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Global well-posedness for KdV in Sobolev spaces of negative index *

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Abstract

The initial value problem for the Korteweg-deVries equation on the line is shown to be globally well-posed for rough data. In particular, we show global well-posedness for initial data in $H^s(\mathbb{R})$ for $-3/10 < s$.

1 Introduction

Consider the initial value problem for the Korteweg-deVries (KdV) equation

$$\begin{aligned} \partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x(u^2) = 0, \quad x \in \mathbb{R}, \\ u(0) = \phi, \end{aligned} \tag{1.1}$$

for rough initial data $\phi \in H^s(\mathbb{R}),\ s < 0$. The initial data ϕ and the solution u are assumed to take values in $\mathbb{R}$. This problem is known [9] to be locally well-posed provided $-3/4 < s$. For $s \geq 0$, the local result and L^2 norm conservation imply (1.1) is globally well-posed [1]. Recently, a direct adaptation [7] of Bourgain's high-low frequency technique [3], [2] showed (1.1) is globally well-posed for $\phi \in H^s \cap \dot{H}^a$ for certain $s, a < 0$. A modification of the high-low frequency technique, first used in [8], is presented in this paper which establishes global well-posedness of (1.1) in $H^s(\mathbb{R})$, $-3/10 < s$.

A subsequent paper [6] will establish that (1.1) is globally well-posed in $H^s(\mathbb{R})$ for $-3/4 < s$. The simplicity of the argument presented here may extend more easily to other situations, such as in our treatment [5] of cubic NLS on $\mathbb{R}^2$ and NLS with derivative in $\mathbb{R}$ [4].

The Multiplier operator I

Let $s < 0$ and $N \gg 1$ be fixed. Define the Fourier multiplier operator

$$\widehat{Iu}(\xi) = m(\xi)\widehat{u}(\xi),\ m(\xi) = \begin{cases} 1, & |\xi| < N, \\ N^{-s}|\xi|^s, & |\xi| \geq 10N \end{cases} \tag{1.2}$$

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with m smooth and monotone. The operator I (barely) maps $H^s(\mathbb{R}) \longmapsto L^2(\mathbb{R})$. Observe that on low frequencies $\{\xi : |\xi| < N\}$, I is the identity operator. Note also that I commutes with differential operators. The operator I^{-1} is the Fourier multiplier operator with multiplier $\frac{1}{m(\xi)}$.

An almost L^2 conservation property of (1.1)

Let $\phi \in H^s(\mathbb{R})$, $-3/4 < s < 0$ in (1.1). There is a $\delta = \delta(\|\phi\|_{H^s}) > 0$ such that (1.1) is well-posed for $t \in [0, \delta]$. We observe using the Fundamental Theorem of Calculus, the equation, and integration by parts that

$$\begin{aligned}
\|Iu(\delta)\|_{L^2}^2 &= \|Iu(0)\|_{L^2}^2 + \int_0^\delta \frac{d}{d\tau}(Iu(\tau), Iu(\tau))d\tau, \\
&= \|Iu(0)\|_{L^2}^2 + 2\int_0^\delta (I\dot{u}(\tau), Iu(\tau))d\tau, \\
&= \|Iu(0)\|_{L^2}^2 + 2\int_0^\delta (I(-u_{xxx} - \frac{1}{2}\partial_x[u^2])(\tau), Iu(\tau))d\tau \\
&= \|Iu(0)\|_{L^2}^2 + \int_0^\delta (I(-\partial_x[u^2]), Iu)d\tau.
\end{aligned}$$

Finally, we add $0 = \int_0^\delta \int \partial_x (I(u)^2) I(u) d\tau$ to observe

$$\|Iu(\delta)\|_{L^2}^2 = \|Iu(0)\|_{L^2}^2 + \int_0^\delta \int \partial_x \left\{ (I(u))^2 - I(u^2) \right\} \ Iu \ dxd\tau. \tag{1.3}$$

This last step enables us to take advantage of some internal cancellation. We apply Cauchy-Schwarz as in [10] and bound the integral above by

$$\left\| \partial_x \{(I(u))^2 - I(u^2)\} \right\|_{X^\delta_{0,-\frac{1}{2}-}} \|Iu\|_{X^\delta_{0,\frac{1}{2}+}}. \tag{1.4}$$

The space $X^\delta_{s,b}$ of functions of space-time is defined via the Fourier restriction norm $\|u\|_{X^\delta_{s,b}} = \inf\{\|w\|_{X_{s,b}} := \|(1+|k|)^s(1+|\tau - k^3|)^b \widehat{w}(k,\tau)\|_{L^2_{k,\tau}} : w = u \text{ for } t \in [0,\delta]\}$.

Remark 1 An effort to find a term providing more cancellation than $\int_0^\delta \int \partial_x (I(u)^2) I(u) d\tau$ used above led to the general procedure described in [6].

Proposition 1 (A variant of local well-posedness) *The initial value problem* (1.1) *is locally well-posed in the Banach space* $I^{-1}L^2 = \{\phi \in H^s$ *with norm* $\|I\phi\|_{L^2}\}$ *with existence lifetime* δ *satisfying*

$$\delta \gtrsim \|I\phi\|_{L^2}^{-\alpha}, \ \textit{for some } \alpha > 0, \tag{1.5}$$

and moreover

$$\|Iu\|_{X^\delta_{0,\frac{1}{2}+}} \leq C\|I\phi\|_{L^2}. \tag{1.6}$$

This proposition is not difficult to prove using the argument in [9]. Using Duhamel's formula and $X_{s,b}$ space properties reduces matters to proving the bilinear estimate

$$\|\partial_x I(uv)\|_{X_{0,-\frac{1}{2}+}} \leq C\|Iu\|_{X_{0,\frac{1}{2}+}}\|Iv\|_{X_{0,\frac{1}{2}+}} \tag{1.7}$$

to obtain the contraction. The space-time norm bound is then implied by the contraction estimate. The estimate (1.7) follows from the next proposition and the bilinear estimate of Kenig, Ponce and Vega [9].

Proposition 2 (Extra smoothing) *The bilinear estimate*

$$\|\partial_x\{I(u)I(v) - I(uv)\}\|_{X^\delta_{0,-\frac{1}{2}-}} \leq CN^{-\frac{3}{4}+}\|Iu\|_{X^\delta_{0,\frac{1}{2}+}}\|Iv\|_{X^\delta_{0,\frac{1}{2}+}}. \tag{1.8}$$

holds.

Recall the bilinear estimate $\|\partial_x(uv)\|_{X_{0,-\frac{1}{2}+}} \leq C\|u\|_{X_{0,\frac{1}{2}+}}\|v\|_{X_{0,\frac{1}{2}+}}$ from [9]. Proposition 2 reveals a smoothing beyond the recovery of the first derivative for the particular quadratic expression encountered above in (1.3). We prove Proposition 2 in the next section.

The required pieces are now in place for us to give the proof of global well-posedness of (1.1) in $H^s(\mathbb{R})$, $-3/10 < s$. Global well-posedness of (1.1) will follow if we show well-posedness on $[0,T]$ for arbitrary $T > 0$. We re-normalize things a bit via scaling. If u solves (1.1) then $u_\lambda(x,t) = (\frac{1}{\lambda})^2 u(\frac{x}{\lambda}, \frac{t}{\lambda^3})$ solves (1.1) with initial data $\phi_\lambda(x,t) = (\frac{1}{\lambda})^2\phi(\frac{x}{\lambda})$. Note that u exists on $[0,T]$ if and only if u_λ exists on $[0,\lambda^3 T]$. A calculation shows that

$$\|I\phi_\lambda\|_{L^2} \leq C\lambda^{-\frac{3}{2}-s}N^{-s}\|\phi\|_{H^s}. \tag{1.9}$$

Here $N = N(T)$ will be selected later but we choose $\lambda = \lambda(N)$ right now by requiring

$$C\lambda^{-\frac{3}{2}-s}N^{-s}\|\phi\|_{H^s} \sim 1 \implies \lambda \sim N^{-\frac{2s}{3+2s}}. \tag{1.10}$$

We now drop the λ subscript on ϕ by assuming that

$$\|I\phi\|_{L^2} = \epsilon_0 \ll 1 \tag{1.11}$$

and our goal is to construct the solution of (1.1) on the time interval $[0,\lambda^3 T]$.

The local well-posedness result of Proposition 1 shows we can construct the solution for $t \in [0,1]$ if we choose ϵ_0 small enough. The almost L^2 conservation property shows $\|Iu(1)\|_2^2 \leq \|Iu(0)\|_2^2 + N^{-\frac{3}{4}+}\|Iu\|^3_{X_{0,\frac{1}{2}+}}$. Using (1.6) and (1.11) gives

$$\|Iu(1)\|_2^2 \leq \epsilon_0^2 + N^{-\frac{3}{4}+}.$$

We can iterate this process $N^{\frac{3}{4}-}$ times before doubling $\|Iu(t)\|_{L^2}$. Therefore, we advance the solution by taking $N^{\frac{3}{4}-}$ time steps of size $O(1)$. We now restrict s by demanding that

$$N^{\frac{3}{4}-} \gtrsim \lambda^3 T = N^{\frac{-6s}{3+2s}}T \tag{1.12}$$

is ensured for large enough N, so $s > -3/10$.

2 Proof of the bilinear smoothing estimate

This section establishes Proposition 2. We distinguish the **very low frequencies** $\{\xi : |\xi| \lesssim 1\}$, the **low frequencies** $\{\xi : 1 \lesssim |\xi| \lesssim \frac{1}{2}N\}$ and the **high frequencies** $\{\xi : \frac{1}{2}N \lesssim |\xi|\}$. Decompose the factor u in the bilinear estimate by writing $u = u_{vl} + u_l + u_h$ with $\widehat{u_l}$ supported on the low frequencies and similarly for the very low and high frequency pieces. We decompose v the same way. Since I is the identity operator on the low and very low frequencies, we can assume one of the factors u, v in the estimate to be shown has its Fourier transform supported in the high frequencies. Symmetry allows us to assume $u = u_h$ and we need to consider the three possible interactions of u_h with v_{vl}, v_l and v_h. Finally, since we are considering (weighted) L^2 norms, we can replace $\widehat{u}$ and $\widehat{v}$ by $|\widehat{u}|$ and $|\widehat{v}|$. Assume therefore that $\widehat{u}, \widehat{v} \geq 0$.

Very low/high interaction

An explicit calculation shows that

$$\mathcal{F}\left(\partial_x\{I(u_h v_{vl}) - I(u_h)v_{vl}\}\right)(\xi) = \int_{\xi=\xi_1+\xi_2} i\xi[m(\xi) - m(\xi_1)]\widehat{u_h}(\xi_1)\widehat{v_{vl}}(\xi_2), \tag{2.1}$$

where $\mathcal{F}$ denotes the Fourier transform. The mean value theorem gives

$$|m(\xi) - m(\xi_1)| \leq |m'(\tilde{\xi_1})||\xi_2|,$$

which may be interpolated with the trivial estimate to give

$$|m(\xi) - m(\xi_1)| \leq CN^{-s}|\xi_1|^s|\xi_1|^{-\theta}|\xi_2|^\theta \tag{2.2}$$

for $0 \leq \theta \leq 1$. Recall that m was defined to be smooth and monotone in (1.2).

Therefore, upon defining $\mathcal{F}(\nabla^\theta f)(\xi) = |\xi|^\theta \widehat{f}(\xi)$, we can write

$$|\mathcal{F}(\partial_x\{I(u_h v_{vl}) - I(u_h)v_{vl}\})(\xi)| \leq |\mathcal{F}(\partial_x(\nabla^{-\theta}I(u_h)(\nabla^\theta v_{vl}))(\xi)|.$$

We now estimate the left side of the bilinear estimate in this interaction by

$$\left\|\partial_x(\nabla^{-\theta}I(u_h))(\nabla^\theta v_{vl})\right\|_{X_{0,\frac{1}{2}+}} \tag{2.3}$$

and by the bilinear estimate of Kenig, Ponce and Vega

$$\leq C\left\|\nabla^{-\theta}I(u_h)\right\|_{X_{0,\frac{1}{2}+}}\left\|\nabla^\theta v_{vl}\right\|_{X_{0,\frac{1}{2}+}}. \tag{2.4}$$

The frequency support of v_{vl} shows that $\left\|\nabla^\theta v_{vl}\right\|_{X_{0,\frac{1}{2}+}} \lesssim \|v_{vl}\|_{X_{0,\frac{1}{2}+}}$. A moments thought shows

$$\left\|\nabla^{-\theta}I(u_h)\right\|_{X_{0,\frac{1}{2}+}} \leq N^{-\theta}\|I(u_h)\|_{X_{0,\frac{1}{2}+}} \tag{2.5}$$

and the claim of the Proposition follows for the (very low)(high) interaction by choosing $\theta > 3/4$.

Low/high interaction

The preceding calculations reduce matters to controlling

$$\left\|\partial_x \nabla^{-\theta} I(u_h) \nabla^{\theta} v_l\right\|_{X_{0,\frac{1}{2}+}} \tag{2.6}$$

and we know that $\widehat{u_h}$ and $\widehat{v_l}$ are supported outside the very low frequencies.

Lemma 1 *Assume $\widehat{u}$ and $\widehat{v}$ are supported outside $\{|\xi| < 1\}$. Then*

$$\|\partial_x(uv)\|_{X_{\alpha,-\frac{1}{2}+}} \leq C\|u\|_{X_{-\gamma_1,\frac{1}{2}+}} \|v\|_{X_{-\gamma_2,\frac{1}{2}+}} \tag{2.7}$$

provided

$$\begin{aligned} \alpha - (\gamma_1 + \gamma_2) &< \frac{3}{4}, \\ \alpha - \gamma_i &< \frac{1}{2}, \ i = 1, 2. \end{aligned}$$

We will apply the lemma momentarily with $\alpha = 0, \gamma_1 = \gamma_2 = -3/8+$.

The proof of the lemma is contained in the proof of Theorem 2 in [7]. In particular, the support properties on $\widehat{u}$, $\widehat{v}$ reduce matters to considering Cases A.3, A.4, A.6, B.3, B.4, B.5 and B.6 in [7]. The restriction $\alpha - (\gamma_1 + \gamma_2) < 3/4$ arises in Case A.4.c.ii of [7] which is the region containing the counterexample of [9]. Case B.4.b of [7] requires the other condition $\alpha - \gamma_i < \frac{1}{2}$.

The lemma applied to (2.6) gives

$$\leq C\left\|\nabla^{-\theta} I(u_h)\right\|_{X_{-\frac{3}{8}+,\frac{1}{2}+}} \left\|\nabla^{\theta} v_l\right\|_{X_{-\frac{3}{8}+,\frac{1}{2}+}}.$$

Setting $\theta = \frac{3}{8}-$ leaves

$$C\left\|\nabla^{-\frac{3}{4}+} I(u_h)\right\|_{X_{0,\frac{1}{2}+}} \|v_l\|_{X_{0,\frac{1}{2}+}} \leq CN^{-\frac{3}{4}+} \|I(u_h)\|_{X_{0,\frac{1}{2}+}} \|v_l\|_{X_{0,\frac{1}{2}+}}$$

which was to be shown.

High/high interaction

In this region of the interaction, we do not take advantage of any cancellation and estimate the difference with the triangle inequality

$$\|\partial_x\{I(u_h)I(v_h)\}\|_{X_{0,-\frac{1}{2}+}} + \|\partial_x\{I(u_h v_h)\}\|_{X_{0,-\frac{1}{2}+}}.$$

For the first contribution we use the lemma to get

$$\|I(u_h)\|_{X_{-\frac{3}{8}+,\frac{1}{2}+}} \|I(v_h)\|_{X_{-\frac{3}{8}+,\frac{1}{2}+}} \leq N^{-\frac{3}{4}+} \|I(u_h)\|_{X_{0,\frac{1}{2}+}} \|I(v_h)\|_{X_{0,\frac{1}{2}+}}. \tag{2.8}$$

The second contribution is bounded by throwing away I and applying the lemma,

$$\begin{aligned}
\|\partial_x\{u_h v_h\}\|_{X_{0,-\frac{1}{2}+}} &\leq \|u_h\|_{X_{-\frac{3}{8}+,\frac{1}{2}+}}\|u_h\|_{X_{-\frac{3}{8}+,\frac{1}{2}+}} \\
&\leq N^{-\frac{3}{8}+s+}\|u_h\|_{X_{s,\frac{1}{2}+}}N^{-\frac{3}{8}+s+}\|v_h\|_{X_{s,\frac{1}{2}+}} \\
&\leq N^{-\frac{3}{4}+}\|u_h\|_{X_{0,\frac{1}{2}+}}\|v_h\|_{X_{0,\frac{1}{2}+}}.
\end{aligned}$$

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