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# Local and global estimates for solutions of systems involving the p-Laplacian in unbounded domains *

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**Abstract**

In this paper, we study the local and global behavior of solutions of systems involving the p-Laplacian operator in unbounded domains. We extend some Serrin-type estimates which are known for simple equations to systems of equations.

## 1 Introduction

We consider the system

$$-\Delta_p u = f(x,u,v) \quad x \in \Omega, \tag{1.1}$$

$$-\Delta_q v = g(x,u,v) \quad x \in \Omega, \tag{1.2}$$

$$u = v = 0 \quad x \in \partial\Omega. \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is an exterior domain, $f, g$ are a given functions depending of the variables $x, u, v$ and $\Delta_p$ is the $p$-Laplacian operator; for $1 < p < +\infty$ $\Delta_p$ is defined by $\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$. Here, we study the local and global behavior of solutions of System (1.1)–(1.3). we follow the work of Serrin [4] concerning the quasilinear equation

$$\operatorname{div} \mathcal{A}(x,u,u_x) = \mathcal{B}(x,u,u_x), \tag{1.4}$$

where $\mathcal{A}$ and $\mathcal{B}$ are a given functions depending of the variables $x, u, u_x$ and $u_x = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n})$. In particular, (1.4) generalizes the equation

$$-\Delta_p u = f(x,u) \quad x \in \Omega. \tag{1.5}$$

In [4], Serrin proves that if the function $f$ is bounded by the term $a|u|^{p-1} + g$, where $p > 1$ is a fixed exponent, $a$ is a positive constant and $g$ is a measurable function, then for each $y \in \Omega$ and $R > 0$ we have the estimate

$$\sup_{B_R(y)} u(x) \le cR^{-\frac{N}{p}} \left( \|u\|_{L^p(B_{2R}(y))} + R^{\frac{N}{p}} (R^{\epsilon} \|g\|_{L^{\frac{N}{p-\epsilon}}(B_{2R}(y))})^{\frac{1}{p-1}} \right) \tag{1.6}$$

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for all $0 < \epsilon \leq 1$.

In many cases, especially for unbounded domain, when we wish to show that the solution decay at infinity, the estimate (1.6) requires that the function $f$ belongs to $L^\alpha(\Omega)$ with $\alpha > N/p$, which is not trivial to prove in some cases. To avoid this difficulty Yu [5], Egnell [1] and others have proved that the solution of (1.5) have a regularity $L^q(\Omega)$ for each $q \geq p^*$, and this for all function $f$ bounded by a sublinear, superlinear or an homogeneous terms. We note that in the case of a mixed terms this last technique cannot be adapted. For the case of an homogeneous system see the paper of Fleckinger, Manàsevich, Stavrakakis and de Thélin [2].

The first part of this paper is devoted to the local behavior of solutions of System (1.1)–(1.3). We obtain an estimate of Serrin type in the following cases:
1) $f$ and $g$ are bounded by a sum of homogeneous and critical terms.
2) $f$ and $g$ are bounded by a sum of homogeneous and constant terms.
Thus, we extend the results of [5], [1] concerning Equation and those of [2] concerning System.

In the second part, we obtain a global estimates of solutions of System (1.1)–(1.3) in the particular case $f = A|u|^{\alpha-1}u|v|^{\beta+1}$ and $g = B|u|^{\alpha+1}|v|^{\beta-1}v$ under some conditions on $\alpha, \beta, p$ and $q$. Also we obtain another global estimate when $f$ and $g$ satisfy 2).

We recall that $\mathcal{D}^{1,p}(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}.$$

$p' = \frac{p}{p-1}$ is the conjugate of $p$, $p* = \frac{Np}{N-p}$ is the Sobolev exponent and we define $S_p$ by

$$\frac{1}{S_p} = \inf\left\{\frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}} \quad u \in W^{1,p}(\Omega)\backslash\{0\}\right\}.$$

# 2 Local estimates for solutions of (1.1)–(1.3)

**Theorem 2.1** *Let* $(u, v) \in \mathcal{D}^{1,p}(\mathbb{R}^N) \times \mathcal{D}^{1,q}(\mathbb{R}^N)$ *be a solution of* (1.1) − (1.3) *and* $\tau = \frac{N}{N-p}$, $\bar\tau = \frac{N}{N-q}$. *Assume that* $\max\{p, q\} < N$, $q \geq p$ *and*

$$|f(x, u, v)| \leq C\left(|u|^{p-1} + |u|^{p^*-1} + |v|^{q/p'} + |v|^{\frac{\tau q}{(\tau p)'}}\right), \tag{2.1}$$

*and*

$$|g(x, u, v)| \leq C\left(|v|^{q-1} + |v|^{\tau q-1} + |u|^{p/q'} + |u|^{\frac{\tau p}{(\tau q)'}}\right), \tag{2.2}$$

*where $m'$ is the conjugate of $m$ and $C$ is a constant. Then*
*1) For any $R > 0$ and $x \in \mathbb{R}^N$ satisfying*

$$\begin{aligned} &C \max\left\{2^p S_p \tau^{p-1}, 2^{2q-p} S_q |B_1|^{\frac{q-p}{N}} R^{q-p} \tau^{q-1}\right\} \\ &\quad\times\left(\|u\|^{p(\tau-1)}_{L^{p^*}(B_{2R}(x))} + \|v\|^{q(\tau-1)}_{L^{q\tau}(B_{2R}(x))}\right) < 1 \end{aligned} \tag{2.3}$$

*where $S_p$ and $S_q$ are the Sobolev constants, we have*

$$\begin{aligned}&\|u\|_{L^\infty(B_{\frac{R}{2}}(x))}\\ &\leq c\,(1+R^q)^{\frac{N(N-p)}{p^3}}\max\Big\{R^{\frac{p-N}{p}}\|u\|_{L^{p^*}(B_R(x))},R^{\frac{q-N}{p}}\|v\|^{q/p}_{L^{q^*}(B_R(x))}\Big\}.\end{aligned}$$

*and*

$$\begin{aligned}&\|v\|_{L^\infty(B_{\frac{R}{2}(x)})}\\ &\leq c\,(1+R^q)^{\frac{N(N-p)}{qp^2}}\max\Big\{R^{\frac{q-N}{q}}\|v\|_{L^{q^*}(B_R(x))},R^{\frac{p-N}{q}}\|u\|^{\frac{p}{q}}_{L^{p^*}(B_R(x))}\Big\}.\end{aligned}$$

*witch $c$ independent of $u, v, x$ and $R$.*
*2) Moreover,*

$$\lim_{|x|\to+\infty}u(x)=\lim_{|x|\to+\infty}v(x)=0.$$

**Remark 2.2** There exists an $R_0$ such that for all $R < R_0$, (2.3) is satisfied uniformly for all $x \in \Omega$. This follows from the absolute continuity of the functionals $A \mapsto \int_A |u|^{p^*}dx$ and $A \mapsto \int_A |v|^{q\tau}dx$. To be more specific, for each $\epsilon > 0$ there exists $\eta > 0$ such that for all $R > 0$ and $x \in \mathbb{R}^N$ satisfying $|B_R(x)| \leq \eta$, we have $\int_{B_R(x)} |u|^{p^*}dx < \epsilon$ and $\int_{B_R(x)} |v|^{q\tau}dx < \epsilon$.

**Proof** Let $x \in \mathbb{R}^N$ be fixed. For $y \in B_{2R}(x)$ and any function $h$ defined on $B_{2R}(x)$ we define

$$\tilde{h}(t)=h(y),\quad t=\frac{y-x}{R}.$$

Since $(u, v)$ is a solution for (1.1)–(1.3), then $(\tilde{u}, \tilde{v})$ satisfies

$$-\Delta_p\tilde{u}=R^pf(y,\tilde{u},\tilde{v}),\tag{2.4}$$

$$-\Delta_q\tilde{v}=R^qg(y,\tilde{u},\tilde{v}).\tag{2.5}$$

In this proof $c$ denotes a positive constant independent of $u, v, x$ and $R$. For any ball $B \subset B_2(0)$, we have

$$\forall w\in\mathcal{W}^{1,p}_0(B)\quad\|w\|^p_{L^{p\tau}(B)}\leq S_p\|\nabla w\|^p_{L^p(B)},$$

$$\forall w\in\mathcal{W}^{1,q}_0(B)\quad\|w\|^q_{L^{q\tau}(B)}\leq 2^{q-p}|B_1(0)|^{\frac{q-p}{N}}S_q\|\nabla w\|^q_{L^q(B)}.\tag{2.6}$$

$S_p$ and $S_q$ are the Sobolev constants. Let $(m_n)_n$ be a sequence of positive numbers satisfying $\sigma < \infty$ where $\sigma$ is defined below and $(r_n)_n$ a decreasing sequence defined by

$$r_0=2,\quad r_n=2-\frac{1}{\sigma}\sum_{i=0}^{n-1}\Big(\frac{m_i+p}{p}\Big)^{-1/p'},$$

where $R$ is positive and $\sigma = \sum_{i=0}^{\infty} \left(\frac{m_i+p}{p}\right)^{-1/p'}$. We denote by $B_n = B(0, r_n)$ and we define $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ so that $0 \le \eta \le 1$, $\eta = 1$ in $B_{n+1}$, $supp(\eta) \subset B_n$ and

$$|\nabla\eta| \le c\left(\frac{m_n+p}{p}\right)^{1/p'}. \tag{2.7}$$

We multiply (2.4) by $|\tilde{u}|^{m_n}\tilde{u}\eta^q$, and integrate over $B_n$. Using (2.1), we obtain

$$I_1 + I_2 \le R^p\left(I_3 + I_4 + I_5 + I_6\right), \tag{2.8}$$

where

$$\begin{aligned}
I_1 &= (1+m_n)\int_{B_n} \eta^q|\tilde{u}|^{m_n}|\nabla\tilde{u}|^p dx,\\
I_2 &= q\int_{B_n} \eta^{q-1}\nabla\eta.\nabla\tilde{u}|\nabla\tilde{u}|^{p-2}|\tilde{u}|^{m_n}\tilde{u}dx,\\
I_3 &= C\int_{B_n} |\tilde{u}|^{p+m_n}\eta^q dx,\\
I_4 &= C\int_{B_n} |\tilde{u}|^{p^*+m_n}\eta^q dx,\\
I_5 &= C\int_{B_n} |\tilde{u}|^{m_n}\tilde{u}|\tilde{v}|^{q/p'}\eta^q dx,\\
I_6 &= C\int_{B_n} |\tilde{u}|^{m_n}\tilde{u}|\tilde{v}|^{\frac{\tau q}{(\tau p)'}}\eta^q dx.
\end{aligned}$$

Since $1+m_n = \frac{(p-1)m_n}{p} + \frac{m_n+p}{p}$, we deduce from Young inequality and the facts $p \le q$, $|\eta| \le 1$, that for any $s > 0$

$$\begin{aligned}
|I_2| \le{}& \frac{qs^{p'}}{p'}\left(\frac{m_n+p}{p}\right)\int_{B_n} \eta^q|\nabla\tilde{u}|^p|\tilde{u}|^{m_n}dx\\
&+\frac{q}{ps^p}\left(\frac{m_n+p}{p}\right)^{-\frac{p}{p'}}\int_{B_n} |\nabla\eta|^p|\tilde{u}|^{m_n+p}dx
\end{aligned}$$

Choosing $s$ such that $\frac{qs^{p'}}{p'} \le \frac{1}{2}$, and using (2.7), we have

$$|I_2| \le \frac{1}{2}I_1 + c\int_{B_n} |\tilde{u}|^{m_n+p}dx. \tag{2.9}$$

We deduce from (2.8) and (2.9)

$$I_1 \le 2R^p\sum_{i=3}^{6} I_i + c\int_{B_n} |\tilde{u}|^{m_n+p}dx. \tag{2.10}$$

Using Sobolev inequality and observing that for any $a \geq 0$ and $b \geq 0$ $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, we have

$$\left\| \eta^{q/p} \tilde{u}^{\frac{m_n+p}{p}} \right\|^p_{L^{p\tau}(B_n)} \leq 2^{p-1} S_p \left( I_7 + I_8 \right), \tag{2.11}$$

where

$$I_7 = (\frac{q}{p})^p \int_{B_n} \eta^{q-p} |\nabla \eta|^p |\tilde{u}|^{m_n+p} dx \leq c \left( \frac{m_n+p}{p} \right)^{p-1} \int_{B_n} |\tilde{u}|^{m_n+p} dx,$$

and

$$I_8 = \left( \frac{m_n+p}{p} \right)^p \int_{B_n} \eta^q |\tilde{u}|^{m_n} |\nabla \tilde{u}|^p dx \leq \left( \frac{m_n+p}{p} \right)^{p-1} I_1,$$

thus we deduce from (2.10) that

$$\left\| \eta^{q/p} \tilde{u}^{\frac{m_n+p}{p}} \right\|^p_{L^{p\tau}(B_n)} \leq \left( \frac{m_n+p}{p} \right)^{p-1} \left( c \int_{B_n} |\tilde{u}|^{m_n+p} dx + 2^p S_p R^p \sum_{i=3}^{6} I_i \right). \tag{2.12}$$

**First step.** We construct the sequences $(p_n)_n$ and $(q_n)_n$ by

$$p_n = p\tau^n, \quad q_n = q\tau^n,$$

and we set

$$m_n = p(\tau^n - 1), and \quad l_n = q(\tau^n - 1).$$

We show that if the condition

$$C \max \left\{ 2^p S_p R^p \tau^{n(p-1)}, 2^{2q-p} |B_1|^{\frac{q-p}{N}} S_q R^q \tau^{n(q-1)} \right\} \times \left( \|\tilde{u}\|^{p(\tau-1)}_{L^{p^*}(B_2)} + \|\tilde{v}\|^{q(\tau-1)}_{L^{q\tau}(B_2)} \right) < 1,$$

is satisfied, the solution $(\tilde{u}, \tilde{v})$ belongs to $L^{p_{n+1}}(B_{n+1}) \times L^{q_{n+1}}(B_{n+1})$.
First, we start by estimating the integrals $(I_i), i = 3, \dots, 6$. We have

$$I_3 = C \int_{B_n} |\tilde{u}|^{p+m_n} \eta^q dx \leq c \|\tilde{u}\|^{p_n}_{L^{p_n}(B_n)}. \tag{2.13}$$

Remarking that $\frac{m_n+1}{p_n} + \frac{\frac{q}{p'}}{q_n} = 1$, we deduce from Hölder inequality that

$$I_5 = C \int_{B_n} |\tilde{u}|^{m_n} \tilde{u} |\tilde{v}|^{q/p'} \eta^q dx \leq c \|\tilde{u}\|^{m_n+1}_{L^{p_n}(B_n)} \|\tilde{v}\|^{q/p'}_{L^{q_n}(B_n)}. \tag{2.14}$$

We write $m_n + p^* = p(\tau - 1) + m_n + p \quad , q = \tau q(\frac{m_n+p}{p_{n+1}})$. Observing that $\frac{m_n+p}{p_{n+1}} + \frac{p(\tau-1)}{p^*} = 1$, we deduce from Hölder inequality

$$\begin{aligned} I_4 =& C \int_{B_n} |\tilde{u}|^{p^*+m_n} \eta^q dx \leq C \int_{B_n} |\tilde{u}|^{p(\tau-1)} |\tilde{u}|^{p+m_n} \eta^{\tau q(\frac{m_n+p}{p_{n+1}})} dx \\ \leq& C \|\tilde{u}\|^{p(\tau-1)}_{L^{p^*}(B_n)} \|\eta^{q/p} \tilde{u}^{\tau^n}\|^p_{L^{p\tau}(B_n)}. \end{aligned} \tag{2.15}$$

Remark that

$$\frac{\tau q}{(\tau p)'} - \frac{q}{p'} = q(\tau-1), \quad \frac{q(\tau-1)}{\tau q} + \frac{m_n+1}{p_{n+1}} + \frac{\frac{q}{p'}}{q_{n+1}} = 1, \tag{2.16}$$

and

$$\tau\frac{m_n+1}{p_{n+1}} + \tau\frac{\frac{q}{p'}}{q_{n+1}} = 1,$$

then from Hölder inequality, we have

$$\begin{aligned} I_6 =&C\int_{B_n} |\tilde{u}|^{m_n}\tilde{u}|\tilde{v}|^{\frac{\tau q}{(\tau p)'}}\eta^q dx \\ \le&C\int_{B_n} |\tilde{v}|^{q(\tau-1)}\eta^{\tau q(\frac{m_n+1}{p_{n+1}})}|\tilde{u}|^{m_n+1}\eta^{\tau q(\frac{\frac{q}{p'}}{q_{n+1}})}|\tilde{v}|^{q/p'}dx \\ \le&C\|\tilde{v}\|^{q(\tau-1)}_{L^{\tau q}(B_n)}\|\eta^{q/p}\tilde{u}^{\tau^n}\|^{p(\frac{1+m_n}{p_n})}_{L^{p\tau}(B_n)}\|\eta\tilde{v}^{\tau^n}\|^{q(\frac{\frac{q}{p'}}{q_n})}_{L^{q\tau}(B_n)}. \end{aligned} \tag{2.17}$$

Substituting $m_n$ by $p(\tau^n-1)$ in (2.12), we obtain

$$\begin{aligned} &\|\eta^{q/p}\tilde{u}^{\tau^n}\|^p_{L^{p\tau}(B_n)} - \tau^{n(p-1)}2^pS_pR^p(I_4+I_6) \\ &\le \tau^{n(p-1)}\left(c\int_{B_n}|\tilde{u}|^{p_n}dx + 2^pS_pR^p(I_3+I_5)\right). \end{aligned} \tag{2.18}$$

It follows from (2.13) - (2.17) and the fact $p\le q$ that

$$\begin{aligned} &\|\eta^{q/p}\tilde{u}^{\tau^n}\|^p_{L^{p\tau}(B_n)} - C2^pS_pR^p\tau^{n(p-1)}\Big(\|\tilde{u}\|^{p(\tau-1)}_{L^{p^*}(B_n)}\|\eta^{q/p}\tilde{u}^{\tau^n}\|^p_{L^{p\tau}(B_n)} \\ &+\|\tilde{v}\|^{q(\tau-1)}_{L^{\tau q}(B_n)}\|\eta^{q/p}\tilde{u}^{\tau^n}\|^{p\frac{(1+m_n)}{p_n}}_{L^{p\tau}(B_n)}\|\eta\tilde{v}^{\tau^n}\|^{q(\frac{\frac{q}{p'}}{q_n})}_{L^{q\tau}(B_n)}\Big) \\ &\le c(1+R^q)\tau^{n(q-1)}\left(\|\tilde{u}\|^{p_n}_{L^{p_n}(B_n)} + \|\tilde{u}\|^{m_n+1}_{L^{p_n}(B_n)}\|\tilde{v}\|^{q/p'}_{L^{q_n}(B_n)}\right). \end{aligned} \tag{2.19}$$

Similarly, we have

$$\begin{aligned} &\|\eta\tilde{v}^{\tau^n}\|^q_{L^{q\tau}(B_n)} - C2^{2q-p}S_q|B_1|^{\frac{q-p}{N}}R^q\tau^{n(q-1)}\Big(\|\tilde{v}\|^{q(\tau-1)}_{L^{q\tau}(B_n)}\|\eta\tilde{v}^{\tau^n}\|^q_{L^{q\tau}(B_n)} \\ &+\|\tilde{u}\|^{p(\tau-1)}_{L^{\tau p}(B_n)}\|\eta\tilde{v}^{\tau^n}\|^{q\frac{(1+l_n)}{q_n}}_{L^{q\tau}(B_n)}\|\eta^{q/p}\tilde{u}^{\tau^n}\|^{p(\frac{\frac{p}{q'}}{p_n})}_{L^{p\tau}(B_n)}\Big) \\ &\le c(1+R^q)\tau^{n(q-1)}\|\tilde{v}\|^{q_n}_{L^{q_n}(B_n)} + cR^q\tau^{n(q-1)}\|\tilde{v}\|^{l_n+1}_{L^{q_n}(B_n)}\|\tilde{u}\|^{p/q'}_{L^{p_n}(B_n)}. \end{aligned} \tag{2.20}$$

Next, we define $\theta_{n+1} = \max\{\|\eta^{q/p}\tilde{u}^{\tau^n}\|^p_{L^{p\tau}(B_n)}, \|\eta\tilde{v}^{\tau^n}\|^q_{L^{q\tau}(B_n)}\}$, and $E_n = \max\{\|\tilde{u}\|^{p_n}_{L^{p_n}(B_n)}, \|\tilde{v}\|^{q_n}_{L^{q_n}(B_n)}\}^{1/p_n}$. Simple computations using Hölder inequality and the definition of $E_n$ and $\theta_n$, show that

$$\begin{aligned} &\theta_{n+1} - C\max\left\{2^pS_pR^p\tau^{n(p-1)}, 2^{2q-p}|B_1|^{\frac{q-p}{N}}S_qR^q\tau^{n(q-1)}\right\} \\ &\times\left(\|\tilde{u}\|^{p(\tau-1)}_{L^{p^*}(B_n)} + \|\tilde{v}\|^{q(\tau-1)}_{L^{q\tau}(B_n)}\right)\theta_{n+1} \le c(1+R^q)\tau^{n(q-1)}E_n^{p_n}. \end{aligned} \tag{2.21}$$

We know that there exists $R_0 > 0$ such that for any $R < R_0$

$$\begin{aligned} C\max\Big\{2^pS_pR^p\tau^{n(p-1)}, 2^{2q-p}|B_2|^{\frac{q-p}{N}}S_qR^q\tau^{n(q-1)}\Big\} \\ \times\Big(\|\tilde u\|^{p(\tau-1)}_{L^{p^*}(B_2)} + \|\tilde v\|^{q(\tau-1)}_{L^{q^\tau}(B_2)}\Big) < 1. \end{aligned} \tag{2.22}$$

Also, remark that

$$\begin{aligned} \theta_{n+1} &\geq \max\{\|\tilde u\|^{p_n}_{L^{p_{n+1}}(B_{n+1})}, \|\tilde v\|^{q_n}_{L^{q_{n+1}}(B_{n+1})}\} \\ &\geq \max\{\|\tilde u\|^{p_{n+1}}_{L^{p_{n+1}}(B_{n+1})}, \|\tilde v\|^{q_{n+1}}_{L^{q_{n+1}}(B_{n+1})}\}^{1/\tau} \\ &= E^{p_n}_{n+1}. \end{aligned} \tag{2.23}$$

Therefore, from (2.21) - (2.23), and the fact $p \leq q$

$$E^{p_n}_{n+1} \leq c(1+R^q)\tau^{n(q-1)}E^{p_n}_n.$$

So

$$E_{n+1} \leq (c(1+R^q))^{1/p_n}\,\tau^{\frac{n(q-1)}{p_n}}E_n.$$

This implies that

$$\|\tilde u\|_{L^{p_{n+1}}(B_{n+1})} \leq E_{n+1} \leq (c(1+R^q))^{\sum_{i=0}^\infty \frac{1}{p\tau^i}}\,\tau^{\sum_{i=0}^\infty \frac{i(q-1)}{p\tau^i}}E_0.$$

Since $\sum_{i=0}^\infty \frac{1}{p\tau^i} = \frac{N}{p^2}$ and $\sum_{i=0}^\infty \frac{i(q-1)}{p\tau^i} < \infty$, we deduce that $\tilde u \in L^{p_{n+1}}(B_{n+1})$. Similarly, we have

$$\|\tilde v\|^{q/p}_{L^{q_{n+1}}(B_{n+1})} \leq \|\tilde v\|^{\frac{q_{n+1}}{p_{n+1}}}_{L^{q_{n+1}}(B_{n+1})} \leq E_{n+1} \leq (c(1+R^q))^{\sum_{i=0}^\infty \frac{1}{p\tau^i}}\,\tau^{\sum_{i=0}^\infty \frac{i(q-1)}{p\tau^i}}E_0,$$

therefore $v \in L^{q_{n+1}}(B_{n+1})$.
**Second step** We remark that hypothesis (2.3) is equivalent to

$$C\max\Big\{2^pS_pR^p\tau^{p-1}, 2^{2q-p}|B_1|^{\frac{q-p}{N}}S_qR^q\tau^{q-1}\Big\}\Big(\|\tilde u\|^{p(\tau-1)}_{L^{p^*}(B_2)} + \|\tilde v\|^{q(\tau-1)}_{L^{q^*}(B_2)}\Big) < 1.$$

We assume that $R, u$ and $v$ satisfy (2.3), which by the first step implies that $(\tilde u, \tilde v) \in L^{p\tau^2}(B_1) \times L^{q\tau^2}(B_1)$. We let $\delta = \frac{\tau^2}{\tau^2-\tau+1}$ and $\chi = \frac{\tau}{\delta}$. It is clear that $1 < \delta < \tau$, and so $\chi > 1$. We construct a sequences $(s_n)_n$ and $(t_n)_n$ by

$$s_n = p\chi^n, \quad t_n = q\chi^n.$$

In this step $m_n$ and $r_n$ are defined by

$$m_n = p\left(\frac{\chi^n}{\delta} - 1\right),$$

and

$$r_0 = 1, \quad r_n = 1 - \frac{1}{2\sigma}\sum_{i=0}^{n-1}\left(\frac{m_i+p}{p}\right)^{-1/p'},$$

which implies $m_n + p = s_n/\delta$. Now, we estimate the integrals $(I_i)_{i=3,\dots,6}$. We have

$$I_3 \le c\|\tilde u\|^{s_n/\delta}_{L^{\frac{s_n}{\delta}}(B_n)} \le c\|\tilde u\|^{s_n/\delta}_{L^{s_n}(B_n)}. \tag{2.24}$$

Remarking that $\frac{m_n+1}{s_n/\delta} + \frac{q/p'}{t_n/\delta} = 1$, it follows from Hölder inequality that

$$I_5 \le c\|\tilde u\|^{m_n+1}_{L^{\frac{s_n}{\delta}}(B_n)}\|\tilde v\|^{q/p'}_{L^{\frac{t_n}{\delta}}(B_n)} \le \|\tilde u\|^{m_n+1}_{L^{s_n}(B_n)}\|\tilde v\|^{q/p'}_{L^{s_n}(B_n)}. \tag{2.25}$$

We have $\frac{p(\tau-1)}{p\tau^2} + \frac{m_n+p}{s_n} = 1$, thus from Hölder inequality we have

$$I_4 \le c\|\tilde u\|^{p(\tau-1)}_{L^{p\tau^2}(B_n)}\|\tilde u\|^{s_n/\delta}_{L^{s_n}(B_n)} \le c\|\tilde u\|^{s_n/\delta}_{L^{s_n}(B_n)}. \tag{2.26}$$

Observing that $\frac{q(\tau-1)}{q\tau^2} + \frac{m_n+1}{s_n} + \frac{q/p'}{t_n} = 1$, it follows from Hölder inequality that

$$\begin{aligned} I_6 &\le c\int_{B_n} |\tilde v|^{q(\tau-1)}|\tilde u|^{m_n+1}|\tilde v|^{q/p'}\,dx \\ &\le c\|\tilde v\|^{q(\tau-1)}_{L^{q\tau^2}(B_n)}\|\tilde u\|^{m_n+1}_{L^{s_n}(B_n)}\|\tilde v\|^{q/p'}_{L^{t_n}(B_n)} \\ &\le c\|\tilde u\|^{m_n+1}_{L^{s_n}(B_n)}\|\tilde v\|^{q/p'}_{L^{t_n}(B_n)}. \end{aligned} \tag{2.27}$$

We deduce from (2.12), (2.24)–(2.27) and the fact $p \le q$ that

$$\left\|\eta^{q/p}\tilde u^{\chi^n/\delta}\right\|^p_{L^{p\tau}(B_n)} \le c\chi^{n(q-1)}\left(1+R^q\right)\left(\|\tilde u\|^{s_n/\delta}_{L^{s_n}(B_n)} + \|\tilde u\|^{m_n+1}_{L^{s_n}(B_n)}\|\tilde v\|^{q/p'}_{L^{t_n}(B_n)}\right) \tag{2.28}$$

Similarly, we have

$$\left\|\eta\tilde v^{\chi^n/\delta}\right\|^q_{L^{q\tau}(B_n)} \le c\chi^{n(q-1)}\left(1+R^q\right)\left(\|\tilde v\|^{t_n/\delta}_{L^{t_n}(B_n)} + \|\tilde v\|^{l_n+1}_{L^{t_n}(B_n)}\|\tilde u\|^{p/q'}_{L^{s_n}(B_n)}\right) \tag{2.29}$$

As in the first step, we let $\Lambda_n = \max\left\{\|\tilde u\|^{s_n}_{L^{s_n}(B_n)}, \|\tilde v\|^{t_n}_{L^{t_n}(B_n)}\right\}^{1/s_n}$
$\Gamma_n = \max\left\{\|\eta^{q/p}\tilde u^{\chi^n/\delta}\|^p_{L^{p\tau}(B_n)}, \|\eta\tilde v^{\chi^n/\delta}\|^q_{L^{q\tau}(B_n)}\right\}$ and
$\Upsilon_n = \max\left\{\|\tilde u\|^{s_n}_{L^{s_n}(B_n)}, \|\tilde v\|^{t_n}_{L^{t_n}(B_n)}\right\}^{\frac{1}{t_n}}$. Simple computations show that

$$\|\tilde u\|^{m_n+1}_{L^{s_n}(B_n)}\|\tilde v\|^{q/p'}_{L^{t_n}(B_n)} \le \min\left\{\Lambda_n^{s_n/\delta}, \Upsilon_n^{t_n/\delta}\right\}, \tag{2.30}$$

and

$$\|\tilde v\|^{l_n+1}_{L^{t_n}(B_n)}\|\tilde u\|^{p/q'}_{L^{s_n}(B_n)} \le \min\left\{\Lambda_n^{s_n/\delta}, \Upsilon_n^{t_n/\delta}\right\}. \tag{2.31}$$

Also, remark that

$$\Gamma_n \ge \max\left\{\|\tilde u\|^{s_n/\delta}_{L^{s_{n+1}}(B_{n+1})}, \|\tilde v\|^{t_n/\delta}_{L^{t_{n+1}}(B_{n+1})}\right\} = \Lambda_{n+1}^{s_n/\delta} = \Upsilon_n^{t_n/\delta}. \tag{2.32}$$

Thus, we deduce from (2.28)–(2.32) that

$$\Lambda_{n+1}^{s_n/\delta} \le c\chi^{n(q-1)}\left(1+R^q\right)\Lambda_n^{s_n/\delta},$$

and so

$$\Lambda_{n+1} \le c^{\delta/s_n}\chi^{\frac{n(q-1)\delta}{s_n}}\left(1+R^q\right)^{\delta/s_n}\Lambda_n.$$

Which implies that

$$\|\tilde{u}\|_{L^{s_n}(B_n)} \le \Lambda_n \le c^{\sum_{i=0}^{\infty}\frac{\delta}{s_i}}\chi^{\sum_{i=0}^{\infty}\frac{i(q-1)\delta}{s_i}}\left(1+R^q\right)^{\sum_{i=0}^{\infty}\frac{\delta}{s_i}}\Lambda_0.$$

Since $\sum_{i=0}^{\infty}\frac{\delta}{s_i} = \frac{\delta\tau}{p(\tau-\delta)}$, and $\sum_{i=0}^{\infty}\frac{i(q-1)\delta}{s_i} < \infty$, then

$$\begin{aligned}\|\tilde{u}\|_{L^{\infty}(B_{\frac{1}{2}})} &\le \lim_{n\to+\infty}\sup\|\tilde{u}\|_{L^{s_n}(B_n)}\\ &\le c\left(1+R^q\right)^{\frac{\delta\tau}{p(\tau-\delta)}}\max\left\{\|\tilde{u}\|_{L^p(B_1)}, \|\tilde{v}\|_{L^q(B_1)}^{q/p}\right\}.\end{aligned}$$

Similarly, we have

$$\Upsilon_{n+1} \le c^{\frac{\delta}{t_n}}\chi^{\frac{n(q-1)\delta}{t_n}}\left(1+R^q\right)^{\frac{\delta}{t_n}}\Upsilon_n$$

As $n$ tends to infinity, we obtain

$$\begin{aligned}\|\tilde{v}\|_{L^{\infty}(B_{\frac{1}{2}})} &\le \lim_{n\to+\infty}\sup\|\tilde{v}\|_{L^{t_n}(B_n)}\\ &\le c\left(1+R^q\right)^{\frac{\delta\tau}{q(\tau-\delta)}}\max\left\{\|\tilde{v}\|_{L^p(B_1)}, \|\tilde{u}\|_{L^q(B_1)}^{\frac{p}{q}}\right\}.\end{aligned}$$

By the imbeddings

$$L^{p^*}(B_1) \subset L^p(B_1) \quad \text{and} \quad L^{q^*}(B_1) \subset L^q(B_1),$$

and the fact

$$\frac{\delta\tau}{\tau-\delta} = \frac{\tau}{(\tau-1)^2} = \frac{N(N-p)}{p^2},$$

we have

$$\|\tilde{u}\|_{L^{\infty}(B_{\frac{1}{2}})} \le c\left(1+R^q\right)^{\frac{N(N-p)}{p^3}}\max\left\{\|\tilde{u}\|_{L^{p^*}(B_1)}, \|\tilde{v}\|_{L^{q^*}(B_1)}^{q/p}\right\},$$

and

$$\|\tilde{v}\|_{L^{\infty}(B_{\frac{1}{2}})} \le c\left(1+R^q\right)^{\frac{N(N-p)}{qp^2}}\max\left\{\|\tilde{v}\|_{L^{p^*}(B_1)}, \|\tilde{u}\|_{L^{q^*}(B_1)}^{\frac{p}{q}}\right\}.$$

Coming back to $(u, v)$ by a simple change of variables, we find

$$\begin{aligned}&\|u\|_{L^{\infty}(B_{\frac{R}{2}(x)})}\\ &\le \quad c\left(1+R^q\right)^{\frac{N(N-p)}{p^3}}\max\left\{R^{\frac{p-N}{p}}\|u\|_{L^{p^*}(B_R(x))}, R^{\frac{q-N}{p}}\|v\|_{L^{q^*}(B_R(x))}^{q/p}\right\}.\end{aligned}$$

and

$$\|v\|_{L^\infty(B_{\frac{R}{2}(x)})} \le c\,(1+R^q)^{\frac{N(N-p)}{qp^2}} \max\left(R^{\frac{q-N}{q}}\|v\|_{L^{q^*}(B_R(x))}, R^{\frac{p-N}{q}}\|u\|^{\frac{p}{q}}_{L^{p^*}(B_R(x))}\right\}.$$

The proof of 2) follows from 1) and Remark 2.2 ♢

**Proposition 2.3** *Let $(u,v)\in\mathcal{D}^{1,p}(\mathbb{R}^N)\times\mathcal{D}^{1,q}(\mathbb{R}^N)$ a solution of (1.1)–(1.3). We assume $q\ge p$,*

$$|f(x,u,v)|\le C\left(|u|^{p-1}+|v|^{q/p'}+1\right), \tag{2.33}$$

*and*

$$|g(x,u,v)|\le C\left(|v|^{q-1}+|u|^{p/q'}+1\right), \tag{2.34}$$

*where $m'$ is the conjugate of $m$. Then*

$$\|u\|_{L^\infty(B_1)}\le c\,(1+R^q)^{\frac{N}{p^2}}\max\left\{1, R^{\frac{p-N}{p}}\|u\|_{L^{p^*}(B_2)}, R^{\frac{q-N}{p}}\|v\|^{q/p}_{L^{q^*}(B_2)}\right\}, \tag{2.35}$$

*and*

$$\|v\|_{L^\infty(B_1)}\le c\,(1+R^q)^{\frac{N}{pq}}\max\left\{1, R^{\frac{p-N}{q}}\|u\|^{\frac{p}{q}}_{L^{p^*}(B_2)}, R^{\frac{q-N}{q}}\|v\|_{L^{q^*}(B_2)}\right\}. \tag{2.36}$$

**Proof** We use the same change of variables as in the proof of Theorem 2.1. Thus, we obtain that $(\tilde u,\tilde v)$ satisfies (2.4) and (2.5). Also we keep the same sequences $(m_n)_n$, $(r_n)_n$, $(B_n)_n$ and the same function $\eta$. We multiply Equation (2.4) by $|\tilde u|^{m_n}\tilde u\eta^q$, and integrate over $B_n$. Using (2.33), we have

$$I_1+I_2\le R^p\left(I_3+I_4+I_5\right), \tag{2.37}$$

where

$$\begin{aligned}
I_1 &= (1+m_n)\int_{B_n}\eta^q|\tilde u|^{m_n}|\nabla\tilde u|^p dx,\\
I_2 &= q\int_{B_n}\eta^{q-1}\nabla\eta.\nabla\tilde u|\nabla\tilde u|^{p-2}|\tilde u|^{m_n}\tilde u dx,\\
I_3 &= C\int_{B_n}|\tilde u|^{p+m_n}\eta^q dx,\\
I_4 &= C\int_{B_n}|\tilde u|^{m_n}\tilde u|\tilde v|^{q/p'}\eta^q dx,\\
I_5 &= C\int_{B_n}|\tilde u|^{m_n}\tilde u\eta^q dx.
\end{aligned}$$

The integrals $I_1, I_2, I_3$ and $I_4$ are the same to those obtained in Theorem 2.1. Simple computations used before show that

$$\left\|\eta^{q/p}\tilde u^{\frac{m_n+p}{p}}\right\|^p_{L^{p\tau}(B_n)}\le\left(\frac{m_n+p}{p}\right)^{p-1}\left(c\int_{B_n}|\tilde u|^{m_n+p}dx+2^pS_pR^p\sum_{i=3}^{5}I_i\right). \tag{2.38}$$

Now, we define $(p_n)_n$ and $(q_n)_n$ by

$$p_n = p\tau^n, \quad q_n = q\tau^n,$$

and let $m_n = p(\tau^n - 1)$, and , $l_n = q(\tau^n - 1)$. Then we estimate the integrals $I_i, i = 3, \dots, 5$. It is clear from (2.13) and (2.14) that

$$I_3 \le c\|\tilde{u}\|^{p_n}_{L^{p_n}(B_n)} \quad \text{and} \quad I_4 \le c\|\tilde{u}\|^{m_n+1}_{L^{p_n}(B_n)}\|\tilde{v}\|^{q/p'}_{L^{q_n}(B_n)}. \tag{2.39}$$

On the other hand

$$\begin{aligned} I_5 &\le C\int_{B_n}|\tilde{u}|^{m_n+1}dx = c\|\tilde{u}\|^{m_n+1}_{L^{m_n+1}(B_n)} \le c|B_n|^{(\frac{1}{m_n}-\frac{1}{p_n})(m_n+1)}\|\tilde{u}\|^{m_n+1}_{L^{p_n}(B_n)} \\ &\le c|B_2|^{\frac{p-1}{p\tau^n}}\|\tilde{u}\|^{m_n+1}_{L^{p_n}(B_n)} \\ &\le c\|\tilde{u}\|^{m_n+1}_{L^{p_n}(B_n)}. \end{aligned} \tag{2.40}$$

We deduce from (2.38)–(2.40) that

$$\begin{aligned} \|\tilde{u}\|^{p_n}_{L^{p_{n+1}}(B_{n+1})} &\le \|\eta^{q/p}\tilde{u}^{\tau^n}\|^p_{L^{p\tau}(B_n)} \\ &\le c\tau^{n(p-1)}\Big(\|\tilde{u}\|^{p_n}_{L^{p_n}(B_n)} \\ &\quad + R^p\left(\|\tilde{u}\|^{p_n}_{L^{p_n}(B_n)} + \|\tilde{u}\|^{m_n+1}_{L^{p_n}(B_n)}\|\tilde{v}\|^{q/p'}_{L^{q_n}(B_n)} + \|\tilde{u}\|^{m_n+1}_{L^{p_n}(B_n)}\right)\Big). \end{aligned} \tag{2.41}$$

Similarly, we have

$$\begin{aligned} \|\tilde{v}\|^{q_n}_{L^{q_{n+1}}(B_{n+1})} &\le \|\eta\tilde{v}^{\tau^n}\|^q_{L^{q\tau}(B_n)} \\ &\le c\tau^{n(q-1)}\Big(\|\tilde{v}\|^{q_n}_{L^{q_n}(B_n)} \\ &\quad + R^q\left(\|\tilde{v}\|^{q_n}_{L^{q_n}(B_n)} + \|\tilde{v}\|^{l_n+1}_{L^{q_n}(B_n)}\|\tilde{u}\|^{p/q'}_{L^{p_n}(B_n)} + \|\tilde{v}\|^{l_n+1}_{L^{q_n}(B_n)}\right)\Big). \end{aligned} \tag{2.42}$$

Following the proof of Theorem 2.1 we let
$E_n = \max\left\{1, \|\tilde{u}_n\|^{p_n}_{L^{p_n}(B_n)}, \|\tilde{v}_n\|^{q_n}_{L^{q_n}(B_n)}\right\}^{1/p_n}$ and
$F_n = \left\{1, \|\tilde{u}_n\|^{p_n}_{L^{p_n}(B_n)}, \|\tilde{v}_n\|^{q_n}_{L^{q_n}(B_n)}\right\}^{\frac{1}{q_n}}$. We obtain

$$\begin{aligned} \|\tilde{u}\|_{L^\infty(B_1)} &\le \lim_{n\to+\infty}\sup\|\tilde{u}\|_{L^{p_n}(B_n)} \le E_n \\ &\le c\left(1+R^q\right)^{\frac{N}{p^2}} E_0 \\ &= c\left(1+R^q\right)^{\frac{N}{p^2}}\max\left\{1, \|\tilde{u}\|_{L^p(B_2)}, \|\tilde{v}\|^{q/p}_{L^q(B_2)}\right\}. \end{aligned} \tag{2.43}$$

$$\begin{aligned}\|\tilde{v}\|_{L^\infty(B_1)} &\le \lim_{n\to+\infty}\sup\|\tilde{v}\|_{L^{q_n}(B_n)} \le F_n \\ &\le c\,(1+R^q)^{\frac{N}{pq}}\,F_0 \\ &= c\,(1+R^q)^{\frac{N}{pq}}\max\left\{1, \|\tilde{u}\|^{\frac{p}{q}}_{L^p(B_2)}, \|\tilde{v}\|_{L^q(B_2)}\right\}.\end{aligned} \tag{2.44}$$

Using a simple change of variables in (2.43) and (2.44) we obtain (2.35) and (2.36).

# 3 Global estimates for solutions of (1.1)–(1.3)

**Proposition 3.1** *Let* $(u, v) \in \mathcal{D}^{1,p}(\Omega) \times \mathcal{D}^{1,q}(\Omega)$ *a solution of (1.1)–(1.3). We assume that there exist a functions* $a, b \in L^1(\Omega) \cap L^\infty(\Omega)$ *and a constant* $C$ *such that*

$$|f(x, u, v)| \le a(x) + C(|u|^{p-1} + |v|^{q/p'}), \tag{3.1}$$

$$|g(x, u, v)| \le b(x) + C(|v|^{q-1} + |v|^{p/q'}), \tag{3.2}$$

*where* $p > 1$, $q > 1$. *Then*
*1)* $(u, v) \in L^\sigma(\Omega) \times L^\eta(\Omega)$ *for all* $(\sigma, \eta) \in [p^*, +\infty) \times [q^*, +\infty)$.
*2)* $\lim_{|x|\to+\infty} u(x) = \lim_{|x|\to+\infty} v(x) = 0$.

**Proof** 1) Let $p_n = p\tau^n$, $q_n = q\tau^n$, $m_n = \tau^n - 1$, $t_n = \tau^n - 1$, $T_k(u) = \max\{-k, \min\{k, u\}\}$ and $w = |T_k(u)|^{pm_n}T_k(u)$, with $k > 0$. Multiplying the equation (1.1) by $w$ and integrating over $\Omega$, we obtain

$$(pm_n + 1)\int_\Omega |\nabla T_k(u)|^p |T_k(u)|^{pm_n}\,dx = \int_\Omega f(x, u, v)w\,dx.$$

Observing that

$$(\frac{1}{m_n+1})^p|\nabla(T_k(u))^{m_n+1}|^p = T_k(u)^{pm_n}|\nabla T_k(u)|^p, \tag{3.3}$$

we deduce from Hölder and Sobolev inequalities that for any $0 < \gamma < 1$, we have

$$\begin{aligned}&\int_\Omega |T_k(u)|^{\tau(pm_n+p)})^{1/\tau} \\ &\le c\left(\|a\|_\infty^{1-\gamma}\|a\|^\gamma_{L^1(\Omega)}\|u\|^{pm_n+1}_{L^{p_n}(\Omega)} + \|u\|^{p_n}_{L^{p_n}(\Omega)} + \|v\|^{q/p'}_{L^{q_n}(\Omega)}\|u\|^{m_n+1}_{L^{p_n}(\Omega)}\right).\end{aligned} \tag{3.4}$$

with $c$ depending from $n$. Letting $k$ tend to infinity in (3.4), we obtain

$$\|u\|^{p_n}_{L^{p_{n+1}}(\Omega)} \le c\left(\|u\|^{pm_n+1}_{L^{p_n}(\Omega)} + \|u\|^{p_n}_{L^{p_n}(\Omega)} + \|v\|^{q/p'}_{L^{q_n}(\Omega)}\|u\|^{m_n+1}_{L^{p_n}(\Omega)}\right). \tag{3.5}$$

We derive from (3.5) that $u \in L^{p_n}(\Omega)$ for all $n \in \mathbb{N}$. Similarly, we prove that $v \in L^{q_n}(\Omega)$ for all $n \in \mathbb{N}$. By interpolation inequality (see [3]) we prove that

$(u,v) \in L^\sigma(\Omega) \times L^\eta(\Omega)$, for all $(\sigma,\eta) \in [p^*,+\infty) \times [q^*,+\infty)$. The proof of 2) follows from Serrin inequality [4] and 1). ♢

Next, we study the sub-homogeneous system

$$-\Delta_p u = B(x)|u|^{\alpha-1}u|v|^{\beta+1}, \tag{3.6}$$

$$-\Delta_q v = C(x)|u|^{\alpha+1}|v|^{\beta-1}v, \tag{3.7}$$

in $\Omega$ an exterior domain or $\mathbb{R}^N$.

**Proposition 3.2** *Assume that $B, C \in L^\infty(\Omega)$ and*

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1, \quad p > 1, \quad q > 1\,.$$

*Then each solution $(u,v) \in \mathcal{D}^{1,p}(\Omega) \times \mathcal{D}^{1,q}(\Omega)$ of the system (3.6), (3.7) satisfies*
***1.*** $(u,v) \in L^\sigma(\Omega) \times L^\eta(\Omega)$ *for all* $(\sigma,\eta) \in [p^*,+\infty[\times[q^*,+\infty[$.
***2.*** $\lim_{|x|\to+\infty} u(x) = 0$ *and* $\lim_{|x|\to+\infty} v(x) = 0$.

**Proof** Let $\tau = \frac{N}{N-p}, \bar\tau = \frac{N}{N-q}$ and $L = 1 - \frac{\alpha+1}{p^*} - \frac{\beta+1}{q^*}$. Assume $q \geq p$, which implies that $\bar\tau \geq \tau$. We define the sequences $(p_n)_n$, $(q_n)_n$ and $(f_n)_n$ by

$$f_{n+1} = \tau(f_n + L - 1) + 1, \quad f_0 = 1,$$
$$p_n = p^* f_n, \quad q_n = q^* f_n.$$

Let $T_k(u) = max\{-k, \min\{k,u\}\}$ for $k > 0$ and $\omega = |T_k(u)|^{pm_n}T_k(u)$, with

$$m_n = (1 - \frac{\alpha+1}{p_n} - \frac{\beta+1}{q_n})\frac{p_n}{p} = f_{n+1} - 1 \tag{3.8}$$

Multiplying (3.6) by $\omega$ and integrating over $\Omega$, we obtain from (3.3) and Sobolev inequality

$$\frac{1}{S_p}(pm_n+1)(\frac{1}{m_n+1})^p(\int_\Omega |T_k(u)|^{\tau(pm_n+p)})^{1/\tau} \leq \|B\|_{L^\infty(\Omega)} \int_\Omega |u|^\alpha |v|^{\beta+1}\omega dx.$$

From the definition of $m_n$ and Hölder inequality, we deduce that

$$(\int_\Omega |T_k(u)|^{p^*(m_n+1)})^{1/\tau} \leq S_p \frac{(m_n+1)^p}{(pm_n+1)} \|B\|_{L^\infty(\Omega)} \|u\|^{\alpha+1+pm_n}_{L^{p_n}(\Omega)} \|v\|^{\beta+1}_{L^{q_n}(\Omega)}.$$

Let $k$ tends to infinity, we have

$$(\int_\Omega |u|^{p^*(m_n+1)})^{1/\tau} \leq S_p \frac{(m_n+1)^p}{(pm_n+1)} \|B\|_{L^\infty(\Omega)} \|u\|^{\alpha+1+pm_n}_{L^{p_n}(\Omega)} \|v\|^{\beta+1}_{L^{q_n}(\Omega)}.$$

$p^*(m_n+1) = p^*(f_{n+1}) = p_{n+1}$, therefore $u \in L^{p_{n+1}}(\Omega)$. To show that $v \in L^{q_{n+1}}(\Omega)$, We consider $\bar w = |T_k(v)|^{qt_n}T_k(v)$, with

$$\begin{aligned} t_n &= (1 - \frac{\alpha+1}{p_n} - \frac{\beta+1}{q_n})\frac{q_n}{q} \\ &= \bar\tau(f_n + L - 1). \end{aligned} \tag{3.9}$$

Proceeding as above, we obtain

$$(\int_\Omega |v|^{q^*(t_n+1)})^{\frac{1}{\bar\tau}} \le S_q \frac{(t_n+1)^q}{(qt_n+1)} \|C\|_{L^\infty(\Omega)} \|u\|^{\alpha+1}_{L^{p_n}(\Omega)} \|v\|^{\beta+1+qt_n}_{L^{q_n}(\Omega)}.$$

Let $\bar q_n = q^*(t_n+1)$. It is clear that $v \in L^{\bar q_n}$, and since

$$\begin{aligned} \bar q_n &= q^*(t_n+1) \\ &= q^*(\bar\tau(f_n+L-1)+1) \\ &\ge q_{n+1}, \end{aligned}$$

then $q_n \le q_{n+1} \le \bar q_n$. By interpolation inequality (see [3]), we deduce that $v \in L^{q_{n+1}}(\Omega)$. 2) follows from Serrin inequality [4] and 1).

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