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Exponential stability of nonlinear time-varying differential equations and applications *

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Abstract

In this paper, we give sufficient conditions for the exponential stability of a class of nonlinear time-varying differential equations. We use the Lyapunov method with functions that are not necessarily differentiable; hence we extend previous results. We also provide an application to exponential stability for nonlinear time-varying control systems.

1 Introduction

The investigation of stability analysis of nonlinear systems using the second Lyapunov function method has produced a vast body of important results and been widely studied [4, 5, 8, 10, 20]. This is due to the theoretical interests in powerful tools for system analysis and control design. It is recognized that the Lyapunov function method serves as a main technique to reduce a given complicated system into a relatively simpler system and provides useful applications to control theory [6, 11, 13, 14, 20]. There have been a number of interesting developments in searching the stability criteria for nonlinear differential systems, but most have been restricted to finding the asymptotic stability conditions [1, 3, 7, 15]. Unlike the linear systems, where the asymptotic stability implies the exponential stability, the exponential stability for nonlinear differential systems, in general, may not be easily verified. Only a few investigations have dealt with exponential stability conditions for nonlinear time-varying systems [17, 18]. Moreover, the problem of Lyapunov characterization of exponential stability of nonlinear time-varying differential equations with non-smooth Lyapunov functions has remained open.

The purpose of this paper is to establish sufficient conditions for the exponential stability of a class of nonlinear time-varying systems. In the spirit of a result of [18], we develop the exponential stability with more general assumptions on the Lyapunov function $V(t, x)$ in two aspects:
(i) Proposing a class of Lyapunov-like functions, we prove new sufficient conditions for the exponential stability of nonlinear time-varying systems with more

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general comparable conditions.
(ii) The results are extended to the systems with non-smooth Lyapunov functions, which need not be differentiable in t and in x, and then the stability results are applied to some stabilization problems of nonlinear time-varying control systems.

The paper is organized as follows. In Section 2, we introduce notation, definitions, and other preliminaries. Section 3 gives new sufficient conditions for the exponential stability with the extended Lyapunov-like functions. An application to exponential stability of a class of nonlinear time-varying control systems is given in Section 4.

2 Preliminaries

The following notation will be used this paper: $\mathbb{R}^n$ is the n-dimensional Euclidean vector space; $\mathbb{R}^+$ is the set of all non-negative real numbers; $\|x\|$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$.

Consider the nonlinear system described by the time-varying differential equations

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)), \quad t \geq 0, \\ x(t_0) &= x_0, \quad t_0 \geq 0 \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $f(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear function satisfying $f(t,0) = 0$ for all $t \in \mathbb{R}^+$. We shall assume that conditions are imposed on system (1) such that the existence of its solutions is guaranteed.

Definition 2.1 The zero solution of system (1) is exponentially stable if any solution $x(t, x_0)$ of (1) satisfies

$$\|x(t, x_0)\| \leq \beta(\|x_0\|, t_0) e^{-\delta(t-t_0)}, \quad \forall t \geq t_0,$$

where $\beta(h,t) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a non-negative function increasing in $h \in \mathbb{R}^+$, and δ is a positive constant.

If the function $\beta(.)$ in the above definition does not depend on t_0, the zero solution is called uniformly exponentially stable. From now on, to shorten expressions, instead of saying the zero solution is stable, we say that the system is stable.

Associated with system (1) we consider a nonlinear time-varying control system

$$\dot{x}(t) = f(t, x(t), u(t)), \quad t \geq 0, \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f(t,x,u) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

Definition 2.2 Control system (2) is exponentially stabilizable by the feedback control $u(t) = h(x(t))$, where $h(x) : \mathbb{R}^n \to \mathbb{R}^m$, if the closed-loop system

$$\dot{x}(t) = f(t, x(t), h(x(t)))$$

is exponentially stable.

Let $D \subset \mathbb{R}^n$ be an open set containing the origin, and let $V(t,x) : \mathbb{R}^+ \times D \to R$ be a given function. Then we define $W = \mathbb{R}^+ \times D$ and

$$D_f^+ V(t,x) = \limsup_{h \to 0^+} \frac{V(t+h, x+hf) - V(t,x)}{h},$$

where $f(.)$ is the right-hand side function of (1). $D_f^+ V$ is called the upper Dini derivative of $V(.)$ along the trajectory of (1). Let $x(t)$ be a solution of (1) and denote by $d^+ V(t,x)$ the upper right-hand derivative of $V(t, x(t))$, i.e.

$$d^+ V(t, x(t)) = \limsup_{h \to 0^+} \frac{V(t+h, x(t+h)) - V(t, x(t))}{h}.$$

Definition 2.3 A function $V(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to R$ is Lipschitzian in x (uniformly in $t \in \mathbb{R}^+$) if there is a number $L > 0$ such that for all $t \in \mathbb{R}^+$,

$$|V(t,x_1) - V(t,x_2)| \le L\|x_1 - x_2\|, \quad \forall (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n.$$

In the sequel we assume that $V(t,x)$ is continuous in t and Lipschitzian in x (uniformly in t) with the Lipschitz constant $L > 0$. In which case, d^+V and $D_f^+ V$ related as follows:

$$\begin{aligned} & V(t+h, x(t+h)) - V(t, x(t)) \\ & \quad = \quad V(t+h, x(t+h)) - V(t+h, x+hf(t,x)) \\ & \qquad \quad + V(t+h, x+hf(t,x)) - V(t, x(t)). \end{aligned}$$

Also

$$\begin{aligned} & \limsup_{h \to 0^+} \frac{V(t+h, x+hf(t,x))) - V(t, x(t))}{h} \\ & \quad \le \quad \limsup_{h \to 0^+} \frac{V(t+h, x(t+h)) - V(t, x(t))}{h} \\ & \qquad \quad + L\{ \lim_{h \to 0^+} \frac{\|x(t+h) - x(t)\|}{h} - f(t, x(t))\}, \end{aligned}$$

which gives

$$d^+ V(t,x) \le \limsup_{h \to 0^+} \frac{V(t+h, x+hf)) - V(t,x)}{h} = D_f^+ V(t,x). \tag{3}$$

As shown in [12], if $D_f^+ V(t,x) \le 0$ and consequently, by (3), $d^+ V(t,x) \le 0$, the function $V(t, x(t))$ is a non-increasing function of t, which means that $V(t,x)$ is non-increasing along a solution of (1). To study exponential stability of (2) we need the following comparison theorem presented in [10, 19]. Consider a scalar differential equation

$$\dot{u}(t) = g(t,u), \quad t \ge 0, \tag{4}$$

where $g(t,u)$ is continuous in (t,u).

Proposition 2.1 *Let $u(t)$ be the maximal solution of (4) with $u(t_0) = u_0$. If a continuous function $v(t)$ with $v(t_0) = u_0$ satisfies*

$$d^+v(t) \leq g(t, u(t)), \quad \forall t \geq t_0,$$

then

$$v(t) - v(t_0) \leq \int_{t_0}^{t} g(s, u(s))ds, \quad \forall t \geq t_0.$$

Let us set

$$D_f V(t,x) = \frac{dV(t,x)}{dt} + \frac{dV(t,x)}{dx} f(t,x).$$

Definition 2.4 A function $V(t,x) : W \to R$ is called a Lyapunov-like function for (1) if $V(t,x)$ is continuously differentiable in $t \in \mathbb{R}^+$ and in $x \in D$, and there exist positive numbers $\lambda_1, \lambda_2, \lambda_3, K, p, q, r, \delta$ such that

$$\lambda_1 \|x\|^p \leq V(t,x) \leq \lambda_2 \|x\|^q, \quad \forall (t,x) \in W\,, \tag{5}$$

$$D_f V(t,x) \leq -\lambda_3 \|x\|^r + Ke^{-\delta t}, \quad \forall t \geq 0,\, x \in D \setminus \{0\} \tag{6}$$

Definition 2.5 A function $V(t,x) : W \to R$ is called a generalized Lyapunov-like function for (1) if $V(t,x)$ is continuous in $t \in \mathbb{R}^+$ and Lipschitzian in $x \in D$ (uniformly in t) and there exist positive functions $\lambda_1(t), \lambda_2(t), \lambda_3(t)$, where $\lambda_1(t)$ is non-decreasing, and there exist positive numbers K, p, q, r, δ such that

$$\lambda_1(t) \|x\|^p \leq V(t,x) \leq \lambda_2(t) \|x\|^q, \quad \forall (t,x) \in W\,, \tag{7}$$

$$D_f^+ V(t,x) \leq -\lambda_3(t) \|x\|^r + Ke^{-\delta t}, \quad \forall t \geq 0, x) \in D \setminus \{0\} \tag{8}$$

3 Main results

We start this section by giving a result from [18] on the exponential stability of (1), with the existence of a uniform Lyapunov function.

Theorem 3.1 ([17]) *Assume that (1) admits a Lyapunov-like function, where $p = q = r$. The system (1) is uniformly exponentially stable if*

$$\delta > \frac{\lambda_3}{\lambda_2}.$$

In the two theorems below, we give sufficient conditions for the exponential stability of (1) with a more general Lyapunov-like function.

Theorem 3.2 *The system (1) is uniformly exponentially stable if it admits a Lyapunov-like function and the following two conditions hold for all $(t,x) \in W$:*

$$\delta > \frac{\lambda_3}{[\lambda_2]^{r/q}}\,, \tag{9}$$

$$\exists \gamma > 0 \text{ such that } V(t,x) - V(t,x)^{r/q} \leq \gamma e^{-\delta t}. \tag{10}$$

Proof. Consider any initial time $t_0 \geq 0$, and let $x(t)$ be any solution of (1) with $x(t_0) = x_0$. Let us set

$$Q(t,x) = V(t,x)e^{M(t-t_0)}, \quad M = \frac{\lambda_3}{[\lambda_2]^{r/q}}.$$

Then

$$\dot{Q}(t,x(t)) = D_f V(t,x)e^{M(t-t_0)} + MV(t,x)e^{M(t-t_0)}.$$

Taking (6) into account, for all $t \geq t_0, x \in D$, we have

$$\dot{Q}(t,x) \leq (-\lambda_3)\|x\|^r + Ke^{-\delta t})e^{M(t-t_0)} + MV(t,x)e^{M(t-t_0)}.$$

By the condition (5) we have $\|x\|^q \geq \frac{V(t,x)}{\lambda_2}$, equivalently

$$-\|x\|^r \leq -[\frac{V(t,x)}{\lambda_2}]^{r/q}.$$

Therefore, we have

$$\dot{Q}(t,x) \leq \{-V(t,x)^{r/q}\frac{\lambda_3}{[\lambda_2]^{r/q}} + Ke^{-\delta t}\}e^{M(t-t_0)} + MV(t,x)e^{M(t-t_0)}.$$

Since

$$\frac{\lambda_3}{[\lambda_2]^{r/q}} = M, \quad \forall t \geq 0,$$

we have

$$\dot{Q}(t,x) \leq M\{V(t,x) - V(t,x)^{r/q}\}e^{M(t-t_0)} + Ke^{(M-\delta)(t-t_0)}.$$

Using (10), we obtain

$$\dot{Q}(t,x) \leq (K + M\gamma)e^{(M-\delta)(t-t_0)}.$$

Integrating both sides of the above inequality from t_0 to t, we obtain

$$\begin{aligned} Q(t,x(t)) - Q(t_0,x_0) &\leq \int_{t_0}^{t}(K+M\gamma)e^{()M-\delta)(s-t_0)}ds, \\ &= (K+M\gamma)\frac{1}{M-\delta}\{e^{(M-\delta)(t-t_0)} - 1\}. \end{aligned}$$

Setting $\delta_1 = -(M-\delta)$, by (9) we have $\delta_1 > 0$ and

$$\begin{aligned} Q(t,x(t)) &\leq Q(t_0,x_0) + \frac{K+M\gamma}{\delta_1} - \frac{K+M\gamma}{\delta_1}e^{(M-\delta)(t-t_0)} \\ &\leq Q(t_0,x_0) + \frac{K+M\gamma}{\delta_1}. \end{aligned}$$

Since $Q(t_0,x_0) = V(t_0,x_0) \leq \lambda_2\|x_0\|^q$, we have

$$Q(t,x(t)) \leq \lambda_2\|x_0\|^q + \frac{K+M\gamma}{\delta_1}.$$

Setting

$$\lambda_2\|x_0\|^q + \frac{K+M\gamma}{\delta_1} = \beta(\|x_0\|) > 0,$$

we have

$$Q(t,x(t)) \le \beta(\|x_0\|), \quad \forall t \ge t_0. \tag{11}$$

On the other hand, from (5) it follows that

$$\lambda_1\|x(t)\|^p \le V(t,x(t)),$$

$$\|x(t)\| \le \{\frac{V(t,x(t))}{\lambda_1}\}^{1/p}.$$

Substituting $V(t,x) = Q(t,x)/e^{M(t-t_0)}$ into the last inequality, we obtain

$$\|x(t)\| \le \{\frac{Q(t,x(t))}{e^{M(t-t_0)}\lambda_1)}\}^{1/p}. \tag{12}$$

Combining (11) and (12) gives

$$\|x(t)\| \le \{\frac{\beta(\|x_0\|}{e^{M(t-t_0)}\lambda_1}\}^{1/p} = \{\frac{\beta(\|x_0\|}{\lambda_1}\}^{1/p} e^{-\frac{M}{p}(t-t_0)}, \quad \forall t \ge t_0, \tag{13}$$

This inequality shows that (1) is uniformly exponentially stable. Therefore, the present proof is complete. $\diamondsuit$

Note that Theorem 3.1 is a special case of Theorem 3.2: $p = q = r$.

Example 3.1 Consider a nonlinear differential equation

$$\dot{x} = -\frac{1}{4}x^{\frac{3}{5}} + xe^{-2t}, \quad t \ge 0. \tag{14}$$

Let us take a Lyapunov function $V(t,x) : \mathbb{R}^+ \times D \to \mathbb{R}^+$, $V(t,x) = x^6$, where $D = \{x : |x| \le 1\}$. Note that

$$|x|^7 \le V(t,x) \le |x|^6, \quad \forall x \in D.$$

Then condition (5) holds with $\lambda_1 = \lambda_2 = 1$, $p = 7$, $q = 6$. On the other hand, we have

$$\dot{V}(t,x) = 6x^5\dot{x} = 6x^5(-\frac{1}{4}x^{\frac{3}{5}} + xe^{-2t}) = -\frac{3}{2}x^{\frac{28}{5}} + 6x^6e^{-2t}.$$

Therefore,

$$\dot{V}(t,x) \le -\frac{3}{2}x^{\frac{28}{5}} + 6e^{-2t}, \quad \forall x \in D.$$

Conditions (9), (10) of Theorem 3.2 are also satisfied with $\lambda_3 = 3/2$, $K = 6$, $\delta = 2$, $r = 28/5$. Moreover, we also have

$$V(t,x) - V(t,x)^{r/q} = x^6 - x^{\frac{28}{5}} = x^{\frac{28}{5}}(x^{\frac{2}{5}} - 1) \le 0 \le e^{-2t}, \quad \forall x \in D\,.$$

Therefore, (14) is exponentially stable.

We now give a sufficient condition for the exponential stability of (1) admitting a generalized Lyapunov-like function.

Theorem 3.3 *System (1) is exponentially stable if it admits a generalized Lyapunov-like function and the following two conditions hold for all* $(t,x) \in W$*:*

$$\delta > \inf_{t\in\mathbb{R}^+} \frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}} > 0\,. \tag{15}$$

$$\exists \gamma > 0 \text{ such that } V(t,x) - [V(t,x)]^{r/q} \le \gamma e^{-\delta t}. \tag{16}$$

Proof. We consider the function $Q(t,x(t)) = V(t,x(t))e^{M(t-t_0)}$, where

$$M = \inf_{t\in\mathbb{R}^+} \frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}}$$

We see that $M < \delta$ and

$$D_f^+ Q(t,x) = D_f^+ V(t,x)e^{M(t-t_0)} + MV(t,x(t))e^{M(t-t_0)}.$$

By the same arguments used in the proof of Theorem 3.2, we arrived at the fact that

$$D_f^+ Q(t,x) \le (-\lambda_3(t)\|x\|^r + Ke^{-\delta t})e^{M(t-t_0)} + MV(t,x)e^{M(t-t_0)}.$$

Taking condition (7) into account and since, by the assumption, $\lambda_2(t) > 0$ for all $t \in \mathbb{R}^+$, we have

$$\|x\|^q \ge \frac{V(t,x)}{\lambda_2(t)},$$

equivalently

$$-\|x\|^r \le -[\frac{V(t,x)}{\lambda_2(t)}]^{r/q}.$$

Therefore, we have

$$D_f^+ Q(t,x) \le \{-V(t,x)^{r/q}\frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}} + Ke^{-\delta t}\}e^{M(t-t_0)} + MV(t,x)e^{M(t-t_0)}.$$

Since

$$\frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}} \ge M, \quad \forall t \ge 0,$$

and by the condition (16) we obtain

$$\begin{aligned} D_f^+ Q(t,x) &\le M\{V(t,x) - V(t,x)^{r/q}\}e^{M(t-t_0)} + Ke^{(M-\delta)(t-t_0)} \\ &\le M\gamma e^{-\delta t}e^{M(t-t_0)} + Ke^{-\delta t}e^{M(t-t_0)} \\ &= (K + M\gamma)e^{-\delta t}e^{M(t-t_0)} \\ &\le (K + M\gamma)e^{-\delta(t-t_0)}e^{M(t-t_0)}. \end{aligned}$$

Therefore, $D_f^+ Q(t,x) \le (K + M\gamma)e^{(M-\delta)(t-t_0)}$. Thus, applying Proposition 2.1 to the case

$$v(t) = Q(t,x(t)), \quad g(t,u(t)) = (K + M\gamma)e^{(M-\delta)(t-t_0)},$$

we obtain

$$\begin{aligned} Q(t,x(t)) - Q(t_0,x_0) &\le \int_{t_0}^{t} (K+M\gamma)e^{(M-\delta)(s-t_0)}ds \\ &= (K+M\gamma)\frac{1}{M-\delta}\{(e^{(M-\delta)(t-t_0)}-1\}. \end{aligned}$$

Setting $\delta_1 = -(M-\delta)$, by condition (15) we have $\delta_1 > 0$ and

$$\begin{aligned} Q(t,x(t)) &\le Q(t_0,x_0) + \frac{K+M\gamma}{\delta_1} - \frac{K+M\gamma}{\delta_1}e^{(M-\delta)(t-t_0)} \\ &\le Q(t_0,x_0) + \frac{K+M\gamma}{\delta_1}. \end{aligned}$$

Since $Q(t_0,x_0) = V(t_0,x_0) \le \lambda_2(t_0)\|x_0\|^q$, we get

$$Q(t,x(t)) \le \lambda_2(t_0)\|x_0\|^q + \frac{K+M\gamma}{\delta_1}.$$

Letting

$$\lambda_2(t_0)\|x_0\|^q + \frac{K+M\gamma}{\delta_1} = \beta(\|x_0\|,t_0) > 0,$$

we have

$$Q(t,x(t)) \le \beta(\|x_0\|,t_0), \quad \forall t \ge t_0. \tag{17}$$

Furthermore, from condition (7), it follows that

$$\begin{gathered} \lambda_1(t)\|x(t)\|^p \le V(t,x(t)), \\ \|x(t)\| \le \{\frac{V(t,x(t))}{\lambda_1(t)}\}^{1/p}. \end{gathered}$$

Since $\lambda_1(t)$ is non-decreasing, $\lambda_1(t) \ge \lambda_1(t_0)$, we have

$$\|x(t)\| \le \{\frac{V(t,x(t))}{\lambda_1(t_0)}\}^{1/p}.$$

Substituting

$$V(t,x) = \frac{Q(t,x)}{e^{M(t-t_0)}},$$

into the last inequality, we obtain

$$\|x(t)\| \le \{\frac{Q(t,x(t))}{e^{M(t-t_0)}\lambda_1(t_0)}\}^{1/p}. \tag{18}$$

Combining (17) and (18),

$$\|x(t)\| \le \{\frac{\beta(\|x_0\|,t_0)}{e^{M(t-t_0)}\lambda_1(t_0)}\}^{1/p} = \{\frac{\beta(\|x_0\|,t_0)}{\lambda_1(t_0)}\}^{\frac{1}{p}} e^{-\frac{M}{p}(t-t_0)}, \quad \forall t \ge t_0, \tag{19}$$

The relation (19) shows that system (1) is exponentially stable. Theorem is proved. $\diamondsuit$

Remark 3.1. Note that in Theorem 3.3 we assume that the function $\lambda_1(t)$ is non-decreasing. In the case if the function $\lambda_1(t)$ satisfies the condition

$$\exists a > 0 : a < M, \quad \lambda_1(t) \geq e^{-at}, \quad \forall t \geq 0, \tag{20}$$

then we can replace the non-decreasing assumption by the above condition (20), where

$$M = \inf_{t \in \mathbb{R}^+} \frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}}.$$

The examples below illustrate our results in the case the Lyapunov function satisfies either more general comparable conditions or non-differentiability conditions, which include, as a special case, the results of [17, 18].

Example 3.2 Consider the system

$$\dot{x} = -\frac{1}{6}e^t x^{\frac{3}{5}} + \frac{x}{12} + e^{-\frac{3t}{2}} \sin x, \quad t \geq 0. \tag{21}$$

We take the Lyapunov function $V(t,x) : \mathbb{R}^+ \times D \to \mathbb{R}^+$, where D is defined as in Example 3.1, given by

$$V(t,x) = e^{-t/2}x^6.$$

In this case, we have $p = q = 6$, $\lambda_1(t) = e^{-t/2}$, $\lambda_2(t) = 1$, and

$$\begin{aligned} \dot{V}(t,x) &= -\frac{1}{2}e^{-t/2}x^6 + 6e^{-t/2}x^5(-\frac{1}{6}e^t x^{\frac{3}{5}} + \frac{x}{12} + e^{-\frac{3t}{2}} \sin x) \\ &= -e^{\frac{t}{2}}x^{\frac{28}{5}} + 6x^5 e^{-2t} \sin x, \\ &\quad \dot{V}(t,x) \leq -e^{\frac{t}{2}}x^{\frac{28}{5}} + 6e^{-2t}, \quad \forall x \in D. \end{aligned}$$

Therefore, we see that $r = 28/5$, $\delta = 2$, $K = 6$, $\lambda_3(t) = e^{t/2}$, and

$$M = \inf_{t \in \mathbb{R}^+} \frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}} = 1 < \delta = 2,$$

Moreover, we see that (20) holds: $a = \frac{1}{2} < M = 1$, and we can check condition (16) of Theorem 3.3 as follows:

$$\begin{aligned} V(t,x) - [V(t,x)]^{r/q} &= e^{-t/2}x^6 - \{e^{-t/2}x^6\}^{28/30} \\ &= e^{-t/2}x^6 - e^{-14t/30}x^{28/5} \leq 0 \leq e^{-2t}, \quad \forall x \in D, \end{aligned}$$

because of $x \in D = \{x : \|x\| \leq 1\}$, and

$$e^{-t/2}x^6 \leq e^{-t/2}x^{\frac{28}{5}} \leq e^{-\frac{14t}{30}}x^{\frac{28}{5}}.$$

System (21) is exponentially stable.

Example 3.3 Consider the system

$$\dot{x} = -\frac{1}{5}x^{1/3} + xe^{-2t} \tag{22}$$

with the Lyapunov function $V(t,x) = |x|^5$, where D is defined the same as in Example 3.1. We have

$$V(t,x) = \begin{cases} x^5, & \text{if } x \geq 0 \\ -x^5 & \text{if } x < 0\,. \end{cases}$$

Then we calculate

$$D_f^+ V(t,x) = \begin{cases} 5x^4(-\frac{1}{5}x^{1/3} + xe^{-2t}) = -x^{-13/3} + 5x^5e^{-2t} & \text{if } x \geq 0 \\ -5x^4(-\frac{1}{5}x^{1/3} + xe^{-2t}) = x^{14/3} - 5x^5e^{-2t} & \text{if } x < 0 \end{cases}$$

Therefore,

$$D_f^+ V(t,x) = -|x|^{13/3} + 5|x|^5e^{-2t} \leq -|x|^{13/3} - 5e^{-2t}, \quad \forall x \in D\,.$$

It follows that the conditions (8) and (15) hold for $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $p = q = K = 5$, $\delta = 2$, $r = 13/3$, $\delta > \lambda_3/\lambda_2^{r/q} = 1$. Condition (16) of Theorem 3.3 is also true because of:

$$V(t,x) - [V(t,x)]^{r/q} = |x|^5 - |x|^{13/3} = |x|^{13/3}(|x|^{2/3} - 1) \leq 0, \quad \forall x \in D.$$

Then system (22) is uniformly exponentially stable.

Example 3.4 For the system

$$\dot{x} = -x^{1/3} + x^3e^{-2t},$$

we take $V(t,x) = |x|$ with $x \in D$. We have $\lambda_1 = \lambda_2 = p = q = 1$, and

$$D_f^+ V(t,x) = \begin{cases} -x^{1/3} + x^3e^{-2t} & \text{if } x \geq 0, \\ x^{1/3} - x^3e^{-2t} & \text{if } x < 0\,. \end{cases}$$

Then, for all $x \in D$, we have

$$D_f^+ V(t,x) = -|x|^{1/3} + |x|^3e^{-2t} \leq -|x|^{1/3} + e^{-2t}.$$

All the conditions of Theorem 3.3 hold with $\lambda_3 = 1$, $r = 1/3$, $\delta = 2$, $K = 1$. The system is uniformly exponentially stable.

4 Applications to control systems

Consider the nonlinear control system (2), assuming that $f(t,0,0) = 0$, for all $t \geq 0$. We recall that system (2) is asymptotically stabilizable by a feedback control $u(t) = h(x,t)$, where $h(x) : \mathbb{R}^n \to \mathbb{R}^m$, $h(0) = 0$, if the zero solution of the system without control

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), h(x)), \quad t \geq 0, \\ x(t_0) &= x_0, \quad t_0 \geq 0 \end{aligned} \tag{23}$$

is asymptotically stable in the Lyapunov sense [11, 19]. If the zero solution of (23) is exponentially stable, we say that (2) is exponentially stabilizable.

Stabilization problem of system (2) has attracted a lot of attention from many researches in control theory in the last decade [11, 13, 14, 20]. Some sufficient conditions bellow for stability using Lyapunov functions were obtained for a class of time-invariant systems of the form

$$\dot{x}(t) = f(x(t), u(t)), \quad t \geq 0, \tag{24}$$

using smooth Lyapunov functions.

Theorem 4.1 ([11]) *Consider time-invariant system (24). If there exist a function $h(x) : \mathbb{R}^n \to \mathbb{R}^m$, $h(0) = 0$, where $h(x)$ is continuously differentiable in x and a positive definite function $V(x) : \mathbb{R}^n \to \mathbb{R}^+$, which is continuously differentiable in x such that*
(i) $V(x) \to \infty$ as $\|x\| \to \infty$.
(ii) $\frac{\partial V}{\partial x_i} f^i(x, h(x)) < 0$, $i = 1, 2, \dots, n$, for all $x \neq 0$. Then the system is asymptotically stabilizable by the feedback control $u(t) = h(x(t))$.

Some other sufficient conditions for stabilization of (24) using Lyapunov control functions can be found in [2, 9, 16], where the Lyapunov function $V(x)$ is assumed to be proper (i.e. the condition (i) in Theorem 4.1 holds) and has a negative lower Dini derivative along a solution of the system. Based on the stability results obtained in the previous section, we can derive the following sufficient conditions for the exponential stability of nonlinear control system (2) with non-smooth Lyapunov functions.

Theorem 4.2 *Assume that there is a function $h(x) : \mathbb{R}^n \to \mathbb{R}^m$, $h(0) = 0$ with $h(x)$ continuous in x, such that system (23) admits a Lyapunov-like function satisfying (9) and (10). Then the nonlinear control system (2) is exponentially stabilizable by feedback $u(t) = h(x(t))$.*

Theorem 4.3 *Assume that there is a function $h(x) : \mathbb{R}^n \to \mathbb{R}^m$ with $h(0) = 0$ and $h(x)$ continuous in x, such that system (23) admits a generalized Lyapunov-like function satisfying (15), (16). Then the nonlinear control system (2) is exponentially stabilizable by feedback control $u(t) = h(x(t))$.*

Conclusions Exponential stability of a class of nonlinear time-varying systems by the second Lyapunov method has been studied. New sufficient conditions for the exponential stability and applications to exponential stabilization problem of nonlinear control systems were given.

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