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Semilinear elliptic problems on unbounded subsets of the Heisenberg group *

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Abstract

In this paper we discuss the applications of an abstract version of concentration compactness to minimax problems. In particular, we prove the existence of solutions to semilinear elliptic problems on unbounded subsets of the Heisenberg group.

1 Introduction

Heisenberg Laplacian is a subelliptic differential operator defined as follows. Let $\mathbb{H}^N$ be the space $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, whose elements we denote as $\eta = (\alpha, \beta, \tau)$, $\eta = (x, y, t)$, etc, equipped with the group operation

$$\eta \circ \eta' = (\alpha + \alpha', \beta + \beta', \tau + \tau' + 2(\alpha\beta' - \beta\alpha')). \tag{1.1}$$

This group multiplication endows $\mathbb{H}^N$ with a structure of a Lie group. The Laplacian $\Delta_{\mathbb{H}}$ is obtained from the vector fields $X_i = \partial_{x_i} + 2y_i\partial_t$, $Y_i = \partial_{y_i} - 2x_i\partial_t$, $i = 1, \dots, N$, as

$$\Delta_{\mathbb{H}} := \sum_{i=1}^{N} X_i \circ X_i + Y_i \circ Y_i = \sum_{i=1}^{N} \partial_{x_i}^2 + \partial_{y_i}^2 + 4y_i\partial_{x_i}\partial_t - 4x_i\partial_{y_i}\partial_t + 4(x_i^2 + y_i^2)\partial_t^2. \tag{1.2}$$

It can be also defined as an operator associated with a quadratic form $-a$ in $L^2(\mathbb{H}^N)$, where

$$a(u) = \int_{\mathbb{H}^N} |Xu|^2 + |Yu|^2 \tag{1.3}$$

(the left Haar measure of the Heisenberg group is the Lebesgue measure; we also consider $\mathbb{H}^N$ as endowed with the topology of $\mathbb{R}^{2N+1}$). Let D be the group of linear operators on $L^2(\mathbb{H}^N)$ defined by left shifts:

$$(g_\eta u)(x) = u(\eta \circ x), \eta \in \mathbb{H}^N. \tag{1.4}$$

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These operators commute with $\Delta_{\mathbb{H}}$, the form a and the $L^p(\mathbb{H}^N)$-norms are invariant under D. Following [4] we define the following Hilbert space associated with the norm given by $a(u) + \|u\|^2_{L^2}$: if Ω is an open subset of the Heisenberg group, $\overset{\circ}{S}{}^1_2(\Omega)$ is a closure of $C_0^\infty(\Omega)$ in the norm given by (1.3). The space $S^1_2(\mathbb{H}^N)$([3]) is identical with $\overset{\circ}{S}{}^1_2(\Omega)$ for $\Omega = \mathbb{H}^N$. Due to the Folland-Stein ([3]) inequality

$$a(u) \geq S\|u\|^2_{2^*}, S > 0, 2^* = 2 + 2/N. \tag{1.5}$$

and the Hölder inequality, $\overset{\circ}{S}{}^1_2(\Omega)$ is imbedded into L^p with $2 \leq p \leq 2^*$. For unbounded domains the imbedding is generally not compact.

In this paper we study the existence of solutions in the Dirichlet problem for the equation $-\Delta_{\mathbb{H}} u = f(u)$ on (generally unbounded) open subsets of $\mathbb{H}^N$. Existence results for semilinear problems on bounded subsets of Heisenberg group with subcritical nonlinearities (where compactness of imbedding into L^p is used) can be found in [4], [2], [1] and [12]. This paper deals with the non-compact case by using the concentration compactness approach ([5],[6]) in its abstract version ([9]).

Section 2 deals with the concentration compactness for Heisenberg shifts, structure of Palais-Smale sequences and geometric conditions for unbounded domains to which we extend the concentration-compactness argument (the class of "asymptotically contractive trace domains"). In Section 3 the results of Section 2 are applied to the semilinear problems.

2 Abstract concentration compactness in application to the Heisenberg group

Definition 2.1. Let H be a separable Hilbert space. We say that a group D of unitary operators on H is a *group of dislocations* if

$$u_k \in H, u_k \rightharpoonup 0, g_k \in D, g_k \not\rightharpoonup 0 \Rightarrow \exists \{k_j\} \subset \mathbb{N}, g_{k_j} u_{k_j} \rightharpoonup 0. \tag{2.1}$$

Definition 2.2. Let $u, u_k \in H$. We will say that u_k converges to u *weakly with concentration* (under dislocations D), which we will denote as

$$u_k \overset{D}{\rightharpoonup} u,$$

if for all $\varphi \in H$,

$$\lim_{k\to\infty} \sup_{g\in D} (g(u_k - u), \varphi) = 0. \tag{2.2}$$

Theorem 2.3 ([9]). *Let $u_k \in H$ be a bounded sequence. Then there exists $w^{(n)} \in H$, $g_k^{(n)} \in D$, $k, n \in \mathbb{N}$ such that for a renumbered subsequence*

$$g_k^{(1)} = id,\ {g_k^{(n)}}^{-1} g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m, \tag{2.3}$$

$$w^{(n)} = \text{w-lim}\, {g_k^{(n)}}^{-1} u_k \tag{2.4}$$

$$\sum_{n\in\mathbb{N}} \|w^{(n)}\|^2 \le \limsup \|u_k\|^2 \tag{2.5}$$

$$u_k - \sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)} \overset{D}{\rightharpoonup} 0. \tag{2.6}$$

Let $G \in C^1(H)$ satisfy the following conditions:

(i) G is invariant under D: $\forall g \in D,\ G \circ g = G$,

(ii) G' is weak-to-weak continuous: $u_k \rightharpoonup u \Rightarrow G'(u_k) \rightharpoonup G'(u)$.

We recall that a sequence u_k is called a $(PS)_c$-sequence if $G(u_k) \to c$ and $G'(u_k) \to 0$. The critical set of G will be denoted as $K := \{u \in H : G'(u) = 0\}$.

Lemma 2.4. *Assume that G satisfies (i),(ii). Then any bounded $(PS)_c$-sequence for G has a subsequence satisfying (2.6) with $w^{(n)} \in K$.*

Proof. Let u_k be a PS_c-sequence. We apply Theorem 2.3 to u_k and in what follows use the resulting renumbered subsequence. By (i),(ii)

$${g_k^{(n)}}^{-1} G'(u_k) \rightharpoonup G'(w^{(n)}). \tag{2.7}$$

This implies that $G'(w^{(n)}) = 0$. ☐

This statement does not provide sufficient information for problems on unbounded subsets of $\mathbb{H}^N$ that are not invariant with respect to the Heisenberg group or its subgroups. This matter will be addressed later in this section.

Let D be the group of left actions of $\mathbb{H}^N$ on $\overset{\circ}{S}{}_2^1(\Omega)$:

$$(g_\eta u)(z) = u(\eta \circ z), \eta \in \mathbb{H}^N. \tag{2.8}$$

We will also use the notation $\mathbb{H}_{\mathbb{Z}}^N$ for the subgroup of the Heisenberg group whose elements have integer coordinates and $D_{\mathbb{Z}}$ the correspondent group of actions.

To use Theorem 2.3 we have to show that D (resp $D_{\mathbb{Z}}$) is a group of dislocations. Indeed, note first that if $g_{\eta_k} \not\rightharpoonup 0$ then $\eta_k \in \mathbb{H}^N$ has a bounded subsequence, otherwise, let $\varphi, \psi \in C_0^\infty$ and observe that $g_{\eta_k}\varphi$ and ψ will have disjoint supports for k sufficiently large (the left shifts by Heisenberg group either move one of x_i, y_i to infinity, or, if there is a subsequence where none of the first $2N$ components of η_k goes to infinity, then $t_k \to \infty$). Once η_k has a bounded subsequence, it also has a convergent (renamed) subsequence, and then, with any $\varphi \in C_0^\infty$,

$$(g_{\eta_k} u_k, \varphi) = (g_\eta u_k, \varphi) + (u_k, (g_{-\eta_k} - g_{-\eta})\varphi) \to 0. \tag{2.9}$$

Lemma 2.5. *Let $B = \{(\alpha, \beta, \tau) \in G : |\alpha_i| < 1/2, |\beta_i| < 1/2, i = 1, \dots, N, |\tau| < 1/2\}$. Then the sets $\{\eta \circ \overline{B}, \eta \in \mathbb{H}^N_{\mathbb{Z}}\}$ cover $\mathbb{H}^N$ with multiplicity not exceeding 2^{2N+1}.*

Proof. It suffices to show that for each point $\eta \in \mathbb{H}^N$ there is at least one and at most 2^{2N+1} points $\eta' \in \mathbb{H}^N_{\mathbb{Z}}$ such that $\eta' \circ \eta \in \overline{B}$. Indeed, each of the inequalities $|\alpha_i' + \alpha_i| \le 1/2$, $|\beta_i' + \beta_i| \le 1/2$ are satisfied with at least one and at most two integer values α' (resp. β'), which implies that the remaining inequality $|\tau' + \tau + 2(\alpha'\beta - \alpha\beta'|) \le 1/2$ is also satisfied with at least one and, for each choice of α, β two integer values of τ'. ☐

Lemma 2.6. *Let $\Omega \subset \mathbb{H}^N$ be an invariant domain with respect to the subgroup of $\mathbb{H}^N$ with integer coordinates. Let $q \in (2, 2^*)$ and let $u_k \in \overset{\circ}{S}{}^1_2(\Omega)$ be a bounded sequence. Then*

$$u_k \overset{D_{\mathbb{Z}}}{\rightharpoonup} 0 \Leftrightarrow u_k \to 0 \text{ in } L^q(\Omega). \tag{2.10}$$

Same conclusion holds for $\Omega = \mathbb{H}^N$ if $D_{\mathbb{Z}}$ is replaced by D.

Proof. We prove the lemma for the case of the group of integer shifts $D_{\mathbb{Z}}$. Since $D_{\mathbb{Z}} \subset D$, the concentrated weak convergence in the case of D is not weaker than for $D_{\mathbb{Z}}$ and therefore implies L^q-convergence. The proof of the converse statement for D is a literal repetition of that for $D_{\mathbb{Z}}$.

First, assume that $u_k \to 0$ in $L^q(\Omega)$. Then for every sequence $\eta_k \in \mathbb{H}^N$, $g_{\eta_k} u_k \rightharpoonup 0$ in $L^q(\Omega)$. However, since u_k is bounded in S^1_2-norm, $g_{\eta_k} u_k \rightharpoonup 0$ in $\overset{\circ}{S}{}^1_2(\Omega)$.

Assume now that $u_k \overset{D_{\mathbb{Z}}}{\rightharpoonup} 0$. In what follows we consider elements of $\overset{\circ}{S}{}^1_2(\Omega)$ extended by zero as elements of $\overset{\circ}{S}{}^1_2(\mathbb{H}^N)$. By the Folland-Stein embedding for bounded domains, there is a $C > 0$ such that

$$\int_{\eta \circ B} |u_k|^q \le C \|u_k\|^2_{S^1_2(\eta \circ B)} \Big(\int_{\eta \circ B} |u_k|^q\Big)^{1-2/q}, \eta \in \mathbb{H}^N_{\mathbb{Z}}. \tag{2.11}$$

Due to Lemma 2.5, the sets $\eta \circ \overline{B}$, $\eta \in \mathbb{H}^N_{\mathbb{Z}}$ form a covering of finite multiplicity for Ω, so by adding terms in (2.11) over $\eta \in \mathbb{H}^N_{\mathbb{Z}}$, we obtain

$$\int_{\Omega} |u_k|^q \le C \|u_k\|_{\overset{\circ}{S}{}^1_2(\Omega)} \sup_{\eta \in \mathbb{H}^N_{\mathbb{Z}}} \Big(\int_B |g_{-\eta} u_k|^q\Big)^{1-2/q} \le 2C\Big(\int_B |g_{\eta_k} u_k|^q\Big)^{1-2/q} \tag{2.12}$$

for an appropriately chosen "near-supremum" sequence $\eta_k \in \mathbb{H}^N_{\mathbb{Z}}$. It remains to note that by compactness of imbedding of $\overset{\circ}{S}{}^1_2\,(B)$ into $L^q(B)$, one has $g_{\eta_k} u_k \to 0$ in $L^q(\Omega)$, so that the assertion of the lemma follows from (2.12). ☐

Let B_R denote a closed Euclidean ball in R^{2N+1} of radius R, centered at the origin. We now introduce the following class of domains.

Definition 2.7. An open set $\Omega \subset \mathbb{H}^N$ will be called strongly asymptotically contractive if for every unbounded sequence $\eta_k \in \mathbb{H}^N_Z$ either $|\liminf(\eta_k \circ \Omega)| = 0$ or there is a $\beta \in \mathbb{H}^N$ and a set $Z \subset \mathbb{H}^N$ of zero measure such that, on a renumbered subsequence, for every $R > 0$ the set

$$\overline{B_R} \cap \overline{\limsup(\eta_k \circ \Omega) \setminus Z} \tag{2.13}$$

is a compact subset of $\beta \circ \Omega$.

(We recall that for a sequence of sets X_k, $\liminf X_k := \cup_n \cap_{k\geq n} X_k$ and $\limsup X_k := \cap_n \cup_{k\geq n} X_k$.)

This is a stronger requirement than asymptotic contractiveness defined in a similar context in [9] for the Euclidean case. Open bounded sets are strongly asymptotically contractive: if Ω is bounded and η_k is an unbounded sequence, then $\limsup(\eta_k \circ \Omega) = \emptyset$. Another example of a strongly asymptotically contractive domain will be $\Omega = \{(x, y, t) \in \mathbb{H}^N : x^2 + y^2 < f(t)\}$ with a continuous function f, such that $f(t) < f_\infty := \lim_{s\to\pm\infty} f(s)$. In this case all the correspondent lim sup lie in the closure of $\{(x, y, t) \in \mathbb{H}^N : x^2+y^2 \leq f_\infty\}$ up to a constant group shift. If one modifies the definition of strong asymptotic contractivenes by using different groups on $\mathbb{R}^{2N+1}$, the property obviously becomes dependent on the group. For example, the set $\{(x, y, t) \in \mathbb{R}^3 : y^2 + t^2 < x^2/(1 + x^2)\}$ is not asymptotically contractive with respect to parallel translations, but all its limit sets with respect to unbounded sequences of $\mathbb{H}^1$-shifts have zero measure.

Remark 2.8. Although $\overset{\circ}{S}{}^1_2(\Omega)$ is not necessarily $D_\mathbb{Z}$-invariant, dislocated weak limits for sequences on $\overset{\circ}{S}{}^1_2(\Omega)$ are well defined (up to extraction of subsequence and a constant group shift), since by definition, $\varphi \in C_0^\infty(\Omega) \Rightarrow \varphi((-\beta)\circ\eta_k^{(n)}\circ\cdot) \in C_0^\infty(\Omega)$ for sufficiently large k, and moreover, regarding u_k as a sequence in $\overset{\circ}{S}{}^1_2(\mathbb{H}^N)$, $B_R \cap \operatorname{supp} \text{w-lim}\, u_k((-\eta_k^{(n)}) \circ \beta \circ \cdot)$ is a compact subset of Ω, so that the dislocated weak limit is an element of $\overset{\circ}{S}{}^1_2(\Omega)$.

3 Application to elliptic problems

Let $F \in C^1(\mathbb{R})$, $f(s) = F'(s)$,

$$a_1(u) := a(u) + \int u^2,\ g(u) := \int F(u), u \in \overset{\circ}{S}{}^1_2(\Omega) \tag{3.1}$$

and let

$$\sigma := \sup_{u\in \overset{\circ}{S}{}^1_2(\Omega)} g(u)/a_1(u). \tag{3.2}$$

Assume that

$$\lim_{|s|\to\infty} |f(s)|/|s|^{2^*} \to 0 \tag{3.3}$$

and

$$\lim_{s\to 0}\frac{f(s)}{s}=0, \tag{3.4}$$

Let

$$G(u):=a_1(u)-2g(u), u\in \overset{\circ}{S}{}_2^1(\Omega). \tag{3.5}$$

It is clear that $G\in C^1(\overset{\circ}{S}{}_2^1(\Omega))$.

Lemma 3.1. *Let $\Omega\subset\mathbb{H}^N$ be a strongly asymptotically contractive open set. Then any bounded $(PS)_c$-sequence for G has a convergent subsequence.*

Proof. Let u_k be a bounded PS_c-sequence. Consider the asymptotic decomposition of its extension to $\overset{\circ}{S}{}_2^1(\mathbb{H}^N)$ by Theorem 2.3 and note that $w^{(1)}=\text{w-lim}\,u_k$ in the sense of $\overset{\circ}{S}{}_2^1(\Omega)$, while for $n>1$, the essential support of $w^{(n)}$ intersected with any closed ball is a compact subset of Ω. Let $\phi\in C_0^\infty(\Omega)$. Then $\phi(\eta_k^{n)}\circ\cdot)\in C_0^\infty(\Omega)$ for k sufficiently large. Then, understanding G' as an extended map $\overset{\circ}{S}{}_2^1(\Omega)\to\overset{\circ}{S}{}_2^1(\mathbb{H}^N)$, due to Remark 2.8 we have $(G'(u_k),\phi(\eta_k^{n)}\circ\cdot))\to(G'(w^{(n)}),\phi)=0$, where the terms in the last scalar product can be identified as elements of $\overset{\circ}{S}{}_2^1(\Omega)$. However, for all $n>1$, $\operatorname{supp}w^{(n)}$ has an open complement in Ω, which contradicts to the maximum principle ([4]), unless $w^{(n)}=0$. Since all dislocated weak limits $w^{(n)}$, $n>1$, of u_k equal zero, by Theorem 2.3 and Lemma 2.6, $u_k\to w^{(1)}$ in $L^p(\mathbb{H}^N)$ for any $p\in(2,2^*)$. Since $u_k\in\overset{\circ}{S}{}_2^1(\Omega)$, one has also convergence in $L^p(\Omega)$. Since $G'(u_k)\to 0$ in $\overset{\circ}{S}{}_2^1(\Omega)$, u_k is also convergent in $\overset{\circ}{S}{}_2^1(\Omega)$. ☐

Theorem 3.2. *Let $\Omega\subset\mathbb{H}$ be a $\mathbb{H}^N_{\mathbb{Z}}$-invariant or a strongly asymptotically contractive open set. Assume (3.3) and (3.4). If, in addition*

$$\sigma>1, \tag{3.6}$$

then for every $\epsilon>0$ there exists a $\eta\in[1-\epsilon,1]$ and a $u\in\overset{\circ}{S}{}_2^1(\Omega)\setminus\{0\}$ satisfying

$$-\Delta_{\mathbb{H}}u+u=\eta f(u). \tag{3.7}$$

Note that condition (3.6) is satisfied if $F(s)/s^2\to\infty$ when $s\to+\infty$.

Proof. It is easy to see that the functional G has the classical mountain pass geometry (the proof repeats with trivial modifications similar calculations for semilinear equations with the Laplace operator and can be omitted): $G(0)=0$, $G(u)\ge\frac12 a_1(u)$ for all $a_1(u)$ sufficiently small (due to 3.4), and by (3.6) there is a $e\in\overset{\circ}{S}{}_2^1(\Omega)$ such that $G(e)<0$. Let $\lambda>1$ and let $G_\lambda=\lambda a_1-2g$. Let

$$\Phi_\lambda=\{\varphi\in C([0,1]):\varphi(0)=0,\varphi(1)=e,G_\lambda(\varphi([0,1]))\le s\},\ s\in\mathbb{R}. \tag{3.8}$$

We assume that $s = s(\lambda)$ is sufficiently large so that $\Phi \neq \emptyset$. Let

$$c_\lambda := \inf_{\varphi \in \Phi_\lambda} \max G(\varphi([0,1])). \tag{3.9}$$

By Theorem 2.1 in [10] there is a sequence $u_k \in \overset{\circ}{S}{}^1_2(\Omega)$ and $\theta_k \in [0,1]$ satisfying

$$(1-\theta_k)G'(u_k) + \theta_k G'_\lambda(u_k) \to 0, G(u_k) \to c_\lambda > 0, \tag{3.10}$$

and such that

$$G_\lambda(u_k) \leq s. \tag{3.11}$$

Note now that from $G(u_k) \to c_\lambda$ and (3.11) follows immediately that $\|u_k\|^2 \leq (s - c_\lambda)/\lambda$. On a renumbered subsequence such that $\theta_k \to \theta \in [0,1]$ one has therefore

$$u_k - \eta g'(u_k) \to 0, \tag{3.12}$$

with $\eta = (1 + \theta(\lambda - 1))^{-1}$. If Ω is $\mathbb{H}^N_{\mathbb{Z}}$-invariant, then by Lemma 2.4, the dislocated weak limits $w^{(n)}$ of u_k satisfy the equation $w^n = g'(w^{(n)})$. If $w^n = 0$ for all n, then by Theorem 2.3, $u_k \overset{D}{\rightharpoonup} 0$ and by Lemma 2.6 $u_k \to 0$ in $L^q(\Omega)$ for any $q \in (2, 2^*)$. Then $g'(u_k) \to 0$ and so, by (3.12), $u_k \to 0$ in $\overset{\circ}{S}{}^1_2(\Omega)$, in which case $G(u_k) \to 0$ which contradicts (3.10). Therefore at least one of the dislocated limits is a non-zero critical point. The theorem is proved for this case.

In case when Ω is strongly asymptotically contractive, due to Lemma 3.1 u_k converges to its weak limit $w^{(1)}$ and $w^{(1)} = g'(w^{(1)})$. Then $G(w^{(1)}) \neq 0$ due to (3.10). ☐

Corollary 3.3. *Assume in addition to the conditions of Theorem 3.2 that there exists a $\mu > 2$ such that*

$$f(s)s > \mu F(s) \tag{3.13}$$

for s sufficiently large. Then there exists a $u \in \overset{\circ}{S}{}^1_2(\Omega) \setminus \{0\}$ satisfying

$$-\Delta_{\mathbb{H}} u + u = f(u). \tag{3.14}$$

Note that in this case the condition (3.6) follows from (3.13).

Proof. We will now allow λ from the previous proof to vary. Let u_λ be the corresponding critical point. By considering appropriate linear combination of equations $(G'(u_\lambda), u_\lambda) = 0$ and $G(u_\lambda) = c_\lambda \downarrow c_\infty > 0$, it is easy to deduce from (3.13) that u_λ are bounded in the $\overset{\circ}{S}{}^1_2(\Omega)$-norm for all λ close to 1. Let us consider now a sequence $\lambda_k \to 1$ and a corresponding sequence u_{λ_k}. If Ω is $\mathbb{H}^N_{\mathbb{Z}}$-invariant, applying Theorem 2.3 to u_{λ_k} we obtain that all dislocated weak limits of this sequence are critical points for G. If all of them equal zero, then $u_{\lambda_k} \overset{D}{\rightharpoonup} 0$, and repeating the argument at the end of the previous proof we conclude that (on a renamed subsequence) $G(u_{\lambda_k}) \to 0$, a contradiction. In the remaining case, when Ω is strongly asymptotically contractive, we apply Lemma 3.1 to u_{λ_k}, in which case it is a convergent sequence, and so its limit is a non-zero critical point. ☐

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