Periodic solutions for a class of non-coercive Hamiltonian systems

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Abstract

We prove the existence of non-constant $T$-periodic orbits of the Hamiltonian system

$$\begin{align*}
\dot{q} &= H_p(t, p(t), q(t)), \\
\dot{p} &= -H_q(t, p(t), q(t)),
\end{align*}$$

where $H$ is a $T$-periodic function in $t$, non-convex and non-coercive in $(p, q)$, and has the form $H(t, p, q) \sim |q|^{\alpha}(|p|^\beta - 1)$ with $\alpha > \beta > 1$.

1 Introduction

We study the existence of $T$-periodic solutions of the Hamiltonian system

$$\begin{align*}
\dot{q} &= H_p(t, p(t), q(t)), \\
\dot{p} &= -H_q(t, p(t), q(t)).
\end{align*}$$

Here, $H(t, p, q) : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ ($N \geq 3$) is $T$-periodic in $t$ and differentiable in $(p, q)$. We also assume that $H, H_p, H_q$ are continuous.

Most of the existence results use coercivity (i.e., $H(t, p, q) \rightarrow \infty$ as $(p, q) \rightarrow \infty$) or convexity assumptions in $H(t, .)$; see [1, 2, 3, 4, 5] and references therein. The purpose of this paper is to study non-coercive and non-convex Hamiltonians. Typically,

$$H(t, p, q) \sim |q|^{\alpha}(|p|^\beta - 1); \quad \alpha > \beta > 1.$$

To state our existence result, we introduce the following hypotheses. For constants $\alpha > \beta > 1$, $r > 0$, $a_1, \ldots, a_8 > 0$ and functions $A_i, K_i \in C(\mathbb{R}^N, \mathbb{R})$ with $K_i(0) = 0$ ($i = 1, 2, 3$), we assume:

(H1) $H(t + \frac{T}{2}, p, q) = H(t, -p, -q)$ for all $t, p, q$;

(H2) (i) $H(t, p, q) \leq a_1 |q|^{\alpha} |p|^\beta$ for all $t, p, q$;

(ii) $H(t, p, q) \geq a_2 |q|^{\alpha} |p|^\beta - K_i(q)$ for all $t, p, q$;

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(H3) \(-H(t,p,q) + H_p(t,p,q)p \geq a_3|q|^{\alpha}|p|^{\beta} + 1 - a_4\) for all \(t,p,q\);

(H4) \(|H_p(t,p,q)| \leq a_5|q|^{\alpha}|p|^{\beta-1} + 1 + a_6|q|\) for all \(t,p,q\);

(H5) \(|H_q(t,p,q)| \leq A(q)(|p|^{\beta} + 1)\) for all \(t,p,q\);

(H6) (i) \(|H_q(t,p,q)q - H_p(t,p,q)p| \geq a_7H(t,p,q) + K_2(q)\) for all \(t,p,q\);

(ii) \(|H_p(t,p,q)|^{\frac{\beta-1}{\beta}} \leq a_8|q|^{\alpha}(|q|^{\alpha}|p|^{\beta} + K_3(q))\) for all \(t,p,q\)

Our main result is as follows.

**Theorem 1.1** Under assumptions (H1)-(H6), System (1.1) has at least one non-constant \(T\)-periodic solution \((p(t),q(t))\) with \(q(t) \neq 0\) for all \(t\).

**Remark.** If \(H(t,p,q) = a(t)|q|^{\alpha}(|p|^{\beta} - 1)\) with \(\alpha > \beta > 1\) and \(a(t) \in C(\mathbb{R}, \mathbb{R})\) is a \(\frac{T}{2}\)-periodic and positive function, then (H1)-(H6) hold.

**Remark.** The condition \(\alpha > \beta\) is necessarily for the existence of non-constant \(T\)-periodic solution. More precisely, in case

\[H(t,p,q) = |q|^{\alpha}(|p|^{\beta} - 1),\]

if \((p(t),q(t))\) is a non-constant \(T\)-periodic solution of (1.1), then

(i) \(\alpha > \beta\);

(ii) there exists a constant \(C > 0\) such that

\[|q(t)|^{\alpha}(|p(t)|^{\beta} - 1) = C > 0\] for all \(t \in \mathbb{R}\).

In particular, \(q(t) \neq 0\) for all \(t \in \mathbb{R}\).

Indeed, by (1.1) we have

\[\int_0^Tp\dot{q}dt = \beta \int_0^T|q|^{\alpha}|p|^{\beta}dt = \alpha \int_0^T|q|^{\alpha}(|p|^{\beta} - 1)dt.\]

Then

\[(\alpha - \beta) \int_0^T|q|^{\alpha}|p|^{\beta}dt = \alpha \int_0^T|q|^{\alpha}dt.\]

Since \((p,q)\) is non-constant, one can see that \(q \neq 0\) and \(\alpha > \beta\). Also note that (ii) follows from the conservation of the energy.

To show the existence of a \(T\)-periodic solution of (1.1), we use a variational method; we introduce the functional

\[I(p,q) = \int_0^T[p\dot{q} - H(t,p,q)]dt\]

defined on the function space

\[E = \{(p,q) \in L^\gamma(0,T;\mathbb{R}^N) \times W^{1,\frac{\gamma}{\gamma-1}}(0,T;\mathbb{R}^N); \ q(0) = q(T)\}\]
where $\gamma = \alpha + \beta$. Critical points of $I(p, q)$ on $E$ correspond to $T$-periodic solutions of (1.1). We remark that the correspondence is one-to-one.

Since it is difficult to verify the Palais-Smale compactness condition for $I(p, q)$, we introduce in the following section, modified functionals and a finite dimensional approximation. We will use a minimax argument.

### 2 Modified functionals and other preliminaries

As stated in the introduction, we will find a critical point of the functional $I(p, q)$ on $E = P \times Q$ where

$$P = L^\gamma(0, T; \mathbb{R}^N), \quad Q = \{q \in W^{1, \frac{\gamma}{\gamma-1}}(0, T; \mathbb{R}^N); q(0) = q(T)\}.$$

We set

$$\Lambda = \{q \in Q; q(t) \neq 0 \text{ for all } t\}$$

and introduce the modified functionals

$$I_\delta(p, q) = \int_0^T \left| \dot{p} - H(t, p, q) + \frac{\delta}{|q|^{\gamma}} \right| dt,$$

$$I_{\delta, \varepsilon}(p, q) = \int_0^T \left[ \dot{p} - H(t, p, q) + \frac{\delta}{|q|^{\gamma}} + \varepsilon |q|^\gamma - |p|^\gamma \right] dt$$

for $\delta, \varepsilon \in [0, 1]$. Since $\gamma \geq \beta > 1$, by (H2), (H4), and (H5), we can see that $I_{\delta, \varepsilon} \in C^1(P \times \Lambda; \mathbb{R})$.

To get the existence of a $T$-periodic solution for a symmetric Hamiltonians, we have to restrict our functionals to a subsets of $E$. We set

$$E_0 = \{(p, q) \in E; (p, q)(t + \frac{T}{2}) = -(p, q)(t)\}$$

with norm

$$\|(p, q)\|_{E_0} = \|p\|_\gamma + \|\dot{q}\|_{\frac{\gamma}{\gamma-1}}$$

where

$$\|u\|_s = \left( \int_0^T |u(t)|^s dt \right)^{1/s} \text{ for all } s \geq 1.$$ 

For $m \in \mathbb{N}$, we define

$$P_m = Q_m = \bigg\{ p(t) = \sum_{|j| \leq m} \theta_j e^{2\pi ij \frac{t}{T}}; \quad p(t + \frac{T}{2}) = -p(t), \theta_j \in \mathbb{C}^N, \theta_{-j} = \bar{\theta}_j, |j| \leq m \bigg\},$$

$$E_m = P_m \times Q_m,$$

$$\Lambda_m = \{q \in Q_m; q(t) \neq 0 \text{ for all } t\},$$

$$\partial \Lambda_m = \{q \in Q_m; q(t_0) = 0 \text{ for some } t_0\}$$
and we consider the restriction of \( I_{\delta,\varepsilon}(p, q) : \)
\[
I_{\delta,\varepsilon,m} = I_{\delta,\varepsilon}/P_m \times \Lambda_m : P_m \times \Lambda_m \to \mathbb{R}.
\]
The main reason for introducing such subspaces are the following Lemmas.

**Lemma 2.1** For any \( u \in Q \) such that \( u(t + \frac{T}{2}) = -u(t) \), we have
\[
\|u\|_{\infty} \leq \int_0^T |\dot{u}| dt.
\]

**Proof.** Let \( u \in Q \) such that \( u(t + \frac{T}{2}) = -u(t) \). Then for all \( t \in [0, T] \), we have
\[
|u(t)| = \frac{1}{2} |u(t + \frac{T}{2}) - u(t)| = \frac{1}{2} |\int_t^{t+\frac{T}{2}} \dot{u} ds| \leq \int_0^T |\dot{u}| ds.
\]
Thus we obtain the desired result. \( \diamond \)

**Lemma 2.2** Suppose \((p, q) \in P_m \times \Lambda_m \) is such that
\[
I'_{\delta,\varepsilon,m}(p, q)(h, k) = 0 \quad \text{for all } (h, k) \in E_m.
\]
Then \((p, q)\) is a critical point for \( I_{\delta,\varepsilon,m} \).

**Proof.** It is sufficient to remark that, by (H1), \( I'_{\delta,\varepsilon,m}(p, q) \in E_m \). Since \( I'_{\delta,\varepsilon,m}(p, q) \) belongs also to \( E_m^\perp \) from 2.1, we have the conclusion. \( \diamond \)

The proof of Theorem 1.1 will be done as follows: In section 3, we introduce a minimax method to \( I_{\delta,\varepsilon,m} \). For \( \delta, \varepsilon \in [0, 1] \) and \( m \in \mathbb{N} \), we establish the existence of a sequence \((p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) \in P_m \times \Lambda_m \) such that
\[
I'_{\delta,\varepsilon,m}(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) = 0,
\]
\[
I_{\delta,\varepsilon,m}(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) \leq \bar{c}.
\]
where \( \bar{c} > 0 \) is a constant independent of \( \delta, \varepsilon \) and \( m \). From 2.2-3.2, we can find uniform estimates for \((p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) \) and we can extract, in section 4, a subsequence converging to \((p_{\delta,\varepsilon}, q_{\delta,\varepsilon}) \in (P \times \Lambda) \cap E_0 \). Next in Section 5, we pass to the limit as \( \varepsilon \to 0 \) and obtain a critical points \((p, q) \in (P \times \Lambda) \cap E_0 \) of \( I_{\delta} \) such that
\[
I_{\delta}(p, q) \leq \bar{c}.
\]
Finally in Section 6, we pass to the limit as \( \delta \to 0 \). Lemma 2.1 plays a essential role to obtain a non-constant \( T \)-periodic solution \((p, q) = \lim(p_3, q_3) \) of (1.1).

In the sequel, we use the projection operator
\[
\text{proj}_m : L^1(0, T; \mathbb{R}^N) \to \text{span}\{e^{2\pi i j}\mu T; |j| \leq m\},
\]
\[
(\text{proj}_m u)(t) = \sum_{|j| \leq m} \theta_j e^{2\pi i j\mu T} \text{ for } u(t) = \sum_{j \in \mathbb{Z}} \theta_j e^{2\pi i j\mu T}.
\]
Lemma 2.3 For any $s \in [1, +\infty[$, there exists a constant $K_s > 0$ independent of $m \in \mathbb{N}$ such that

$$||\text{proj}_m u||_s \leq K_s ||u||_s \quad \text{for all} \quad u \in L^s(0, T; \mathbb{R}^N).$$

This lemma is a special case of Steckin's theorem [6, Theorem 6.3.5]. In sections 3, 4, 5, and 6, we will assume (H1)-(H6).

3 A minimax method for $I_{\delta, \varepsilon, m}$

In this part, we study the existence of critical points in $P_m \times \Lambda_m$ of $I_{\delta, \varepsilon, m}$ for $\delta, \varepsilon \in [0, 1]$ and $m \in \mathbb{N}$. First, we give some a priori estimates and verify the Palais-Smale condition (PS) for $I_{\delta, \varepsilon, m}$.

Lemma 3.1 (i) For any $M_1 > 0$, there exists a constant $C_0 = C_0(M_1) > 0$ independent of $\delta, \varepsilon \in [0, 1]$ and $m \in \mathbb{N}$ such that: If $(p_j, q_j) \in P_m \times \Lambda_m$ satisfies

$$I_{\delta, \varepsilon, m}(p_j, q_j) \leq M_1,$$

$$I_{\delta, \varepsilon, m}'(p_j, q_j) = 0,$$

then

$$\int_0^T |q|^\alpha |p|^\beta dt + \int_0^T |q|^\alpha dt \leq C_0,$$

$$\varepsilon \int_0^T \left(|q|^{\gamma} + |p|^{\gamma}\right) dt + \delta \int_0^T \frac{1}{|q|^{\gamma}} dt \leq C_0.$$

(ii) For any $\delta, \varepsilon \in [0, 1]$ and $m \in \mathbb{N}$, if $(p_j, q_j)_{j=1}^\infty \subset P_m \times \Lambda_m$ satisfies

$$(p_j, q_j) \rightharpoonup (p_0, q_0) \in P_m \times \partial \Lambda_m,$$

then $I_{\delta, \varepsilon, m}(p_j, q_j) \to +\infty$.

(iii) For any $\delta, \varepsilon \in [0, 1]$ and $m \in \mathbb{N}$, $I_{\delta, \varepsilon, m}$ satisfies the condition (PS) on $P_m \times \Lambda_m$; i.e., if $(p_j, q_j)_{j\in \mathbb{N}} \subset P_m \times \Lambda_m$ satisfies $I_{\delta, \varepsilon, m}(p_j, q_j) \to c > 0$ and $(I_{\delta, \varepsilon, m})'(p_j, q_j) \to 0$, then $(p_j, q_j)$ possesses a subsequence converging in $E_m$ to some $(p, q) \in P_m \times \Lambda_m$.

Proof. (i) Let $\delta, \varepsilon \in [0, 1]$ and $m \in \mathbb{N}$. We assume $(p, q) \in P_m \times \Lambda_m$ satisfies 3.1 and 3.2 for $M_1 > 0$. We have

$$I_{\delta, \varepsilon, m}'(p, q)(p, 0) = \int_0^T [p \dot{q} - H_p(t, p, q)p - \varepsilon \gamma |p|^{\gamma}] dt.$$

Hence,

$$I_{\delta, \varepsilon, m}(p, q) - I_{\delta, \varepsilon, m}'(p, q)(p, 0) = \int_0^T \left[-H(t, p, q) + H_p(t, p, q)p + \frac{\delta}{|q|^{\gamma}} + \varepsilon |q|^{\gamma} + \varepsilon (\gamma - 1)|p|^{\gamma}\right] dt.$$
By the assumptions 3.1 and 3.2, we get
\[
\int_0^T \left[ -H(t, p, q) + H_p(t, p, q)p + \frac{\delta}{|q|^\gamma} + \varepsilon|q|^\gamma + \varepsilon(\gamma - 1)|p|^{\gamma} \right] dt \leq M_1.
\]
From (H3), it follows that
\[
\int_0^T \left[ a_3|q|^\alpha(|p|^\beta + 1) - a_4 + \frac{\delta}{|q|^\gamma} + \varepsilon|q|^\gamma + \varepsilon(\gamma - 1)|p|^{\gamma} \right] dt \leq M_1.
\]
Thus we obtained (i).
(ii) By (H2)(i), we have for all \((p, q) \in P_m \times \Lambda_m\)
\[
I_{\delta,\varepsilon,m}(p, q) \geq \int_0^T \left[ p \dot{q} - a_1|q|^\alpha|p|^{\beta} + \varepsilon(|q|^\alpha - |p|^{\gamma}) \right] dt + \delta \int_0^T \frac{1}{|q|^\gamma} dt. \tag{3.4}
\]
Since \(\delta \int_0^T \frac{1}{|q|^\gamma} dt \to \infty\), we get the conclusion easily.
(iii) Let \((p_j, q_j)_{j \in \mathbb{N}} \subset P_m \times \Lambda_m\) be a sequence satisfying the assumptions of the condition (PS). We may assume that
\[
I_{\delta,\varepsilon,m}(p_j, q_j) \to c, \tag{3.5}
\]
\[
\|I_{\delta,\varepsilon,m}'(p_j, q_j)\|_{E_m^*} \to 0. \tag{3.6}
\]
We prove that \((p_j, q_j)\) possesses a convergent subsequence to some \((p, q) \in P_m \times \Lambda_m\). By (H3) and 3.3-3.6, for large \(j\),
\[
\int_0^T \left[ a_3|q_j|^\alpha(|p_j|^\beta + 1) - a_4 \right] dt + \delta \int_0^T \frac{1}{|q_j|^\gamma} dt + \varepsilon \int_0^T |q_j|^\gamma dt + \varepsilon(\gamma - 1) \int_0^T |p_j|^\gamma dt \leq 2c + \|p_j\|_{\gamma}.
\]
Thus, for some constant \(C_1 > 0\) independent of \(j\),
\[
\int_0^T |q_j|^\alpha dt, \int_0^T |p_j|^\gamma dt \leq C_1 \quad \text{for all } j \in \mathbb{N}.
\]
Since \(\dim E_m < \infty\), we can extract a subsequence - still indexed by \((p_j, q_j)\) - such that \((p_j, q_j) \to (p, q) \in E_m\). By (ii), we necessarily have \(q \in \Lambda_m\).
Next, we apply to \(I_{\delta,\varepsilon,m}\) a minimax argument related to the one in [7]. This argument will play an important role in obtaining a critical points \((p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) \in P_m \times \Lambda_m\) with uniform upper bound of critical values. We define
\[
\Gamma_m = \{ A(p, \xi) \in C(P_m \times S^{N-2}, P_m \times \Lambda_m); A(p, \xi) = (p, \sigma_0(\xi)) \text{for large } \|p\|_{\beta} \}
\]
where
\[
\sigma_0 : S^{N-2} = \{ \xi = (\xi_1, \ldots, \xi_{N-1}) \in \mathbb{R}^{N-1}: \sum_{j=1}^{N-1} |\xi_j|^2 = 1 \} \to Q_m
\]
is given by
\[ \sigma_0(\xi)(t) = \cos^2 \pi t T(\xi_1, \ldots, \xi_{N-1}, 0) + \sin^2 \pi t T(0, \ldots, 0, 1). \]
We remark that \( A_0(p, \xi) = (p, \sigma_0(\xi)) \in \Gamma_m \) and \( \Gamma_m \neq \emptyset \). Then we define the minimax values of \( I_{\delta, \varepsilon, m} \) as follows
\[ c_{\delta, \varepsilon, m} = \inf_{A \in \Gamma_m} \sup_{(p, \xi) \in P_m \times S^{N-2}} I_{\delta, \varepsilon, m}(A(p, \xi)). \]

**Proposition 3.1** For any \( \delta, \varepsilon \in [0, 1] \) and \( m \in \mathbb{N} \), there exists a constant \( c(\delta, \varepsilon) > 0 \) such that
\[ c_{\delta, \varepsilon, m} \geq c(\delta, \varepsilon) > 0. \]
To prove this proposition, we need the following result.

**Lemma 3.2** For any \( A \in \Gamma_m \) and \( \lambda > 0 \), we have
\[ A(P_m \times S^{N-2}) \cap D_{m, \lambda} \neq \emptyset \]
where
\[ D_{m, \lambda} = \left\{ (p, q) \in P_m \times \Lambda_m; p = \lambda \text{proj}_{m}(|q|^{\gamma-1}\dot{q}) \right\}. \]
The proof of this lemma will be given in the appendix.

**Lemma 3.3** For sufficiently small \( \lambda_{\varepsilon} > 0 \), there exists a constant \( c(\delta, \varepsilon) > 0 \) such that
\[ I_{\delta, \varepsilon, m}(p, q) \geq c(\delta, \varepsilon) > 0 \]
for all \( (p, q) \in D_{m, \lambda_{\varepsilon}} \) where \( D_{m, \lambda_{\varepsilon}} \) is given in Lemma 3.2.

**Proof.** Let \( (p, q) \in D_{m, \lambda} \). We recall that \( \gamma = \alpha + \beta \). By the Young’s inequality,
\[ a_1 \int_0^T |q|^\alpha |p|^\beta dt \leq \frac{\alpha}{\gamma} \varepsilon \int_0^T |q|^\gamma dt + \frac{\beta}{\gamma} \left( \frac{a_1}{\varepsilon \gamma} \right)^{\frac{\beta}{\gamma}} \int_0^T |p|^\gamma dt. \]
Thus, from 3.4,
\[ I_{\delta, \varepsilon, m}(p, q) \geq \int_0^T p \dot{q} dt - a(\varepsilon) \int_0^T |p|^\gamma dt + \delta \int_0^T \frac{1}{|q|^\gamma} dt \]
where \( a(\varepsilon) = \varepsilon + \frac{\beta}{\gamma} \left( \frac{a_1}{\varepsilon \gamma} \right)^{\frac{\beta}{\gamma}} > 0 \). Since \( (p, q) \in D_{m, \lambda} \),
\[ \int_0^T p \dot{q} dt = \lambda \int_0^T |q|^{\frac{\gamma}{\gamma-1}} dt. \]
Moreover, by Lemma 2.1 and Lemma 2.3
\[ T^{\frac{1}{\gamma}} \| \dot{q} \|^{\frac{\gamma}{\gamma-1}} \geq \int_0^T |q| dt \geq \| q \|_\infty. \]
\[
\int_0^T |p|^\gamma dt = \lambda^\gamma \|\text{proj}_m(|q|^\gamma^{-1})\|_2 \leq \lambda^\gamma K_\gamma \|q\|_{\gamma^{-1}}. \tag{3.9}
\]

By 3.7 and 3.9, we get
\[
I_{\delta,\varepsilon,m}(p,q) \geq (\lambda - a(\varepsilon) K_\gamma \lambda^\gamma) \|\dot{q}\|_{\gamma^{-1}} \delta T + \int_0^T \frac{1}{|q|^\gamma} dt.
\]

Taking \(\lambda_\varepsilon\) small enough so that 
\[
A_{\varepsilon} = \lambda_\varepsilon - a(\varepsilon) K_\gamma \lambda^\gamma > 0,
\]
from 3.8, for all \((p,q) \in D_{m,\lambda_\varepsilon}\), we have
\[
I_{\delta,\varepsilon,m}(p,q) \geq \inf_{q \in \Lambda} \left( \frac{A_{\varepsilon}}{T^{-1}} \|q\|_{\infty}^{\gamma^{-1}} + \frac{\delta T}{\|q\|_{\infty}} \right) = c(\delta,\varepsilon) > 0.
\]

**Proof of Proposition 3.1** Let \(\lambda_\varepsilon > 0\) be as in Lemma 3.3. By Lemma 3.2, we have
\[
A(P_m \times S^{N-2}) \cap D_{m,\lambda_\varepsilon} \neq \emptyset
\]
for all \(A \in \Gamma_m\).

Thus, we find that
\[
c_{\delta,\varepsilon,m} = \inf_{A \in \Gamma_m} \sup_{(p,\xi) \in P_m \times S^{N-2}} I_{\delta,\varepsilon,m}(A(p,\xi))
\geq \inf_{(p,q) \in D_{m,\lambda_\varepsilon}} I_{\delta,\varepsilon,m}(p,q)
\geq c(\delta,\varepsilon) > 0.
\]

We choose \(c(\delta,\varepsilon) = c(\delta,\varepsilon)\), we get the desired result. \(\diamondsuit\)

Now, we prove an existence result

**Proposition 3.2** For any \(\delta,\varepsilon \in [0,1]\) and \(m \in \mathbb{N}\), we have

(i) \(0 < c(\delta,\varepsilon) \leq c_{\delta,\varepsilon,m} \leq \bar{c}\)

where \(\bar{c}\) is independent of \(\delta,\varepsilon\) and \(m\).

(ii) If \(|p|_\beta\) is sufficiently large, then for all \(\xi \in S^{N-2}\),
\[
I_{\delta,\varepsilon,m}(A_0(p,\xi)) \leq 0.
\]

(iii) There exists a critical point \((p_{\delta,\varepsilon,m},q_{\delta,\varepsilon,m}) \in P_m \times \Lambda_m\) of \(I_{\delta,\varepsilon,m}\) such that
\[
I_{\delta,\varepsilon,m}(p_{\delta,\varepsilon,m},q_{\delta,\varepsilon,m}) = c_{\delta,\varepsilon,m}.
\]

**Proof.** (i) By (H2)(ii), we have
\[
I_{\delta,\varepsilon,m}(A_0(p,\xi)) \leq \int_0^T |p| \frac{d}{dt}\sigma_0(\xi) dt - a_2 \int_0^T |\sigma_0(\xi)|^\alpha |p|^{\beta} dt
+ \int_0^T K_1(\sigma_0(\xi)) dt + \int_0^T \left( \frac{1}{|\sigma_0(\xi)|^\gamma} + |\sigma_0(\xi)|^\gamma \right) dt
\leq k_1 |p|_\beta - k_2 |p|_\beta^\gamma + k_3 \tag{3.10}
\]
for some positive constants $k_1,k_2,k_3$ independent of $\delta, \varepsilon$ and $m$. Since $\beta > 1$, there exists a constant $\bar{c} > 0$ independent of $\delta, \varepsilon$ and $m$ such that

$$c_{\delta, \varepsilon, m} \leq \sup_{(p, \xi) \in P_m \times S^{N-2}} I_{\delta, \varepsilon, m}(A_0(p, \xi)) \leq \bar{c}.$$ 

(ii) follows clearly from 3.10.

(iii) Since $I_{\delta, \varepsilon, m}$ satisfies the (PS) condition and property (ii) of Lemma 3.1, then by a standard argument using the deformation theorem and (ii), we can see that $c_{\delta, \varepsilon, m} > 0$ is a critical value of $I_{\delta, \varepsilon, m}$. By Lemma 2.2, we get (iii). 

As a corollary to (i) of Lemma 3.1 and the uniform estimates of $c_{\delta, \varepsilon, m}$, we have the following statements.

**Corollary 3.1** Let $(p_{\delta, \varepsilon, m}, q_{\delta, \varepsilon, m}) \in P_m \times \Lambda_m$ be a critical point of $I_{\delta, \varepsilon, m}$ obtained by Proposition 3.2. Then, there exists a constant $C_2 > 0$ independent of $\delta, \varepsilon$ and $m$, such that for all $\delta, \varepsilon \in [0,1]$ and $m \in \mathbb{N}$, we have

(i) $$\int_0^T |q_{\delta, \varepsilon, m}|^\alpha |p_{\delta, \varepsilon, m}|^\beta dt + \int_0^T |q_{\delta, \varepsilon, m}|^\alpha dt \leq C_2,$$

(ii) $$\varepsilon \int_0^T (|q_{\delta, \varepsilon, m}|^\gamma + |p_{\delta, \varepsilon, m}|^\gamma) dt \leq C_2,$$

(iii) $$\delta \int_0^T \frac{1}{|q_{\delta, \varepsilon, m}|^\gamma} dt \leq C_2.$$

**4 Limiting process as $m \to \infty$**

**Proposition 4.1** For any $\delta, \varepsilon \in [0,1]$, $(p_{\delta, \varepsilon, m}, q_{\delta, \varepsilon, m})$ possesses a subsequence converging in $E$ to $(p_{\delta, \varepsilon}, q_{\delta, \varepsilon}) \in (P \times \Lambda) \cap E_0$. Moreover,

$$I_{\delta, \varepsilon}(p_{\delta, \varepsilon}, q_{\delta, \varepsilon}) \leq \bar{c}, \quad (4.1)$$

$$I'_{\delta, \varepsilon}(p_{\delta, \varepsilon}, q_{\delta, \varepsilon}) = 0. \quad (4.2)$$

**Proof.** By (ii) of Corollary 3.1, we can extract a subsequence - still indexed by $m$- such that

$$(p_{\delta, \varepsilon, m}, q_{\delta, \varepsilon, m}) \rightharpoonup (p_{\delta, \varepsilon}, q_{\delta, \varepsilon}) \quad \text{weakly in } L^\gamma(0, T; \mathbb{R}^N).$$

We remark that $H(t, p_{\delta, \varepsilon, m}, q_{\delta, \varepsilon, m}) = 0$ is equivalent to

$$\dot{q}_{\delta, \varepsilon, m} = \text{proj}_m[H_p(t, p_{\delta, \varepsilon, m}, q_{\delta, \varepsilon, m}) + \varepsilon \gamma |p_{\delta, \varepsilon, m}|^{\gamma - 2} p_{\delta, \varepsilon, m}], \quad (4.3)$$

$$\dot{p}_{\delta, \varepsilon, m} = -\text{proj}_m[H_q(t, p_{\delta, \varepsilon, m}, q_{\delta, \varepsilon, m}) + \delta \gamma \frac{q_{\delta, \varepsilon, m}}{|q_{\delta, \varepsilon, m}|^\gamma} - \varepsilon \gamma |q_{\delta, \varepsilon, m}|^{\gamma - 2} q_{\delta, \varepsilon, m}]. \quad (4.4)$$

By (H4) and Lemma 2.3, we have from 4.3

$$\left\| \dot{q}_{\delta, \varepsilon, m} \right\|_{\gamma} \leq K \left[ a_5 \| |q_{\delta, \varepsilon, m}|^\alpha |p_{\delta, \varepsilon, m}|^{(\beta - 1)} \|_{\gamma} + a_\gamma \| q_{\delta, \varepsilon, m} \|_{\alpha \gamma} \right] + a_6 \| q_{\delta, \varepsilon, m} \|_{\gamma} + \varepsilon \gamma \| p_{\delta, \varepsilon, m} \|_{\gamma - 1}.$$
Using a Hölder’s inequality and (i)-(ii) of Corollary 3.1, we can find a constant $C_3 > 0$ independent of $m \in \mathbb{N}$, such that

$$\|q_{\delta, \varepsilon, m}\|_{W^{1, \frac{\gamma}{\gamma-1}}(0, T; \mathbb{R}^N)} \leq C_3.$$ 

Thus we can see from (iii) of Corollary 3.1 that

$$q_{\delta, \varepsilon, m} \rightarrow q_{\delta, \varepsilon} \in \Lambda \text{ uniformly in } [0, T]. \quad (4.5)$$

On the other hand, by (H5) and Lemma 2.3, we have from 4.4

$$\|\dot{p}_{\delta, \varepsilon, m}\|_{\frac{\gamma}{\gamma-1}} \leq K\|A(q_{\delta, \varepsilon, m})\|_{\frac{\gamma}{\gamma-1}} + \|A(q_{\delta, \varepsilon, m})\|_{\frac{\gamma}{\gamma-1}} + \gamma||\delta \frac{q_{\delta, \varepsilon, m}}{|q_{\delta, \varepsilon, m}|^{\gamma+2}} - \varepsilon |q_{\delta, \varepsilon, m}|^{-2} q_{\delta, \varepsilon, m}\|_{\frac{\gamma}{\gamma-1}}.$$ 

Using 4.5, we find

$$\|p_{\delta, \varepsilon, m}\|_{W^{1, \frac{\gamma}{\gamma-1}}(0, T; \mathbb{R}^N)} \leq C_4$$

where $C_4 > 0$ is a constant independent of $m$. The injection $W^{1, \frac{\gamma}{\gamma-1}}(0, T; \mathbb{R}^N) \subset L^{\gamma}(0, T; \mathbb{R}^N)$ is compact, thus we have

$$p_{\delta, \varepsilon, m} \rightarrow p_{\delta, \varepsilon} \text{ strongly in } L^{\gamma}(0, T; \mathbb{R}^N) \text{ and uniformly in } [0, T]. \quad (4.6)$$

By (i) and (iii) of Proposition 3.2, we deduce that

$$I_{\delta, \varepsilon}(p_{\delta, \varepsilon}, q_{\delta, \varepsilon}) = \lim_{m \to \infty} I_{\delta, \varepsilon, m}(p_{\delta, \varepsilon, m}, q_{\delta, \varepsilon, m}) \leq \bar{c},$$

$$I'_{\delta, \varepsilon}(p_{\delta, \varepsilon}, q_{\delta, \varepsilon})(h, k) = \lim_{m \to \infty} I'_{\delta, \varepsilon, m}(p_{\delta, \varepsilon, m}, q_{\delta, \varepsilon, m})(h, k) = 0$$

for all sums

$$h = \sum_{|j| \leq n} \theta_j e^{2\pi i j t}, \quad k = \sum_{|j| \leq n} \psi_j e^{2\pi i j t} \quad (\theta_j, \psi_j \in \mathbb{C}^N).$$

Therefore, $I'_{\delta, \varepsilon}(p_{\delta, \varepsilon}, q_{\delta, \varepsilon})(h, k) = 0$ for all $(h, k) \in E$.

5 Limiting process as $\varepsilon \to 0$

We take the limit as $\varepsilon \to 0$ to obtain a critical point $(p_{\delta}, q_{\delta}) \in (P \times \Lambda) \cap E_0$ of $I_{\delta}$ with uniform upper bound for critical values. As a consequence to Corollary 3.1, and 4.5, 4.6 we have the following lemma.

Lemma 5.1 For any $\delta, \varepsilon \in [0, 1]$, $(p_{\delta, \varepsilon}, q_{\delta, \varepsilon}) \in (P \times \Lambda) \cap E_0$ satisfies

(i) \[ \int_0^T |q_{\delta, \varepsilon}|^\alpha |p_{\delta, \varepsilon}|^3 \, dt + \int_0^T |q_{\delta, \varepsilon}|^\alpha \, dt \leq C_2, \]

(ii) \[ \varepsilon \int_0^T (|q_{\delta, \varepsilon}|^\gamma + |p_{\delta, \varepsilon}|^\gamma) \, dt \leq C_2, \]

(iii) \[ \delta \int_0^T \frac{1}{|q_{\delta, \varepsilon}|^\gamma} \, dt \leq C_2. \]
Proposition 5.1 For any $\delta \in [0, 1]$, $(p_{\delta, \varepsilon}, q_{\delta, \varepsilon})$ possesses a subsequence converging in $E$ to $(p_\delta, q_\delta) \in (P \times \Lambda) \cap E_0$. Moreover,

$$I'_\delta(p_\delta, q_\delta) = 0,$$
$$I_\delta(p_\delta, q_\delta) \leq \bar{c}.$$

Proof. Since $I'_\delta(p_{\delta, \varepsilon}, q_{\delta, \varepsilon}) = 0$, we have

$$q_{\delta, \varepsilon} = H_p(t, p_{\delta, \varepsilon}, q_{\delta, \varepsilon}) + \varepsilon \gamma |p_{\delta, \varepsilon}|^{\gamma - 2} p_{\delta, \varepsilon}, \quad (5.1)$$
$$\dot{p}_{\delta, \varepsilon} = -[H_q(t, p_{\delta, \varepsilon}, q_{\delta, \varepsilon}) + \delta \gamma q_{\delta, \varepsilon} |q_{\delta, \varepsilon}|^{\gamma - 2} - \varepsilon \gamma |q_{\delta, \varepsilon}|^{\gamma - 2} q_{\delta, \varepsilon}]. \quad (5.2)$$

By (H4) and 5.1, we can see from (i)-(ii) of Lemma 5.1 that

$$\int_0^T |\dot{q}_{\delta, \varepsilon}| dt \leq a_5 \left[ \int_0^T |q_{\delta, \varepsilon}|^\alpha |p_{\delta, \varepsilon}|^{\beta - 1} dt + \int_0^T |q_{\delta, \varepsilon}|^\gamma dt \right] + a_6 \int_0^T |q_{\delta, \varepsilon}| dt + \varepsilon \gamma \int_0^T |p_{\delta, \varepsilon}|^{\gamma - 1} dt$$
$$\leq C_5$$

where $C_5 > 0$ is a constant independent of $\varepsilon$. Thus, we deduce that $(q_{\delta, \varepsilon})_\varepsilon$ is bounded in $L^\infty(0, T; \mathbb{R}^N)$.

By (H4) and (5.1) again, we have

$$||\dot{q}_{\delta, \varepsilon}||_{\gamma^{\frac{\gamma - 1}{\gamma}}} \leq a_5 ||(q_{\delta, \varepsilon}|^\alpha |p_{\delta, \varepsilon}|^{\beta - 1})||_{\gamma^{\frac{\gamma - 1}{\gamma}}} + a_5 ||q_{\delta, \varepsilon}||_{\alpha^{\frac{\gamma - 1}{\gamma}}}$$
$$+ a_6 ||q_{\delta, \varepsilon}||_{\gamma^{\frac{\gamma - 1}{\gamma}}} + \varepsilon \gamma ||p_{\delta, \varepsilon}||_{\gamma^{\frac{\gamma - 1}{\gamma}}}^{-1}.$$

Here we will apply the Hölder’s inequality

$$||fg||_s \leq ||f||_s ||g||_s$$

with $f(t) = |q_{\delta, \varepsilon}|^\frac{s}{(\gamma - 1)^\frac{\beta - 1}{\alpha}}$, $g(t) = (|q_{\delta, \varepsilon}|^\alpha |p_{\delta, \varepsilon}|^\beta)^{\frac{\alpha - 1}{\alpha}}$, $s = \frac{\gamma}{\gamma - 1}$, $\mu = \frac{(\gamma - 1)\beta}{\alpha}$ and $\nu = \frac{(\gamma - 1)\beta}{(\gamma - 1)^\frac{\beta - 1}{\alpha}}$.

We verify that $\frac{1}{s} + \frac{1}{\mu} = 1$. Then we have

$$||(|q_{\delta, \varepsilon}|^\alpha |p_{\delta, \varepsilon}|^{\beta - 1})||_{\gamma^{\frac{\gamma - 1}{\gamma}}} = ||(|q_{\delta, \varepsilon}|^\frac{s}{(\gamma - 1)^\frac{\beta - 1}{\alpha}})(|q_{\delta, \varepsilon}|^\alpha |p_{\delta, \varepsilon}|^\beta)^{\frac{\alpha - 1}{\alpha}}||_{\gamma^{\frac{\gamma - 1}{\gamma}}}$$
$$\leq ||(|q_{\delta, \varepsilon}|^\frac{s}{(\gamma - 1)^\frac{\beta - 1}{\alpha}})||_{\frac{\gamma}{\gamma - 1}} ||(|q_{\delta, \varepsilon}|^\alpha |p_{\delta, \varepsilon}|^\beta)^{\frac{\alpha - 1}{\alpha}}||_{\frac{\gamma}{\gamma - 1}}$$
$$= ||q_{\delta, \varepsilon}||_{\gamma s} ||(|q_{\delta, \varepsilon}|^\alpha |p_{\delta, \varepsilon}|^\beta)^{\frac{\alpha - 1}{\alpha}}||_{\gamma^{\frac{\gamma - 1}{\gamma}}}$$
$$\leq C_6$$

where $C_6 > 0$ is a constant independent of $\varepsilon$. 
Finally \((q_\delta, \varepsilon)\) is bounded in \(W^{1, \frac{\gamma}{\gamma-1}}(0, T; \mathbb{R}^N)\). That is we can extract a subsequence -still indexed by \(\varepsilon\)- such that
\[
q_\delta, \varepsilon \to q_\delta \in \Lambda \quad \text{uniformly in } [0, T].
\] (5.3)
Since \(\int_0^T |q_\delta, \varepsilon| \alpha |p_\delta, \varepsilon| \beta \, dt \leq C_2\), we get
\[
\int_0^T |p_\delta, \varepsilon| \beta \, dt \leq C_7
\] (5.4)
for some constant \(C_7 > 0\) independent of \(\varepsilon\). By (H5) and 5.2-5.4, there exists a constant \(C_8 > 0\) independent of \(\varepsilon\) such that
\[
\int_0^T |\dot{p}_\delta, \varepsilon| \gamma \, dt \leq C_8
\]
and
\[
\int_0^T |\dot{q}_\delta, \varepsilon| \gamma \, dt \leq C_8.
\]
So we can extract a subsequence -still indexed by \(\varepsilon\)- such that
\[
p_\delta, \varepsilon \to p_\delta \quad \text{strongly in } L^\gamma(0, T; \mathbb{R}) \text{ and uniformly in } [0, T].
\] (5.5)
By 5.3 and 5.5, a passage to the limit on 4.1-4.2 similar as in Section 4 completes the proof.

6 Proof of Theorem 1.1

We take a limit as \(\delta \to 0\) to obtain a \(T\)-periodic solution of (1.1). Let \((p_\delta, q_\delta) \in (P \times \Lambda) \cap E_0\) be a critical point of \(I_\delta(p, q)\) obtained by Proposition 5.1. By Lemma 5.1, 5.3 and 5.5, we have

**Lemma 6.1** For any \(\delta \in [0, 1]\),

(i) \[
\int_0^T |q_\delta| \alpha |p_\delta| \beta \, dt + \int_0^T |q_\delta| \alpha \, dt \leq C_2,
\]

(ii) \[
\delta \int_0^T \frac{1}{|q_\delta| \gamma} \, dt \leq C_2.
\]

By (i) of Lemma 6.1, we can extract a subsequence -still indexed by \(\delta\)- such that
\[
q_\delta \to q \quad \text{weakly in } L^\gamma(0, T; \mathbb{R}^N).
\]

We also remark that \(I_\delta'(p_\delta, q_\delta) = 0\) is equivalent to
\[
\dot{q}_\delta = H_p(t, p_\delta, q_\delta),
\]
\[
\dot{p}_\delta = -H_q(t, p_\delta, q_\delta) + \delta \frac{q_\delta}{|q_\delta|^{\gamma+2}}.
\] (6.1) (6.2)

**Lemma 6.2** \(q_\delta \to q \in \Lambda\) uniformly in \([0, T]\).
Proof. By (H4) and 6.1, we have

\[ \int_0^T |\dot{q}_\delta| dt \leq a_5 \int_0^T |q_\delta|^\alpha |p_\delta|^{\beta-1} dt + a_5 \int_0^T |q_\delta|^\alpha dt + a_6 \int_0^T |q_\delta| dt. \]

Using (i) of Lemma 6.1, we can see that \( \|q_\delta\|_{W^{1,1}(0,T;\mathbb{R}^N)} \) is bounded. Thus we can find a constant \( C_9 > 0 \) independent of \( \delta \), such that

\[ \int_0^T |\dot{q}_\delta|^{\beta-1} dt \leq C_9. \]

Consequently, we obtain \( q_\delta \to q \) uniformly in \([0,T]\).

We now argue indirectly and suppose that \( q(t_0) = 0 \) for some \( t_0 \in [0,T] \).

We may assume \( t_0 = 0 \). By 6.1, for any \( t \in [0,T] \) we have

\[ |\log |q_\delta(t)| - \log |q_\delta(0)|| \leq \int_0^t \frac{|q_\delta(s)|}{|q_\delta|} ds = \int_0^t \frac{|H_p(s,p_\delta,q_\delta)|}{|q_\delta|} ds. \] (6.3)

By (H4),

\[ \int_0^t \frac{|H_p(s,p_\delta,q_\delta)|}{|q_\delta|} ds \leq a_5 \int_0^t |q_\delta|^\alpha |p_\delta|^{\beta-1} ds + a_5 \int_0^t |q_\delta|^\alpha ds + a_6 T. \]

Since \( \alpha > \beta > 1 \) and \( \int_0^T |q_\delta|^\alpha |p_\delta|^{\beta} dt \leq C_2 \), there exists a constant \( C_{10} > 0 \) independent of \( \delta \), such that

\[ \int_0^t \frac{|H_p(s,p_\delta,q_\delta)|}{|q_\delta|} ds \leq C_{10}. \] (6.4)

Passing to the limit in 6.3, we see that \( q_\delta \to 0 \) uniformly in \([0,T]\). By 6.1-6.2, we have

\[ I_\delta(p_\delta,q_\delta) = \int_0^T H_p(t,p_\delta,q_\delta) p_\delta dt - \int_0^T H(t,p_\delta,q_\delta) dt + \delta \int_0^T \frac{1}{|q_\delta|^{\gamma}} dt \]

\[ = \int_0^T H_q(t,p_\delta,q_\delta) q_\delta dt - \int_0^T H(t,p_\delta,q_\delta) dt + \delta(\gamma + 1) \int_0^T \frac{1}{|q_\delta|^{\gamma}} dt. \]

Hence

\[ \int_0^T [H_q(t,p_\delta,q_\delta) q_\delta - H_p(t,p_\delta,q_\delta) p_\delta] dt + \delta \gamma \int_0^T \frac{1}{|q_\delta|} dt = 0. \]

From (H6)(i) and (H2)(ii), it follows that

\[ a_7 a_2 \int_0^T |q_\delta|^\alpha |p_\delta|^{\beta} dt - a_7 \int_0^T K_1(q_\delta) dt + \int_0^T K_2(q_\delta) dt + \delta \gamma \int_0^T \frac{1}{|q_\delta|^{\gamma}} dt \leq 0 \]
for small $\delta$. Since $q_\delta \to 0$ uniformly in $[0, T]$, we find
\[ \int_0^T |q_\delta|^\alpha |p_\delta|^\beta \, dt \to 0 \text{ as } \delta \to 0. \quad (6.5) \]

Thus we can see from 6.1, 6.5 and (H6)(ii),
\[ \int_0^T \frac{|q_\delta|^\alpha}{|q_\delta|^{\frac{\alpha}{\gamma}} - 1} |q_\delta|^\beta \, dt \leq a_8 \int_0^T \frac{\|q_\delta\|^\beta}{\|q_\delta\|_\infty^{\frac{\beta}{\gamma}}} \, dt \to 0 \text{ as } \delta \to 0. \quad (6.6) \]

In other hand, we have from Lemma 2.1
\[ \int_0^T \frac{|\dot{q}_\delta|^\beta}{|q_\delta|^{\frac{\beta}{\gamma}}} \, dt \geq \frac{\left( \int_0^T |\dot{q}_\delta| \, dt \right)^{\frac{\beta}{\gamma}}}{T^{\frac{1}{2}} \|q_\delta\|_\infty^{\frac{\beta}{\gamma}}} \geq \frac{1}{T^{\frac{1}{2}} \|q_\delta\|_\infty^{\frac{\beta}{\gamma}}} \to +\infty \text{ as } \delta \to 0. \]

This is a contradiction to 6.6 which proves the Lemma 6.2.

**Lemma 6.3** There exists a constant $C_{11}$ independent of $\delta \in [0, 1]$ such that
\[ \|p_\delta\|_{W^{1, \gamma}(0, T; \mathbb{R}^N)} \leq C_{11}. \]

**Proof.** Since $q_\delta \to q \in \Lambda$ uniformly in $[0, T]$ and $\int_0^T |q_\delta|^\alpha |p_\delta|^\beta \, dt \leq C_2$, there exists a constant $C_{12} > 0$ independent of $\delta \in [0, 1]$ such that
\[ \int_0^T |p_\delta|^\beta \, dt \leq C_{12}. \]

By (H5) and 6.2, one deduce that $\int_0^T |p_\delta| \, dt$ is bounded. Thus we can see for some constant $C_{11} > 0$ independent of $\delta \in [0, 1]$
\[ \|p_\delta\|_{W^{1, \gamma}(0, T; \mathbb{R}^N)} \leq C_{11}. \]

We complete the proof of Theorem 1.1 as follows: By Lemmas 6.2 and 6.3, we can extract a subsequence -still indexed by $\delta$- such that $p_\delta \to p$ strongly in $L^\gamma(0, T; \mathbb{R}^N)$ and $(p_\delta, q_\delta) \to (p, q) \in (P \times \Lambda) \cap E_0$ uniformly in $[0, T]$. Since $I_\delta'(p_\delta, q_\delta) = 0$, we get
\[ I'(p, q)(h, k) = 0 \quad \text{for all } (h, k) \in E. \]
That is $(p, q) \in (P \times \Lambda) \cap E_0$ is a non-constant $T$-periodic solution of (1.1).
7 Remarks on the prescribed energy problem

If \( H(t,p,q) \) does not depend on \( t \), then the energy surface

\[
S_h = H^{-1}(h) = \{(p,q) \in \mathbb{R}^N \times \mathbb{R}^N; \ H(p,q) = h \} \ (h > 0)
\]

is not compact for such Hamiltonian functions. Moreover, \( S_h \) is equal to

\[
\tilde{H}^{-1}(1) = \{(p,q) \in \mathbb{R}^N \times \mathbb{R}^N \setminus \{0\}; \ \tilde{H}(p,q) = 1\}
\]

where

\[
\tilde{H}(p,q) = \frac{H(p,q) - h}{|q|^\alpha} + 1. \quad (7.1)
\]

It is clear that, if \( H(p,q) \sim |q|^\alpha(|p|^\beta - 1) \), then

\[
\tilde{H}(p,q) \sim |p|^\beta - \frac{h}{|q|^\alpha}. \quad (7.2)
\]

In the last few years, the existence of periodic solutions of singular Hamiltonian systems has been studied via variational methods under the situation related to two-body problem in celestial mechanics. That is, situation \( \tilde{H}(p,q) \) is of the form

\[
\tilde{H}(p,q) = \frac{1}{2} |p|^2 + V(q)
\]

where \( V(q) \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R}) \) and \( V(q) \rightarrow -\infty \) as \( q \rightarrow 0 \). See [8, 9, 10] and references therein. Results dealing with more general singular Hamiltonians of the form (7.2) can be found in [7, 11] for fixed period problems, and in [12, 13] for fixed energy problems.

According to the fundamental lemma of Rabinowitz (see [1] and [14, lemma 3.1]), it follows that the Hamiltonian system (1.1) has, for \( H \) and \( \tilde{H} \) which are related by 7.1, the same orbits on \( S_h \). Therefore, under suitable conditions on \( H \) including \( |q|^\alpha(|p|^\beta - 1) \) with \( \alpha > \beta > 1 \), the theorem of [12] carries a non-collision orbit of the singular Hamiltonian system

\[
\begin{align*}
\dot{q} &= \tilde{H}_p(p(t), q(t)) \\
\dot{p} &= -\tilde{H}_q(p(t), q(t)) \\
\tilde{H}(p,q) &= 1,
\end{align*}
\]

which corresponds to a non-constant periodic solution of (1.1) with energy \( h \).

Appendix: Proof of Lemma 3.2

The proof of Lemma 3.2 is a special case of [7, lemma 3.1]. We fix \( A \in \Gamma_m \) and take \( R > 0 \) such that

\[
R > \lambda \max_{\xi \in S^{N-2}} \| \text{proj}_m \frac{d}{dt}(\sigma_0)(\xi)(t) \|^\alpha \frac{1}{\gamma - 1} \frac{d}{dt}(\sigma_0)(\xi)(t) \|^\beta,
\]
A(p, ξ) = (p, σ_0(ξ)) \quad \text{if} \quad ||p||_β ≥ R.

We note that

\[ A(p, ξ) = (x(p, ξ), y(p, ξ)), \quad (A.1) \]

\[ B(ρ) = \{ p \in P_m; \; ||p||_β ≤ ρ \}, \quad ρ > 0. \]

Then we define the function \( φ(ρ) ∈ C(ℝ, [0, 1]) \) such that

\[ φ(ρ) = \begin{cases} 1, & ρ ≤ R, \\ 0, & ρ ≥ 2R. \end{cases} \]

Using the notation (A.1), we define a mapping \( F: P_m × S^{N-2} × [0, T]/\{0, T\} \sim P_m × S^{N-2} × S^1 \to P_m × S^{N-1} \) by

\[ F(p, ξ, t) = (x(p, ξ) − λφ(||p||_β)proj_m(||\dot{y}(p, ξ)||^{1−1}_γ − 1\dot{y}(p, ξ)), \tilde{σ}(ξ)(t)) \]

where \( \tilde{σ}(ξ)(t) = \frac{σ(ξ)(t)}{|σ(ξ)(t)|} \) and \( σ(ξ)(t) = (3 + \cos\frac{2πt}{T})(ξ_1, \ldots, ξ_{N-1}, 0) − (3, 0, \ldots, 0) + (0, \ldots, 0, \sin\frac{2πt}{T}). \)

We remark that \( F(p, ξ, t) = (p, \tilde{σ}(ξ)(t)) \) for \( ||p||_β ≥ 2R \) and the degree of the map \( \tilde{σ}: S^{N-2} × S^1 \to S^{N-1} \) is not equal to zero.

Thus, there exists \( R’ ≥ 2R \) such that the degree of the mapping

\[ F: (B(R’) × S^{N-2} × S^1; ∂B(R’) × S^{N-2} × S^1) \to (B(R’) × S^{N-1}; ∂B(R’) × S^{N-1}) \]

is not equal to zero. Then it follows the existence of \( (p, ξ) \) such that

\[ x(p, ξ) − λφ(||p||_β)proj_m(||\dot{y}(p, ξ)||^{1−1}_γ − 1\dot{y}(p, ξ)) = 0. \]

By the definition of \( R \), we have necessarily \( ||p||_β ≤ R \). That is

\[ x(p, ξ) = λproj_m(||\dot{y}(p, ξ)||^{1−1}_γ − 1\dot{y}(p, ξ)) \]

and then

\[ A(P_m × S^{N-2}) \bigcap D_m,λ ≠ \emptyset. \]

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References


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