varepsilon^n} \int_G u(1 - |u|^2)(\nu \cdot \nabla u) = - \int_G \text{div}(v^{(n-2)/2} \nabla u)(\nu \cdot \nabla u) \\
= \int_G v^{(n-2)/2} \nabla u \cdot (\nu \cdot \nabla u) - \int_{\partial G} v^{(n-2)/2} |u_n|^2.
\]

Combining this with

\[
\frac{1}{\varepsilon^n} \int_G u(1 - |u|^2)(\nu \cdot \nabla u) = \frac{1}{2\varepsilon^n} \int_G (1 - |u|^2)(\nu \cdot \nabla (|u|^2)) \\
= - \frac{1}{4\varepsilon^n} \int_G (1 - |u|^2)^2 \text{div} \nu
\]

and
\[
\int_G v^{(n-2)/2} \nabla u \cdot \nabla (\nu \cdot \nabla u)
\]
\[
= \int_G v^{(n-2)/2} |\nabla u|^2 \, \text{div} \nu + \frac{1}{n} \int G v^{n/2} - \frac{1}{n} \int \partial G v^{n/2} \, \text{div} \nu.
\]
we obtain
\[
\int_{\partial G} v^{(n-2)/2} |u_n|^2 \leq C \frac{\epsilon}{n} \int_G (1 - |u|^2)^2 + C \int v^{n/2} + \frac{1}{n} \int_{\partial G} v^{n/2}.
\]
Thus
\[
\int_{\partial G} v^{n/2} = \int_{\partial G} v^{(n-2)/2} |u_n|^2 + |g_t|^2 + \tau
\]
\[
\leq C \int_{\partial G} v^{(n-2)/2} + \frac{1}{n} \int_{\partial G} v^{n/2} + C \mathcal{E}_\epsilon^* (u_\epsilon^*, G).
\]
Substituting this into (5.2) yields
\[
\frac{1}{\epsilon^n} \int_G |u|^2 (1 - |u|^2) \leq C \mathcal{E}_\epsilon^* (u_\epsilon^*, G).
\]
Let \( \tau \to 0 \), applying (2.1) and Proposition 2.3 we have
\[
\frac{1}{\epsilon^n} \int_G |u_\epsilon|^2 (1 - |u_\epsilon|^2) \leq C \mathcal{E}_\epsilon (u_\epsilon, G) \leq C (1 + |\ln \epsilon|)
\]
which and (2.6) imply (5.1).

**Theorem 5.2** Denote \( \rho = |u_\epsilon| \). There exists a constant \( C \) which is independent of \( \epsilon \in (0, 1) \) such that
\[
\|\nabla \rho\|_{L^n(G)} \leq C.
\]

**Proof.** Denote \( u = u_\epsilon \). From the Remark in \S 4 we know
\[
\int_{\Omega} |u|^n |\nabla \frac{u}{|u|}|^n \, dx \geq (n-1)^{n/2} |k_j| \frac{n}{S^{n-1}} |\ln \frac{\sigma}{\epsilon} - C.
\]
Thus we may modify (4.5) as
\[
\int_{\bigcup_{j=1}^{N_j} \Omega_j} \rho^n |\nabla \frac{u}{|u|}|^n \geq (n-1)^{n/2} |S^{n-1}| d \ln \frac{\sigma}{\epsilon} - C.
\]
Combining this with
\[
\int_{\bigcup_{j=1}^{N_j} \Omega_j} |\nabla u|^n \geq \int_{\bigcup_{j=1}^{N_j} \Omega_j} \rho^n |\nabla \frac{u}{|u|}|^n + \int_{\bigcup_{j=1}^{N_j} \Omega_j} |\nabla \rho|^n - C.
\]
and Proposition 2.3, we derive
\[ \int_{\cup_{j=1}^{N_2} B_j} |\nabla \rho|^n \leq C. \] (5.4)

On the other hand, from (2.1) and Proposition 2.1 we are led to
\[ \int_{G \cap B(x, h \varepsilon)} |\nabla u_\varepsilon|^n = \lim_{\varepsilon \to 0} \int_{G \cap B(x, h \varepsilon)} |\nabla u_{\varepsilon \tau}|^n \leq C(\lambda \varepsilon)^n (C/\varepsilon)^n \leq C, \]
for \( i \in \Lambda_j \). Summarizing for \( i \) and using (5.4) we can obtain (5.3).

**Theorem 5.3** For the \( \sigma > 0 \) in Theorem 4.4, then as \( \varepsilon \to 0 \),
\[ \frac{1}{\varepsilon^n} \int_{G_{3\sigma}} (1 - \rho^2)^2 \to 0, \] (5.5)
where \( G_{3\sigma} = G \setminus \cup_{j=1}^{N_2} B(a_j, 3\sigma) \).

**Proof.** The regularizable minimizer \( u_\varepsilon \) satisfies
\[ \int_{G_\sigma} |\nabla u|^{n-2} \nabla u \nabla \phi = \frac{1}{\varepsilon^n} \int_{G_\sigma} u \phi (1 - |u|^2), \] (5.6)
where \( \phi \in W^{1,n}_0(G_\sigma, \mathbb{R}^n) \) since \( u_\varepsilon \) is a weak solution of (2.4). Denoting \( u = u_\varepsilon^\beta = \rho w, \rho = |u|, w = \frac{u}{\rho} \) in \( G_\sigma \) and taking \( \phi = \rho w \zeta, \zeta \in W^{1,n}_0(G_\sigma, \mathbb{R}^n) \), we have
\[ \int_{G_\sigma} |\nabla u|^{n-2} (w \nabla \rho + \rho \nabla w)(\rho \zeta \nabla w + \rho w \nabla \zeta + w \zeta \nabla \rho) = \frac{1}{\varepsilon^n} \int_{G_\sigma} \rho^2 \zeta (1 - \rho^2). \] (5.7)
Substituting \( 2w \nabla w = \nabla (|w|^2) = 0 \) into (5.7), we obtain
\[ \int_{G_\sigma} |\nabla u|^{n-2}(\rho \nabla \rho \nabla \zeta + |\nabla u|^2 \zeta) = \frac{1}{\varepsilon^n} \int_{G_\sigma} \rho^2 \zeta (1 - \rho^2). \] (5.8)
Set \( S = \{ x \in G_\sigma : \rho(x) > 1 - \varepsilon^\beta \} \) for some fixed \( \beta \in (0, n/2) \) and \( \overline{\rho} = \max(\rho, 1 - \varepsilon^\beta) \), thus \( \rho = \overline{\rho} \) on \( S \). In (5.8) taking \( \zeta = (1 - \overline{\rho}) \psi \), where \( \psi \in C^\infty(G_\sigma, R) \), \( \psi = 0 \) on \( G_\sigma \setminus G_{2\sigma}, 0 < \psi < 1 \) on \( G_{2\sigma} \setminus G_{3\sigma}, \psi = 1 \) on \( G_{3\sigma} \), we have
\[ \int_{G_\sigma} |\nabla u|^{n-2} \rho \nabla \rho \nabla \psi + \frac{1}{\varepsilon^n} \int_{G_\sigma} t^2 (1 - \rho^2)(1 - \overline{\rho}) \psi \]
\[ = \int_{G_\sigma} |\nabla u|^{n-2} \rho \nabla \rho \nabla \psi (1 - \overline{\rho}) + \int_{G_\sigma} |\nabla u|^n \psi (1 - \overline{\rho}) \] (5.9)
Noticing \( 1/2 \leq t \leq 1 \) in \( G_\sigma \) and applying (4.6) we obtain
\[ \frac{1}{\varepsilon^n} \int_{G_{3\sigma}} (1 - \overline{\rho})(1 - \rho^2) + \int_{S \cap G_{3\sigma}} |\nabla u|^{n-2} |\nabla \rho|^2 \leq C \varepsilon^\beta. \] (5.10)
On the other hand, (2.6) implies
\[ \varepsilon^{2\beta} |G_{\sigma} \setminus S| \leq \int_{G_{\sigma} \setminus S} (1 - l^2)^2 \leq C\varepsilon^n, \]
namely \( |G_{\sigma} \setminus S| \leq C\varepsilon^{n-2\beta} \). Then there exists a small constant \( \varepsilon_0 > 0 \) such that
\[ G_{3\sigma} \subset S \cup E \]
as \( \varepsilon \in (0, \varepsilon_0) \) where \( E \) is a set, the measure of which converges to zero. Thus
\[ \lim_{\varepsilon \to 0} \int_{G_{3\sigma}} (1 - \rho^2)(1 - \rho)^2 = \lim_{\varepsilon \to 0} \int_{G_{3\sigma}} (1 + \rho)(1 - \rho)^2. \]
By (5.10),
\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \int_{G_{3\sigma}} (1 + \rho)^2(1 - \rho)^2 \leq \lim_{\varepsilon \to 0} \frac{2}{\varepsilon^n} \int_{G_{5\sigma}} (1 - \bar{\rho})(1 - \rho^2) = 0 \]
This is our conclusion.

**Theorem 5.4** Assume \( B(x, 2\sigma) \subset G_{\sigma} \) satisfies
\[ \frac{1}{\varepsilon^n} \int_{B(x, \sigma)} (1 - |u_{\varepsilon}|^2)^2 \to 0, \text{ as } \varepsilon \to 0, \quad (5.11) \]
then \( |u_{\varepsilon}| \to 1 \) in \( C(B(x, \sigma), R) \).

**Proof.** Since \( B(x, 2\sigma) \subset G_{\sigma} \), there exists \( \varepsilon_0 \) sufficiently small so that \( B(x, \sigma) \subset G^{2\delta_0} \). We always assume \( \varepsilon < \varepsilon_0 \). For \( x_0 \in B(x, \sigma) \), set \( \alpha = |u_{\varepsilon}(x_0)| \). Proposition 2.2 implies
\[ |u_{\varepsilon}(x) - u_{\varepsilon}(x_0)| \leq C\varepsilon^{-1}\tau\varepsilon, \quad \text{if } x \in B(x_0, \tau\varepsilon), \]
where \( \tau = (1 - \alpha)(NC)^{-1} \), \( C \) is the constant in Proposition 2.2 and \( N \) is a large number such that \( \tau < \delta \). Thus \( B(x_0, \tau\varepsilon) \subset B(x, \sigma) \) and
\[ |u_{\varepsilon}(x)| \leq \alpha + C\tau, \quad \text{if } x \in B(x_0, \tau\varepsilon), \]
\[ \int_{B(x_0, \tau\varepsilon)} (1 - |u_{\varepsilon}(x)|^2)^2 \geq (1 - 1/N)^2(1 - \alpha)^{n+2}\pi\varepsilon^n(\alpha N)^{-n}. \]
Combining this with (5.11) we obtain \( (1 - \alpha)^{n+2} = o(1) \) as \( \varepsilon \to 0 \). Thus it is not difficult to complete the proof of Theorem.
6 The proof of Theorem 1.1

It is known that the singularities of $u_n$ are in $G$ from the discussion in §3. Since $\deg(g, \partial G) > 0$, we can see that the zeroes of $u_\varepsilon$ are also in $G$. Moreover, the zeroes are contained in finite bad balls, i.e. $B(x^*_i, h\varepsilon), i \in J$. As $\varepsilon \to 0, B(x^*_i, h\varepsilon) \to a_j, i \in J$. This implies that the zeroes of $u_\varepsilon$ distribute near these singularities of $u_n$ as $\varepsilon \to 0$. Thus it is necessary to describe these singularities $\{a_j\}, j = 1, 2, \ldots, N_2$.

**Proposition 6.1** $k_j = \deg(u_n, a_j)$.

**Proof.** Denote $\Omega' = G' \cup \bigcup_{j=1}^{N_2} B(a_j, \sigma)$. Combining (4.6) and

$$\int_{G' \setminus G} |\nabla u_\varepsilon|^n = \int_{G' \setminus G} |\nabla \bar{g}|^n \leq C,$$

we have

$$\int_{\Omega'} |\nabla u_\varepsilon|^n \leq C + (n-1)^{n/2}|S^{n-1}| d |\ln \sigma|,$$

(6.1)

where $C$ is a constant which is independent of $\varepsilon$. For $R$ in (4.13), from (6.1) we have

$$\int_{A_{R, \varepsilon}(a_j)} |\nabla u_\varepsilon|^n \leq C.$$

Then we know that there exists a constant $r \in (\sigma, R)$ such that

$$\int_{\partial B(a_j, r)} |\nabla u_\varepsilon|^n \leq C(r)$$

by using integral mean value theorem. Thus there exists a subsequence $u_{\varepsilon_k}$ of $u_\varepsilon$ such that

$$u_{\varepsilon_k} \to u_n, \quad \text{in } C(\partial B(a_j, r))$$

as $\varepsilon_k \to 0$, which implies

$$k_j = \deg(u_\varepsilon, \partial B(a_j, \sigma)) = \deg(u_n, a_j).$$

**Proposition 6.2** $k_j = 0$ or $k_j = 1$.

**Proof.** From the regularity results on $n$-harmonic maps (see [3][5] or [9]), we know $u_n \in C^0(G_\sigma, \mathbb{R}^n)$. Set

$$w = \begin{cases} \bar{g} & \text{on } G' \setminus G, \\ u_n & \text{on } G_\sigma, \end{cases}$$

then $w \in \Pi$. Using Proposition 4.5 and (4.7) we have

$$\int_{\Omega'} |\nabla w|^n \geq (n-1)^{n/2}|S^{n-1}|(\sum_{j=1}^{N_2} k_j^{n-1}) \ln \frac{1}{\sigma} - C(R).$$

(6.2)
On the other hand, (6.1) and (4.9) imply
\[ u_{\varepsilon k} \xrightarrow{w} w, \quad \text{in } W^{1,n}(\Omega', \mathbb{R}^n). \]
Noting this and the weak lower semicontinuity of \( \int_{\Omega'} |\nabla u|^n \), applying (6.1) we have
\[ \int_{\Omega'} |\nabla w|^n \leq \lim_{\varepsilon_k \to 0} \int_{\Omega'} |\nabla u_{\varepsilon k}|^n \leq (n - 1)^{n/2} |S^{n-1}| d \ln \frac{1}{\sigma} + C. \]  
(6.3)
Combining this with (6.2), we obtain
\[ \sum_{j=1}^{N_2} k_j^{n/2} - d \ln \frac{1}{\sigma} \leq C \quad \text{or} \quad \sum_{j=1}^{N_2} k_j^{n/2} \leq d = \sum_{j=1}^{N_2} k_j \]
for \( \sigma \) small enough. Thus \( (k_j^{1/(n-1)} - 1)k_j \leq 0 \) which implies that the Proposition holds.

**Proposition 6.3** \( k_j > 0, \ j = 1, 2, \ldots, N_2. \)

**Proof.** Suppose \( k_1 = 0 \) and \( k_2, k_3, \ldots, k_{N_2} > 0 \). Similar to the proof of Theorem 4.3 we have
\[ \int_{\cup_{j=1}^{N_2} \Omega_j} |\nabla u_{\varepsilon}|^n \geq (n - 1)^{n/2} |S^{n-1}| d \ln \frac{1}{\varepsilon} - C. \]
By this we can rewrite (4.6) as
\[ \int_{G \cup \cup_{j=1}^{N_2} B(a_j, \sigma)} |\nabla u_{\varepsilon}|^n + \frac{1}{4\varepsilon^n} \int_G (1 - |u_{\varepsilon}|^2)^2 \leq C(\sigma). \]
Thus similar to the proof of Theorem 5.3 we may modify (5.5) as
\[ \frac{1}{\varepsilon^n} \int_{G \cap \cup_{j=1}^{N_2} B(a_j, 3\sigma)} (1 - |u_{\varepsilon}|^2)^2 \to 0 \]  
(6.4)
as \( \varepsilon \to 0 \). Noticing
\[ G \cap B(a_1, \sigma) \subset G \cap B(a_1, R) \subset G \setminus \cup_{j=2}^{N_2} B(a_j, R) \subset G \setminus \cup_{j=2}^{N_2} B(a_j, 3\sigma) \]
we have
\[ \frac{1}{\varepsilon^n} \int_{G \cap B(a_1, \sigma)} (1 - |u_{\varepsilon}|^2)^2 \to 0. \]  
(6.5)
On the other hand, the definition of \( a_1 \) implies that there exists at least one bad ball \( B(x_0^\varepsilon, h\varepsilon) \) such that
\[ G \cap B(x_0^\varepsilon, h\varepsilon) \subset G \cap B(a_1, \sigma). \]
Applying the definition of bad ball we obtain
\[ \frac{1}{\varepsilon^n} \int_{G \cap B(a_1, \sigma)} (1 - |u_{\varepsilon}|^2)^2 \geq \frac{1}{\varepsilon^n} \int_{G \cap B(x_0^\varepsilon, h\varepsilon)} (1 - |u_{\varepsilon}|^2)^2 \geq \mu > 0 \]
which is contrary to (6.5). This contradiction shows \( k_1 > 0. \)
Remark We may conclude $k_j = 1, j = 1, 2, \ldots, N_2$ from Proposition 6.2 and Proposition 6.3. Noticing $d = \sum_{j=1}^{N_2} k_j$, we obtain

$$N_2 = d, \quad 1 = k_j = \sum_{i \in \Lambda_j} d_i.$$ 

Thus on one hand, although the number of the singularities of $n-$ harmonic maps is indefinite (see Theorem A and Theorem C in [3]), we can say that for this $n-$ harmonic map $u_n$, the limit of the regularizable minimizer $u_{\varepsilon_k}$ in $W^{1,n}$ as $k \to \infty$, the number of its singularities is just the degree $d$ by applying (4.10).

On the other hand, there exists at least one $i_0 \in \Lambda_j$ such that $d_{i_0} \neq 0$. Then we know that there exists at least one zero of $u_{\varepsilon_k}$ in $B(x_{i_0}^\varepsilon, h\varepsilon)$ by using Kronecker’s theorem.

Theorem 6.4 \( a_j \in G, \quad j = 1, 2, \ldots, d. \)

Proof. Suppose $a_1 \in \partial G, a_2, a_3, \ldots, a_d \in G$. Set

$$\Omega_\sigma = (G' \setminus B(a_1, R)) - \bigcup_{j=2}^d B(a_j, \sigma), \quad w = \begin{cases} u_n & \text{on } G_\sigma, \\ \bar{g} & \text{on } G' \setminus G. \end{cases}$$

Using Proposition 4.5 on $\Omega_\sigma$ we have

$$\int_{\Omega_\sigma} |\nabla w|^n \geq (n - 1)^{n/2} |S^{n-1}| (d - 1) \ln \frac{1}{\sigma} - C(R). \quad (6.6)$$

Taking $u = w, a = a_1$ in Proposition 4.4 we have

$$\int_{\Lambda_{R, \sigma}(a_1)} |\nabla w|^n \geq 2^+(n - 1)^{n/2} |S^{n-1}| \ln \frac{1}{\sigma} - C.$$ 

Combining this with (6.6) yields

$$\int_{\Omega'} |\nabla w|^n \geq (d + 2^+ - 1) (n - 1)^{n/2} |S^{n-1}| \ln \frac{1}{\sigma} - C.$$ 

Compare this to (6.3) we obtain

$$(d + 2^+ - 1 - d) \ln \frac{1}{\sigma} \leq C$$

where $C$ is a constant which is independent of $\sigma$. It is impossible as $\sigma$ small enough, so $a_1 \in G$.

7 The proof of Theorem 1.3

Theorem 7.1 Let $u_{\varepsilon_k}$ be the regularizable minimizer of $E_\varepsilon(u, G)$. Then there exists a subsequence $u_{\varepsilon_k}$ of $u_{\varepsilon_k}$ such that

$$u_{\varepsilon_k} \to u_n, \quad \text{in } W^{1,n}_{\text{loc}}(G \setminus \bigcup_{j=1}^d \{a_j\}, \mathbb{R}^n).$$
Proof. Step 1: Suppose the ball $B(x_0, 2\sigma) \subset G \setminus \bigcup_{j=1}^d \{a_j\}$, where the constant $\sigma$ may be sufficiently small but independent of $\varepsilon$. Since (4.6) implies

$$E_\varepsilon(u_\varepsilon, B(x_0, 2\sigma) \setminus B(x_0, \sigma)) \leq C,$$

we know there is a constant $r \in (\sigma, 2\sigma)$ such that

$$\int_{\partial B(x_0, r)} |\nabla u_\varepsilon|^n + \frac{1}{\varepsilon^n} \int_{\partial B(x_0, r)} (1 - |u_\varepsilon|^2)^2 \leq C(r), \quad (7.1)$$

by applying the integral mean value theorem. Thus, there exists a subsequence $u_{\varepsilon k}$ of $u_\varepsilon$ such that

$$u_{\varepsilon k} \to u_n, \quad \text{in } C(\partial B(x_0, r), \mathbb{R}^n),$$

which leads to

$$\left| \frac{u_{\varepsilon k}}{|u_{\varepsilon k}|} \right| \to u_n, \quad \text{in } C(\partial B(x_0, r), \mathbb{R}^n). \quad (7.2)$$

Step 2: Denote $\rho = |u_\varepsilon|$ on $B = B(x_0, r)$. It is not difficult to prove that the minimizer $w$ of the problem

$$\min\{ \int_B |\nabla u|^n : u \in W^{1,n}_{\|w\|} (B, S^{n-1}) \} \quad (7.3)$$

exists. Noting $u_\varepsilon$ be a minimizer of $E_\varepsilon(u, G)$, we have

$$E_\varepsilon(u_\varepsilon, B) \leq \frac{1}{n} \int_B |\nabla (\rho w)|^n + \frac{1}{4\varepsilon^n} \int_B (1 - \rho^2)^2.$$

Obviously (4.6) and $|u_\varepsilon| \geq 1/2$ on $B$ imply

$$\frac{1}{2^n} \int_B |\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^n \leq \int_B |\nabla u_\varepsilon|^n \leq C,$$

thus

$$\int_B |\nabla w|^n \leq \int_B |\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^n \leq C. \quad (7.4)$$

Applying this we may claim that

$$\int_B |\nabla u_\varepsilon|^n \leq C \varepsilon^\lambda + \int_B |\nabla w|^n, \quad (7.5)$$

for some $\lambda > 0$. Its proof can be seen in §8.

Step 3: Let $w_\tau$ is a solution of

$$\min\{\int_B (|\nabla w|^2 + \tau)^{n/2} : w \in W^{1,n}_{\|w\|} (B, S^{n-1})\}, \quad \tau \in (0, 1). \quad (7.6)$$

It is easy to see that $w_\tau$ solves

$$-\div(\varepsilon^{(n-2)/2} \nabla w) = w |\nabla w|^2 v_\varepsilon^{(n-2)/2}, \quad v_\varepsilon = |\nabla w|^2 + \tau. \quad (7.7)$$
as \( \tau \to 0 \). Noticing \( u_\varepsilon \in W^{1,n}_{\mathbb{R}^n} (B, S^{n-1}) \) we have
\[
\int_{B} |\nabla w_\tau|^n \leq \int_{B} (|\nabla w_\tau|^2 + \tau)^{n/2}
\leq \int_{B} (|\nabla u_\varepsilon|^2 + \tau)^{n/2} \leq \int_{B} (|\nabla u_\varepsilon|^2 + 1)^{n/2} \leq C
\]
by using (7.4), where \( C \) is a constant which is independent of \( \varepsilon, \tau \). Then there exist \( w^* \in W^{1,n}_{\mathbb{R}^n} (B, S^{n-1}) \) and a subsequence of \( w_\tau \) such that
\[
w_\tau \rightharpoonup w^*, \quad \text{in} \ W^{1,n}(B, \mathbb{R}^n).
\] (7.9)
Noting the weak lower semicontinuity of \( \int_{B} |\nabla w|^n \), we have
\[
\int_{B} |\nabla w^*|^n \leq \lim_{\tau \to 0} \int_{B} |\nabla w_\tau|^n \leq \lim_{\tau \to 0} \int_{B} (|\nabla w|^2 + \tau)^{n/2}.
\] (7.10)
The fact that \( w_\tau \) solves (7.6) implies
\[
\lim_{\tau \to 0} \int_{B} (|\nabla w|^2 + \tau)^{n/2} \leq \lim_{\tau \to 0} \int_{B} (|\nabla w_\tau|^2 + \tau)^{n/2} = \int_{B} |\nabla w_*|^n,
\]
where \( w_* \) is a solution of (7.3). This and (7.10) lead to
\[
\int_{B} |\nabla w_*|^n \leq \lim_{\tau \to 0} \int_{B} |\nabla w_\tau|^n \leq \lim_{\tau \to 0} \int_{B} |\nabla w|^n \leq \int_{B} |\nabla w_*|^n.
\] (7.11)
Since \( w^* \in W^{1,n}_{\mathbb{R}^n} (B, S^{n-1}) \), we know \( w^* \) also solves (7.3), namely
\[
\int_{B} |\nabla w_*|^n = \int_{B} |\nabla w^*|^n.
\]
Combining this with (7.11) yields
\[
\lim_{\tau \to 0} \int_{B} |\nabla w_\tau|^n = \int_{B} |\nabla w^*|^n,
\]
which and (7.9) imply
\[
\nabla w_\tau \rightharpoonup \nabla w^*, \quad \text{in} \ L^n(B, \mathbb{R}^n).
\] (7.12)

Step 4: Similar to the discussion of Step 3, we may derive the following conclusion: Let \( u_\tau \) be a solution of
\[
\min\left\{ \int_{B} (|\nabla u|^2 + \tau)^{n/2} : u \in W^{1,n}_{\mathbb{R}^n}(B, S^{n-1}) \right\}, \quad \tau \in (0, 1).
\] (7.13)
Then \( u^\tau \) satisfies
\[
\int_B |\nabla u^\tau|^n \leq C, \tag{7.14}
\]
where \( C \) is which is independent of \( \tau \), and \( u^\tau \) solves
\[
- \text{div}(v^{(n-2)/2} \nabla u) = u|\nabla u|^2 v^{(n-2)/2}, \quad v = |\nabla u|^2 + \tau. \tag{7.15}
\]
As \( \tau \to 0 \), there exists a subsequence of \( u^\tau \) denoted itself such that
\[
\nabla u^\tau \rightharpoonup \nabla u^*, \quad \text{in } L^n(B, \mathbb{R}^n),
\]
where \( u^* \) is a minimizer of \( \int_B |\nabla u|^n \) in \( W^{1,n}_0(B, S^{n-1}) \). It is well-known that \( u^* \) is a map of the least n-energy, and also an n-harmonic map.

Fix \( R > 2\sigma \) such that \( B(x_0, R) \subset G \setminus \bigcup_{j=1}^d \{ a_j \} \). Applying the regularity results on the map of the least n-energy (for example, Theorem 3.1 in \[5\]), we have
\[
\sup_{B(x_0,R)} |\nabla u^*|^n \leq \sup_{B(x_0,R)} |\nabla u|^n := C_0. \tag{7.17}
\]
It is obvious that \( C_0 \) is a constant which is independent of \( r \).

Step 5: From (7.7) subtracts (7.15). Then
\[
- \text{div}(v^{(n-2)/2} \nabla w - v^{(n-2)/2} \nabla u) = w|\nabla w|^2 v^{(n-2)/2} - u|\nabla u|^2 v^{(n-2)/2}. \tag{7.18}
\]
Multiplying both sides of (7.18) by \( w - u \) and integrating over \( B \) we obtain
\[
- \int_{\partial B} (v^{(n-2)/2} w_{v} - v^{(n-2)/2} u_{v})(w - u) \\
+ \int_B (v^{(n-2)/2} \nabla w - v^{(n-2)/2} \nabla u) \nabla (w - u) \\
= \int_B (w|\nabla w|^2 v^{(n-2)/2} - u|\nabla u|^2 v^{(n-2)/2})(w - u),
\]
where \( \nu \) denotes the unit outside-norm vector of \( \partial B \). Thus
\[
|\int_B (v^{(n-2)/2} \nabla w - v^{(n-2)/2} \nabla u) \nabla (w - u)| \\
\leq |\int_{\partial B} (v^{(n-2)/2} w_{\nu} - v^{(n-2)/2} u_{\nu})(w - u)| \\
+ |\int_B (w|\nabla w|^2 v^{(n-2)/2} - u|\nabla u|^2 v^{(n-2)/2})(w - u)| \\
+ |\int_B (w|\nabla w|^2 v^{(n-2)/2} - w|\nabla u|^2 v^{(n-2)/2})(w - u)| \\
= I_1 + I_2 + I_3. \tag{7.19}
\]
First we give an estimate for \( I_1 \). Let \( w = w^\tau \) is a solution of (7.6). Integrating both sides of (7.7) over \( B \), we have
\[
- \int_{\partial B} v^{(n-2)/2} w_{\nu} = \int_B w|\nabla w|^2 v^{(n-2)/2}.
\]
which and (7.8) imply
\[ | \int_{\partial B} v_{\varepsilon}^{(n-2)/2} w_{\nu} | \leq \int_B v_{\varepsilon}^{n/2} \leq C. \]  
(7.20)

An analogous discussion shows that for the solution \( u = u^\tau \) of (7.13) which equips with (7.14), we may also obtain
\[ | \int_{\partial B} v^{(n-2)/2} u_\nu | \leq \int_B |\nabla u|^n \leq C. \]  
(7.21)

Applying (7.20)(7.21) we derive
\[ I_1 \leq \sup_{\partial B} |w - u| (|\int_{\partial B} v_{\varepsilon}^{(n-2)/2} w_\nu | + | \int_{\partial B} v^{(n-2)/2} u_\nu |) \]  
(7.22)

\[ \leq C \sup_{\partial B} |w - u| = C \sup_{\partial B} \frac{u_{\varepsilon}}{|u_{\varepsilon}|} - u_n, \]
where \( C \) is independent of \( \varepsilon, \tau \). For the estimate of \( I_3 \), we have
\[ I_3 \leq \int_B |u - w| ||\nabla u||^2 v^{(n-2)/2} - |\nabla w||^2 v_{\varepsilon}^{(n-2)/2}| \]  
(7.23)

For estimating \( I_2 \), we multiply both sides of (7.15) by \((u - w)\) and integrate over \( B \), then
\[- \int_{\partial B} v^{(n-2)/2} u_\nu (u - w) + \int_B v^{(n-2)/2} \nabla u \nabla (u - w) \]
\[ = \int_B |\nabla u|^2 v^{(n-2)/2} u (u - w) = \int_B |\nabla u|^2 v^{(n-2)/2} (1 - uw). \]
Thus, we have
\[ I_2 \leq \int_B |\nabla u|^2 v^{(n-2)/2} |u - w|^2 = 2 \int_B |\nabla u|^2 v^{(n-2)/2} (1 - uw) \]
\[ \leq 2 \left( \int_{\partial B} v^{(n-2)/2} u_\nu (u - w) \right) + 2 \int_B v^{(n-2)/2} \nabla u \nabla (u - w). \]
Noting (7.21) we may derive
\[ I_2 \leq C \sup_{\partial B} \frac{u_{\varepsilon}}{|u_{\varepsilon}|} - u_n | + 2 \int_B v^{(n-2)/2} \nabla u \nabla (u - w). \]  
(7.24)

Step 6: Substituting (7.22)-(7.24) into (7.19) yields
\[ | \int_B (v_{\varepsilon}^{(n-2)/2} \nabla w - \nabla u) \nabla (w - u)| \]
\[ \leq C \sup_{\partial B} \frac{u_{\varepsilon}}{|u_{\varepsilon}|} - u_n | + 2 \int_B v^{(n-2)/2} \nabla u \nabla (u - w)| \]
\[ + 2 \int_B v_{\varepsilon}^{(n-2)/2} |\nabla w|^2 - v^{(n-2)/2} |\nabla u|^2| \].
Letting $\tau \to 0$ and applying (7.12)(7.16) we obtain

$$\left| \int_B (|\nabla w^*|^{(n-2)/2} \nabla w^* - |\nabla u^*|^{(n-2)/2} \nabla u^*) \nabla (w^* - u^*) \right|$$

$$\leq C \sup_{\partial B} \frac{|u_\varepsilon|}{|u_\varepsilon|} - u_n + 2 \int_B |\nabla u^*|^{n-1} \nabla (u^* - w^*) + 2 \int_B ||\nabla w^*||^{n-1} - ||\nabla u^*||^{n-1}.$$

Using Lemma 1.2 in [4], we have

$$2^{n-1} \int_B |\nabla w^* - \nabla u^*|^n \leq \int_B (|\nabla w^*|^{(n-2)/2} \nabla w^* - |\nabla u^*|^{(n-2)/2} \nabla u^*) \nabla (w^* - u^*).$$

Thus

$$(2^{n-1} - 2) \int_B |\nabla w^* - \nabla u^*|^n \leq C \sup_{\partial B} \left| \frac{u_\varepsilon}{u_\varepsilon} - u_n \right| + 2 \int_B ||\nabla u^*||^{n-1} \nabla (u^* - w^*).$$

Denote $\psi(\varepsilon) = \int_B |\nabla w^* - \nabla u^*|^n$ and let $\varepsilon \to 0$, then

$$(2^{n-1} - 2)\psi(\varepsilon) \leq o(1) + 2(C_0 |B|)^{(n-1)/n} (\psi(\varepsilon))^{1/n}$$

holds by using (7.2), where $C_0$ is the constant in (7.17).

We claim that for some small constant $\sigma > 0$, the following holds:

$$\psi(\varepsilon) \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

(7.26)

Suppose (7.26) is not true, then there exists $\tau > 0$, for any $\varepsilon_0 > 0$, such that as $\varepsilon < \varepsilon_0$ we have $\psi(\varepsilon) \geq 2\tau > \tau$ or

$$(\psi(\varepsilon))^{(n-1)/n} > \tau^{(n-1)/n}, \quad \forall \varepsilon < \varepsilon_0.$$

(7.27)

Taking $\sigma$ small enough so that

$$2(C_0 |B(x_0, r)|)^{(n-1)/n} = (2^{n-2} - 1)\tau^{(n-1)/n},$$

we obtain from (7.25)

$$(\psi(\varepsilon))^{1/n}[(\psi(\varepsilon))^{(n-1)/n} - \frac{2(C_0 |B|)^{(n-1)/n}}{2^{n-1} - 2}]$$

$$= (\psi(\varepsilon))^{1/n}[(\psi(\varepsilon))^{(n-1)/n} - \frac{1}{2} \tau^{(n-1)/n}] = o(1).$$

(7.28)

Substituting (7.27) into (7.28) we derive $(\psi(\varepsilon))^{1/n} = o(1)$, which is contrary to (7.27).

Step 7: Noting the weak lower semicontinuity of the functional $\int_B |\nabla u|^n$, from (4.9) we are led to

$$\int_B |\nabla u_n|^n \leq \lim_{\varepsilon_k \to 0} \int_B |\nabla u_{\varepsilon_k}|^n.$$
Combining this with (7.5) and (7.26) we obtain

\[ \int_B |\nabla u_n|^n \leq \lim_{\varepsilon_k \to 0} \int_B |\nabla u_{\varepsilon_k}|^n \leq \lim_{\varepsilon_k \to 0} \int_B |\nabla u_{\varepsilon_k}|^n \leq \lim_{\varepsilon_k \to 0} \int_B |\nabla w^*|^n = \int_B |\nabla w^*|^n. \]

Recalling the definition of \( u^* \) in Step 4, and noticing \( u_n \in W^{1,n}(B,S^{n-1}) \), we know that \( u_n \) is also a minimizer of \( \int_B |\nabla u|^n \) and

\[ \lim_{\varepsilon_k \to 0} \int_B |\nabla u_{\varepsilon_k}|^n = \int_B |\nabla u_n|^n = \int_B |\nabla u^*|^n, \quad (7.29) \]

which and (4.9) imply

\[ \nabla u_{\varepsilon_k} \to \nabla u_n, \quad \text{in} \quad L^n(B,\mathbb{R}^n). \]

Combining this with the fact

\[ u_{\varepsilon_k} \to u_n, \quad \text{in} \quad L^n(B,\mathbb{R}^n), \]

which can be deduced from (4.6), we derive

\[ u_{\varepsilon_k} \to u_n, \quad \text{in} \quad W^{1,n}(B,\mathbb{R}^n). \]

Then it is not difficult to complete the proof of this theorem.

8 The proof of (7.5)

To prove (7.5), we will introduce a comparison function first. Consider the functional

\[ E(\rho,B) = \frac{1}{n} \int_B (|\nabla \rho|^2 + 1)^{n/2} + \frac{1}{2\varepsilon^n} \int_B (1 - \rho)^2. \]

It is easy to prove that the minimizer \( \rho_1 \) of \( E(\rho,B) \) on \( W^{1,n}_{|u_\varepsilon|}(B,R^+) \) exists and satisfies

\[ -\text{div} (\mu^{(n-2)/2} \nabla \rho) = \frac{1}{\varepsilon^n} (1 - \rho) \quad \text{on} \quad B, \quad (8.2) \]

\[ \rho|_{\partial B} = |u_\varepsilon|, \quad (8.3) \]

where \( \nu = |\nabla \rho|^2 + 1 \). Since \( 1/2 \leq |u_\varepsilon| \leq 1 \) on \( B \), it follows from the maximum principle that

\[ 1/2 \leq |u_\varepsilon| \leq \rho_1 \leq 1 \quad (8.4) \]

on \( \overline{B} \).

Applying (4.6) we see easily that

\[ E(\rho_1,B) \leq E(|u_\varepsilon|,B) \leq CE_\varepsilon(u_\varepsilon,B) \leq C. \quad (8.5) \]
Multiplying (8.2) by \((\nu \cdot \nabla \rho)\), where \(\rho = \rho_1\), and integrating over \(B\), we obtain
\[
- \int_{\partial B} v^{(n-2)/2}(\nu \cdot \nabla \rho)^2 + \int_B v^{(n-2)/2} \nabla \rho \cdot \nabla (\nu \cdot \nabla \rho) = \frac{1}{\varepsilon n} \int_B (1-\rho)(\nu \cdot \nabla \rho),
\]
where \(\nu\) denotes the unit outside norm vector on \(\partial B\). Using (8.8), we obtain for any \(\nu\)
\[
\int_{\partial B} v^{(n-2)/2} \nabla \rho (\nu \cdot \nabla \rho) \leq C \int_B v^{(n-2)/2} (\nabla \rho)^2 + \frac{1}{2} \int_B v^{(n-2)/2} \nu \cdot \nabla v \leq C + \frac{1}{n} \int B |\text{div}(\nu \rho)| - v^{n/2} d\nu
\]
\[
C + \frac{1}{n} \int_{\partial B} v^{n/2}.
\]
Combining (8.3)(7.1) and (8.5) we also have
\[
|\frac{1}{n} \int_{\partial B} (1-\rho)(\nu \cdot \nabla \rho)| \leq \frac{1}{2n} \int_{\partial B} (1-\rho)^2 d\nu + \int_{\partial B} (1-\rho)^2 \leq C.
\]
Substituting this and (8.7) into (8.6) yields
\[
|\int_{\partial B} v^{(n-2)/2} (\nu \cdot \nabla \rho)^2| \leq C + \frac{1}{n} \int_{\partial B} v^{n/2}.
\]
Applying (8.3)(7.1) and (8.8), we obtain for any \(\delta \in (0,1)\),
\[
\int_{\partial B} v^{n/2} = \int_{\partial B} v^{(n-2)/2} [1 + (\tau \cdot \nabla \rho)^2 + (\nu \cdot \nabla \rho)^2]
\]
\[
= \int_{\partial B} v^{(n-2)/2} [1 + (\tau \cdot \nabla |u|)^2 + (\nu \cdot \nabla \rho)^2]
\]
\[
\leq \int_{\partial B} v^{(n-2)/2} + \int_{\partial B} v^{(n-2)/2} (\nu \cdot \nabla \rho)^2 + \int_{\partial B} v^{(n-2)/2} (\tau \cdot \nabla |u_i|)^{2/n}
\]
\[
\leq C(\delta) + \left(\frac{1}{n} + 2\delta\right) \int_{\partial B} v^{n/2},
\]
where \(\tau\) denotes the unit tangent vector on \(\partial B\). Hence it follows by choosing \(\delta > 0\) so small that
\[
\int_{\partial B} v^{n/2} \leq C.
\]
Now we multiply both sides of (8.2) by \((1-\rho)\) and integrate over \(B\). Then
\[
\int_B v^{(n-2)/2} |\nabla \rho|^2 + \frac{1}{\varepsilon n} \int_B (1-\rho)^2 = - \int_{\partial B} v^{(n-2)/2} (\nu \cdot \nabla \rho)(1-\rho).
\]
From this, using (7.1)(8.3) and (8.9) we obtain
\[
E(\rho_1, B) \leq C |(\nu \cdot \nabla \rho)(1-\rho)|
\]
\[
\leq C |\int_{\partial B} v^{n/2} (|u|)^{2/n} |\int_{\partial B} (1-\rho)^2 |^{1/n}
\]
\[
\leq C |\int_{\partial B} (1-|u|)^{2/n} |^{1/n} \leq C\varepsilon
\]
Since $u_\varepsilon$ is a minimizer of $E_\varepsilon(u, B)$, we have

\[ E_\varepsilon(u_\varepsilon, B) \leq E_\varepsilon(\rho_1 w, B) \]

\[ = \frac{1}{n} \int_B (|\nabla \rho_1|^2 + \rho_1^2 |\nabla w|^2)^{n/2} + \frac{1}{2n} \int_B (1 - \rho_1^2)^2, \tag{8.11} \]

where $w$ is a solution of (7.3). On hand,

\[ \int_B (|\nabla \rho_1|^2 + \rho_1^2 |\nabla w|^2)^{n/2} dx - \int_B (\rho_1^2 |\nabla w|^2)^{n/2} dx \]

\[ = \frac{n}{2} \int_B \int_0^1 \left( (|\nabla \rho_1|^2 + \rho_1^2 |\nabla w|^2)^{(n-2)/2} + (\rho_1^2 |\nabla w|^2)^{(n-2)/2}(1-s) \right) ds |\nabla \rho_1|^2 dx \]

\[ \leq C \int_B (|\nabla \rho_1|^n + |\nabla \rho_1|^2 |\nabla w|^{(n-2)/2}) dx. \tag{8.12} \]

On the other hand, by using (8.10) and (7.4) we have

\[ \int_B |\nabla \rho_1|^2 |\nabla w|^{(n-2)/2} \leq \left( \int_B |\nabla \rho_1|^{4+2n/(n+2)} \right)^{(n+2)/2n} \left( \int_B |\nabla w|^n \right)^{(n-2)/2n} \leq C \varepsilon^\lambda. \tag{8.13} \]

Combining (8.11)-(8.13), we can derive

\[ E_\varepsilon(u_\varepsilon, B) \leq \frac{1}{n} \int_B \rho_1^n |\nabla w|^n + C \varepsilon^\lambda, \]

where $\lambda$ is a constant only depending on $n$. Thus (7.5) can be seen by (8.4).

References


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