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DECAY ESTIMATES OF HEAT TRANSFER TO MELTON POLYMER FLOW IN PIPES WITH VISCOUS DISSIPATION

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Abstract. In this work, we compare a parabolic equation with an elliptic equation both of which are used in modeling temperature profile of a power-law polymer flow in a semi-infinite straight pipe with circular cross section. We show that both models are well-posed and we derive exponential rates of convergence of the two solutions to the same steady state solution away from the entrance. We also show estimates for difference between the two solutions in terms of physical data.

1. Introduction

Chemical engineers frequently use the following equation to model temperature distribution of melton polymer flows inside a semi-infinite circular straight pipe with viscous dissipation

$$\rho c_p u \frac{\partial T}{\partial z} = k\frac{\partial^2 T}{\partial r^2} + \frac{k}{r}\frac{\partial T}{\partial r} + \eta(\frac{du}{dr})^2. \tag{1.1}$$

In this equation,

$$\eta = Ae^{-nB(T-T_m)}\big|\frac{du}{dr}\big|^{n-1},$$

ρ, c_p, k, A, B, T_m, and n are positive constants,

$$u = u_{av}(\frac{\nu+2}{\nu})[1-(\frac{r}{R})^\nu]$$

is the flow velocity in the pipe direction with $\nu = \frac{n+1}{n}$, $r = \sqrt{x^2+y^2}$, R is the radius of the pipe, u_{av} is the mean flow velocity, and $T = T(r,z)$ is the unknown temperature of the flow at location (r,z) with $0 \le r \le R$ and $0 \le z < \infty$. The constant n is called the power-law index which satisfies $0 < n < \infty$ and the flows are frequently referred to as power-law flows. The constants are being obtained experimentally. Here the origin of the xy-plane is at the center of the cross section of the pipe at $z = 0$, and the z-axis is in the pipe flow direction. Equation (1.1) is a nonlinear parabolic equation. The initial and boundary conditions for the equation

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are

$$\begin{aligned} T(r,0) = T_0, &\quad u(R,z)=0,\\ T(R,z)=T_w, &\quad \frac{\partial T}{\partial r}(0,z)=0, \end{aligned} \tag{1.2}$$

where T_w is the pipe wall temperature and T_0 the fluid temperature at the pipe entrance. The boundary condition $\frac{\partial T}{\partial r}(0,z)=0$ is due to the assumption that the solution is symmetric about the z-axis. See, e.g., [1, 6, 19, 23] for detail derivation of the model. Introducing the dimensionless parameters

$$t = \frac{\nu k z}{(\nu+2)\rho c_p u_{av} R^2}, \quad \bar{r} = \frac{r}{R}. \tag{1.3}$$

We then have the following nonlinear parabolic initial-boundary value problem

$$\begin{gathered} (1-\bar{r}^\nu)\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial T}{\partial \bar{r}} + ce^{-nBT}\bar{r}^\nu \quad \text{in } (0,1)\times(0,+\infty),\\ T(\bar{r},0)=T_0 \quad \text{in } (0,1),\\ T(1,t)=T_w \quad \text{in } (0,+\infty),\\ \frac{\partial T}{\partial \bar{r}}(0,t)=0 \quad \text{in } (0,+\infty), \end{gathered} \tag{1.4}$$

where $c = \frac{u_{av}^{n+1}}{k} A e^{nBT_m} \frac{(\nu+2)^{n+1}}{R^{n-1}}$. If the thermal resistance of the pipe wall is not ignored, then the boundary condition $T = T_w$ on the wall will be replaced by a mixed boundary condition $-k\frac{\partial T}{\partial r} = h(T-T_w)$, where T_w now represents the ambient temperature in the exterior of the pipe, the positive constant h is the film coefficient. In the new variables, we replace $T(1,t) = T_w$ in (1.4) by $\frac{\partial T}{\partial \bar{r}}(1,t) = \bar{h}(T_w - T(1,t))$, where $\bar{h} = \frac{ah}{k}$. One of the main assumptions used in deriving this model is that heat transfer by conduction in the pipe direction is negligible compared to both convection in the pipe direction and the conduction in the directions perpendicular to the pipe, which leads to the absence of the term $L\frac{\partial^2 T}{\partial t^2}$, where $L = \big[\frac{\nu k}{(\nu+2)\rho c_p u_{av} R^2}\big]^2$, in the right-hand side of (1.4) in engineering literature. This assumption is frequently used when one is not concerned with the entrance effect and when L is very small. The cancellation of the term $L\frac{\partial^2 T}{\partial t^2}$ changes the partial differential equation from an elliptic type to a parabolic type, and therefore allows one to use different analytical and numerical solution techniques to find the temperature profile in the pipe. Finite difference and numerical ODE techniques can be used for the parabolic model, as is done in [1], to produce stable numerical schemes for approximation of the temperature distribution in the pipe. And in some special cases closed form or semi-closed form solutions are available, see, e.g., results in [6] and [15]. One other advantage of using finite difference and the parabolic model is that it requires less effort in discretizing the domain as compared with other numerical methods. However, the well-posedness of the parabolic problem in the classical sense is not automatic due to the degeneracy of the coefficient of $\frac{\partial T}{\partial t}$, since $u = 0$ on the pipe wall. The parabolic model can not provide accurate solution in the entrance region of the pipe nor can it provide such solutions to pipes with large L which are of considerable practical interest in many applications. When the entrance effect is of main concern, or when high thermal diffusivity fluid flows at a low mean velocity, the conduction term $\frac{\partial^2 T}{\partial t^2}$, must be added to the right-hand side of (1.4) which then becomes a nonlinear elliptic equation. This is especially important for modelling of polymer flows in extrusion dies. We show that the elliptic problem is well posed due to the theory of Fredholm alternatives and comparison principle. However the solution is not classical, and there is a possible discontinuity of the solution at the entrance wall. In numerical approximation of the solution to the elliptic model, more storage and computing time may be required in a computer, and therefore is less economical as being compared with the parabolic model. It however provides more accurate physical solutions.

The purpose of this work is to give a mathematical analysis of the two models with comparison, which to our knowledge is not available in literature. We show that both the parabolic and the elliptic problem are well posed and the solutions to both problems on a cross section of the pipe converge exponentially to the same steady state solution as the cross section moves far away from the entrance of the pipe. In the parabolic case, we show that there exists a unique weak solution which is almost a classical solution except on the pipe wall, while the elliptic problem has a unique weak solution which is everywhere regular except at the boundary of the pipe inlet. We derive a analytic steady state solution and give explicit a priori estimates of rates of convergence of the two solutions to the steady state solution in the interior of the pipe cross sections. We also estimate the difference between the two solutions in terms of physical data. In a more general situation, if the cross section of the pipe is not circular, we denote this cross section by Ω and assume that it is a bounded open set in R^2 with smooth boundary. The corresponding problem to (1.4) is then

$$\begin{gathered}\rho c_p u\frac{\partial T}{\partial z} = k(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) + \eta\big|\nabla u\big|^2 \quad \text{in } \Omega\times(0,+\infty),\\ T = T_0 \quad \text{0n } \Omega\times\{0\},\\ T = T_w \quad \text{on } \partial\Omega\times[0,\infty),\end{gathered}\tag{1.5}$$

where $\eta = Ae^{-nB(T-T_m)}\big|\nabla u\big|^{n-1}$.

In this case, although an explicit form of the steady state solution is not available, our main results of existence, uniqueness and rates of convergence are still valid. In the following, we will first consider boundary condition of the mixed type $-k\frac{\partial T}{\partial r} = h(T-T_w)$ for the equations. The corresponding results for the Dirichlet boundary condition $T = T_w$ follows similarly. We will restrict ourself to circular straight pipes. In section 2, we derive the closed form steady state solution of the temperature profile for $t\to\infty$ in the pipe. In section 3, we prove existence and uniqueness of a weak solution to the parabolic problem and derive explicit rate of convergence of the solution to the steady state solution. In section 4, we prove that the elliptic model is well-posed and also derive rate of convergence of the solution to the same steady state solution. In section 5, we estimate the difference between the solutions of the two models in terms of the quantity L, and in section 6 we discuss the implications of the results in application. This type of work has been considered by several mathematicians. See, e.g., [3], [22], and [18] for decay estimates and [8], [13] for existence and regularity results on other equations.

For simplicity in notation, in the rest of this paper, we will replace $\bar{r}$ by r and $\bar{h}$ by h. In this work, we do not consider variable or time dependent boundary conditions.

2. The fully developed temperature profile

Assuming that the solution T of (1.4) is independent of t, then T satisfies the equation:

$$\begin{gathered} \frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + ce^{-nBT}r^{\nu} = 0 \quad \text{in } (0,1), \\ T(1) = T_w\,, \quad \frac{\partial T}{dr}(0) = 0. \end{gathered} \tag{2.1}$$

By using Chambre's method, see, e.g., [2], we find that the solution to (2.1) is

$$T = T_w + \frac{2}{nB} ln \frac{c_1 r^{\nu+2} + 1}{c_1 + 1}, \tag{2.2}$$

where

$$c_1 = \sqrt{[\frac{cnB + (\nu+2)^2 e^{nBT_w}}{cnB}]^2 - 1} - \frac{cnB + (\nu+2)^2 e^{nBT_w}}{cnB}.$$

If the boundary condition $T(1) = T_w$ in (2.1) is replaced by the mixed boundary condition $\frac{\partial T}{dr}(1) + h(T_w - T(1)) = 0$, then the solution is given by

$$T = T_w - \frac{2c_1(\nu+2)}{nBh(c_1+1)} + \frac{2}{nB} ln \frac{c_1 r^{\nu+2} + 1}{c_1 + 1}, \tag{2.3}$$

where the constant c_1 is implictly determined by

$$c_1 = \sqrt{[\frac{cnB + (\nu+2)^2 e^{nB\ell}}{cnB}]^2 - 1} - \frac{cnB + (\nu+2)^2 e^{nB\ell}}{cnB}.$$

with $\ell = T_w - \frac{2c_1(\nu+2)^2}{nBh(c_1+1)}$. We outline the derivation of (2.2) as following: First, let

$$w = r\frac{dT}{dr} \quad \text{and} \quad v = r^{\nu+2}e^{-nBT}.$$

Then from (2.1), we get

$$(\nu + 2 - nBw)\frac{dw}{dv} + c = 0,$$

which can be solved by using $v|_{r=0} = w|_{r=0} = 0$, and give

$$2(\nu+2)w - nBw^2 + 2cv = 0.$$

From this last equation, we have

$$-nBr^2\big(\frac{dT}{dr}\big)^2 + 2(\nu+2)r\frac{dT}{dr} = -2ce^{-nBT}r^{\nu+2}. \tag{2.4}$$

Multiplying both sides of (2.1) by $2r^2$ we have

$$2r^2\frac{d^2T}{dr^2} + 2r\frac{dT}{dr} = -2ce^{-nBT}r^{\nu+2}. \tag{2.5}$$

The right-hand sides of (2.4) and (2.5) are equal. This gives

$$\frac{d^2T}{dr^2} - \frac{\nu+1}{r}\frac{dT}{dr} + \frac{nB}{2}\big(\frac{dT}{dr}\big)^2 = 0, \tag{2.6}$$

which is a Bernoulli's equation. We have

$$\frac{dT}{dr} = \frac{2(\nu+2)r^{\nu+1}}{2C(\nu+2+nBr^{\nu+2}),}$$

which can be rewritten as

$$\frac{dT}{dr} = \frac{2c_1(\nu+2)r^{\nu+1}}{nB(c_1 r^{\nu+2}+1)}, \tag{2.7}$$

where $c_1 = \frac{nB}{2C(\nu+2)}$, and where C is an arbitrary integration constant. Integrating (2.7) and using $T(1) = T_w$ we then obtain (2.2). Substituting (2.2) into (2.1), we get

$$\frac{2[c_1(\nu+2)^2]}{nB(c_1 r^{\nu+2}+1)^2} + ce^{-nBT(r)} = 0.$$

Let $r = 1$ and $T(1) = T_w$, we then have

$$c_1 = \sqrt{[\frac{cnB + (\nu+2)^2 e^{nBT_w}}{cnB}]^2 - 1} - \frac{cnB + (\nu+2)^2 e^{nBT_w}}{cnB}.$$

Here, we choose $c_1 > -1$ so that $ln(c_1+1)$ is well-defined. Similarly, by using $\frac{dT}{dr}(1) + hT(1) = hT_w$, we then have (2.3) and

$$c_1 = \sqrt{[\frac{cnB + (\nu+2)^2 e^{nB\ell}}{cnB}]^2 - 1} - \frac{cnB + (\nu+2)^2 e^{nB\ell}}{cnB}.$$

with $\ell = T_w - \frac{2c_1(\nu+2)}{nBh(c_1+1)}$. Notice that the solution with the mixed boundary condition given in (2.3) reduces to the solution with Dirichlet boundary condition given in (2.2) as $h \to \infty$.

3. The parabolic model

The parabolic problem (1.4) with mixed boundary condition can be rewritten as

$$\begin{aligned} (1-r^\nu)\frac{\partial T}{\partial t} &= \Delta T + ce^{-nBT} r^\nu \quad \text{in } D\times(0,+\infty),\\ \frac{\partial T}{\partial n} &= h(T_w - T) \quad \text{in } \partial D\times(0,+\infty),\\ T &= T_0 \quad \text{in } D\times\{0\}, \end{aligned} \tag{3.1}$$

where $D = \{(x,y)|\sqrt{x^2+y^2} < 1\}$, is the unit disk and $\Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$. Since the coefficient of $\frac{\partial T}{\partial t}$ in (3.1) is degenerate at $r = 1$, we first consider a perturbed version of this equation

$$\begin{aligned} a(r,\epsilon)\frac{\partial T}{\partial t} &= \Delta T + ce^{-nBT} r^\nu \quad \text{in } D\times(0,+\infty),\\ \frac{\partial T}{\partial n} &= h(T_w - T) \quad \text{in } \partial D\times(0,+\infty),\\ T &= T_0 \quad \text{in } D\times\{0\}, \end{aligned} \tag{3.2}$$

where $a(r,\epsilon) = (1+\epsilon-r^\nu)$ and $0 < \epsilon < \frac{1}{2}$ is the perturbation parameter. A similar problem was studied in [8]. Let T_s denote the solution of the steady state problem obtained in (2.3). Let ϕ_1 be the first eigenfunction associated with the first eigenvalue λ_1 of

$$\begin{aligned} -\Delta\phi &= \lambda\phi \quad \text{in} \quad D,\\ \frac{\partial \phi}{\partial n} + h\phi &= 0 \quad \text{on } \partial D, \end{aligned} \tag{3.3}$$

where $D = \{(x,y) : \sqrt{x^2+y^2} < 1\}$. We have $\phi_1(r) = J_0(\sqrt{\lambda_1}r)$, $\sqrt{\lambda_1}$ is the smallest positive root of $\lambda J_0'(\lambda) + hJ_0(\lambda) = 0$, and $J_0(\lambda)$ is the Bessel function of the first kind with 0 order. It is well-known that for more general smooth, convex domains, the first eigenfunction ϕ_1 of (3.3) is smooth and strictly bounded below in $\bar{D}$ for $h > 0$. See, e.g., [20] and [24] for classical results on the eigenvalues and eigenfunctions. We first prove the following lemma:

Lemma 3.1. *For each $\epsilon > 0$, (3.2) has a unique classical solution T_ϵ. Furthermore, let $K = \max_{0\le r\le 1,} \left|\frac{T_0 - T_s}{\phi_1(r)}\right|$, then*

$$|T_\epsilon(r,t) - T_s(r)| \le K e^{-\alpha t}\phi_1(r), \quad (r,t) \in \bar{D} \times [0,\infty),$$

and $||T_\epsilon||_{L^\infty \bar{D}\times[0,\infty)} \le M$, where $\alpha = \frac{\lambda_1}{(1+\epsilon)}$ and M is a constant independent of ϵ.

Proof. Note that T_s satisfies:

$$\begin{gathered} -\Delta T_s = c e^{-nBT_s} r^\nu \quad \text{in } D\times(0,+\infty),\\ \frac{\partial T_s}{\partial n} = h(T_w - T_s) \quad \text{in } \partial D\times(0,+\infty),\\ T_s = T_0 \quad \text{in } D\times\{0\}. \end{gathered} \tag{3.4}$$

Let $W^u = T_s + Ke^{-\alpha t}\phi_1$, where $K > 0$ and $\alpha > 0$ are to be determined. We have

$$\begin{aligned} a(r,\epsilon)W^u_t - \Delta W^u &= -a(r,\epsilon)K\alpha e^{-\alpha t}\phi_1 - \Delta T_s + Ke^{-\alpha t}\Delta\phi_1\\ &= (\lambda_1 - \alpha a(r,\epsilon))Ke^{-\alpha t}\phi_1 + ce^{-nBT_s}r^\nu. \end{aligned}$$

Let $\alpha = \frac{\lambda_1}{(1+\epsilon)}$, then $\lambda_1 - \alpha a(r,\epsilon) \ge 0$, since $0 < a(r,\epsilon) \le (1+\epsilon)$. We have

$$a(r,\epsilon)W^u_t - \Delta W^u \ge ce^{-nBT_s}r^\nu \tag{3.5}$$

Since $\phi_1 > 0$, we have $ce^{-nBW^u}r^\nu - ce^{-nBT_s}r^\nu = ce^{-nBT_s}r^\nu(e^{-nBKe^{-\alpha t}\phi_1} - 1) \le 0$, which gives $ce^{-nBT_s}r^\nu \ge ce^{-nBW^u r^\nu}$. Let $K = \max_{0\le r\le 1}\left|\frac{T_0 - T_s}{\phi_1(r)}\right|$. From (3.5), we get

$$\begin{gathered} a(r,\epsilon)\frac{\partial W^u}{\partial t} - \Delta W^u \ge ce^{-nBW^u}r^\nu \quad \text{in } D\times(0,+\infty),\\ \frac{\partial W^u}{\partial n} = h(T_w - W^u) \quad \text{in } \partial D\times(0,+\infty),\\ W^u \ge T_0 \quad \text{in } D\times\{0\}. \end{gathered} \tag{3.6}$$

Similarly, let $W_l = T_s - Ke^{-\alpha t}\phi_1$, where $\alpha = \frac{\lambda_1}{(1+\epsilon)}$, and K large enough.

$$\begin{gathered} a(r,\epsilon)\frac{\partial W_l}{\partial t} - \Delta W_l \le ce^{-nBW_l}r^\nu \quad \text{in } D\times(0,+\infty),\\ \frac{\partial W_l}{\partial n} = h(T_w - W_l) \quad \text{in } \partial D\times(0,+\infty),\\ W_l \le T_0. \quad \text{in } D\times\{0\}, \end{gathered} \tag{3.7}$$

We have obtained an upper solution W^u and a lower solution W_l of (3.2). By the comparison principle, see, e.g., Theorem 4.1 in [17], there exists a unique classical solution T_ϵ of (3.2) satisfying

$$T_s(r) - Ke^{-\alpha t}\phi_1(r) \le T_\epsilon(r,t) \le T_s(r) + Ke^{-\alpha t}\phi_1(r), \tag{3.8}$$

This completes the proof of Lemma 3.1. □

We now show that the singular parabolic problem (3.1) has a unique solution which is everywhere "regular" except at the boundary $\partial D\times\{0\}$, where degeneracy occurs and the mixed boundary condition is satisfied only in the sense of trace.

Theorem 3.2. *There exists a unique weak solution T of (3.1) with the following interior regularity:*

$$T \in C^{2,1}_{loc}(D \times (0,\infty)) \cap C(D \times [0,\infty)).$$

Proof. Let D_1 be any smooth subdomain of D such that $\bar{D}_1 \subset D$ and let $\delta_1 > 0$. Then, by Lemma 3.1, there exists a positive constant C independent of ϵ such that

$$||T_\epsilon||_{L^\infty(\bar{D}_1 \times [\delta_1,\infty))} \le C.$$

We now use the standard local regularity theory for linear parabolic PDEs (see, e.g. [9] and [14]) and a "bootstrap" argument to prove that $\{T_\epsilon\}_{\epsilon>0}$ has a convergent subsequence which will converge to the solution of the problem. Applying the interior L^p estimates for $p > 1$, we then have,

$$||T_\epsilon||_{W^{2,1}_p(D_1 \times (\delta_1,\infty))} \le C_1.$$

Then, by Sobolev's embedding Theorem, there exists a $\sigma > 0$ such that

$$||T_\epsilon||_{C^{\sigma,\sigma/2}(\bar{D}_1 \times [\delta_1,\infty))} \le C_2.$$

By Schauder's estimates, we have

$$||T_\epsilon||_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar{D}_1 \times [\delta_1,\infty))} \le C_3.$$

The constants C_1, C_2 and C_3 are independent of ϵ since the coefficent $a(r,\epsilon)$ is uniformly boundad away from 0 in $\bar{D}_1 \times [\delta_1,\infty)$. Using the Ascoli-Arzela Theorem, we then can extract a subsequence $\{T_{\epsilon_k}\}_{k=1}^{\infty}$ of $\{T_\epsilon\}_{\epsilon>0}$, such that

$$T_{\epsilon_k} \to \text{ some } \quad T \in C^{2,1}(\bar{D}_1 \times [\delta_1,\infty)) \text{as} \quad k \to \infty$$

uniformly. Therefore, $T \in C^{2,1}_{loc}(D \times (0,\infty))$, and it satisfies (3.1). We now show that the function T obtained above is the weak solution to (3.1). First, we derive some energy estimates. Let $\bar{t} > 0$ be fixed. Multiply both sides of (3.1) by T_ϵ and then use integratition by parts. We have

$$\begin{aligned} &\int_0^{\bar{t}} \int_D 2|\nabla T_\epsilon|^2 dxdt + \int_D a|T_\epsilon|^2 dx = \int_D a|T_0|^2 dx + \\ &\int_0^{\bar{t}} \int_D 2ce^{-nBT_\epsilon} r^\nu T_\epsilon dxdt + \int_0^{\bar{t}} \int_{\partial D} 2h(T_w - T_\epsilon) dsdt. \end{aligned} \tag{3.9}$$

Therefore, by Lemma 3.1 and (3.9),

$$\int_0^{\bar{t}} \int_D 2|\nabla T_\epsilon|^2 dxdt + \int_D a|T_\epsilon|^2 dx \le M, \tag{3.10}$$

and we have $||T_\epsilon||_{L^2(0,\bar{t};H^1(D))} \le M$, where M is independent of ϵ.

Let $\langle \cdot,\cdot \rangle$ denote the duality pairing between $H^1_0(D)$ and $H^{-1}(D)$, $(\cdot,\cdot)$ the inner product of $L^2(D)$, and $(\cdot,\cdot)_{\partial D}$ the inner product of $L^2(\partial D)$. For any $V \in H^1_0(D)$, with $||V||_{H^1_0(D)} \le 1$, we have

$$\langle a\frac{\partial T_\epsilon}{\partial t}, V\rangle + (\nabla T_\epsilon, \nabla V) = (h(T_\epsilon - T_w), V)_{\partial D} + (g(T_\epsilon), V), \tag{3.11}$$

where $g(T_\epsilon) = ce^{-nBT_\epsilon} r^\nu$. By Lemma 3.1 and (3.11),

$$\int_0^{\bar{t}} ||a\frac{\partial T_\epsilon}{\partial t}||^2_{H^{-1}(D)} dt \le C_1 \int_0^{\bar{t}} ||\nabla T_\epsilon||_{H^1(D)} dt + C_2 \int_0^{\bar{t}} ||T_\epsilon||_{L^2(D)} dt + C_3\bar{t}.$$

We have shown that $\{T_\epsilon\}_{\epsilon>0}$ is uniformly bounded in $L^2(0,\bar{t};H^1(D))$, and $\{a\frac{\partial T_\epsilon}{\partial t}\}_{\epsilon>0}$ is uniformly bounded in $L^2(0,\bar{t};H^{-1}(D))$. Consequently there exists a subsequence $\{T_{\epsilon_k}\}_{k=1}^\infty \subset \{T_\epsilon\}_{\epsilon>0}$ and $\{\frac{\partial T_{\epsilon_k}}{\partial t}\}_{k=0}^\infty \subset \{\frac{\partial T_\epsilon}{\partial t}\}_{\epsilon>0}$, such that (see, e.g., p. 356 of [7])

$$\begin{aligned} T_{\epsilon_k} &\rightharpoonup T \quad \text{weakly in} \quad L^2(0,\bar{t};H^1(D)) \\ \frac{\partial T_{\epsilon_k}}{\partial t} &\rightharpoonup \frac{\partial T}{\partial t} \quad \text{weakly in} \quad L^2(0,\bar{t};H^{-1}(D)). \end{aligned}$$

By the Mean Value Theorem and Lemma 3.1, we have

$$|\int_0^{\bar{t}} (g(T_{\epsilon_k}) - g(T), V)dt| \le M \int_0^{\bar{t}} ||T_{\epsilon_k} - T||_{L^2(D)} dt.$$

Since weak convergence in $L^2(0,\bar{t};H^1(D))$ implies strong convergence in the space $L^2(0,\bar{t};L^2(D))$, we have $g(T_{\epsilon_k}) \rightharpoonup g(T)$ weakly in $L^2(0,\bar{t};L^2(D))$. For each $1 \le k < \infty$, T_{ϵ_k} satisfies

$$\begin{aligned} &\langle a\frac{\partial T_{\epsilon_k}}{\partial t}, V\rangle + (\nabla T_{\epsilon_k}, \nabla V) \\ &\quad = (h(T_{\epsilon_k} - T_w), V)_{\partial D} + (g(T_{\epsilon_k}), V) \quad \forall V \in L^2(0,\bar{t};H_0^1(D)). \end{aligned} \tag{3.12}$$

Passing to the limit, we then have

$$\begin{aligned} &\langle a\frac{\partial T}{\partial t}, V\rangle + (\nabla T, \nabla V) \\ &\quad = (h(T - T_w), V)_{\partial D} + (g(T), V) \quad \forall V \in L^2(0,\bar{t};H_0^1(D)), \end{aligned} \tag{3.13}$$

i.e., T is a weak solution of (3.1). To prove uniqueness, let $E = T_1 - T_2$ where T_1 and T_2 be two solutions of (3.1), then we have

$$\begin{aligned} a(r,\epsilon)\frac{\partial E}{\partial t} - \Delta E &= ce^{-nBT_1}r^\nu - ce^{-nBT_2}r^\nu \quad \text{in } D \times (0,+\infty), \\ \frac{\partial E}{\partial n} + hE &= 0 \quad \text{in } \partial D \times (0,+\infty), \\ E &= 0 \quad \text{in } D \times \{0\}. \end{aligned} \tag{3.14}$$

Integrating (3.14) and use the mean value theorem, we get

$$\frac{1}{2}\int_D aE^2 dxdy + \int_0^t \int_D |\nabla E|^2 dxdyd\tau = -\int_0^t \int_D cnBe^{-cnB\theta}E^2 dxdyd\tau,$$

where $\theta = \alpha T_1 + (1-\alpha)T_2$ for some $0 \le \alpha \le 1$. Therefore $E = 0$, since the right-hand side of this last equation is less than or equal to zero.

We now use the barrier function method to show that T satisfies the initial condition in the classical sense, i.e., T is continuous up to the boundary $D \times \{0\}$. Let $Q_0 = (x_0, y_0, 0)$ where $(x_0, y_0) \in D$ and $V_\epsilon = T_\epsilon - T_0$. Then, V_ϵ satisfies

$$\begin{aligned} a(r,\epsilon)\frac{\partial V_\epsilon}{\partial t} - \Delta V_\epsilon &= \Delta T_0 + ce^{-nB(T_0+V_\epsilon)}r^\nu \quad \text{in } D \times (0,+\infty), \\ V_\epsilon &= T_w - T_0 \quad \text{in } \partial D \times (0,+\infty), \\ V_\epsilon &= 0 \quad \text{in } D \times \{0\}. \end{aligned} \tag{3.15}$$

Choose D_1 so that $Q_0 \in D_1 \subset D$ and $\delta_1 = dist(\bar{D}_1, \partial D) > 0$. Then

$$a(r,\epsilon) \ge (1 - (1-\delta_1)^\nu) = \delta_2 > 0, \forall (x,y) \in \bar{D}_2.$$

By Lemma 3.1, the exists some $M_1 > 0$ such that

$$|\Delta T_0 + ce^{nB(T_0+V_\epsilon)} r^\nu| \le M_1.$$

Let $W_{Q_0}(x,y,t) = ((x-x_0)^2 + (y-y_0)^2 + At)e^t$. Then

$$a(r,\epsilon)\frac{\partial W_{Q_0}}{\partial t} - \Delta W_{Q_0} \ge (\delta_3 A - 4)e^t \text{in} \quad \bar{D}_1 \times [0,\infty).$$

By choosing A large enough, so that $a(r,\epsilon)\frac{\partial W_{Q_0}}{\partial t} - \Delta W_{Q_0} \ge M_1$ in $\bar{D}_1 \times [0,\infty)$, $W_{Q_0} \ge T_w - T_0$ on $\partial D_1 \times (0,\infty)$, and $W_{Q_0} \ge T_0$ in $D_1 \times \{0\}$. By comparison principle, see, e.g., Theorem 2.1 in [17], we have $|V_\epsilon(P)| \le W_{Q_0}(P), P \in \bar{D}_1 \times [0,\infty)$, which implies that $|T_\epsilon(P) - T_0(P)| \le W_{Q_0}(P)$. By letting $\epsilon \to 0$, it follows that, $\lim_{P\to P_0} T(P) = T_0(P)$. ☐

Theorem 3.3. *Let (λ_1, ϕ_1) be the first eigenpair of the problem*

$$-\Delta\phi = \lambda\phi \quad \text{in } D$$
$$\frac{\partial \phi}{\partial n} + h\phi = 0 \quad \text{on } \partial D,$$

where $D = \{(x,y) : x^2 + y^2 < 1\}$. Let $K = \max_{0\le r\le 1} \left|\frac{T_0 - T_s}{\phi_1(r)}\right|$. Then,

$$|T(r,t) - T_s(r)| \le Ke^{-\lambda_1 t}\phi_1(r) \quad \forall (r,t) \in D \times (0,\infty),$$

where T is the solution of (3.1) and T_s the steady state solution given by (2.3).

Proof. The result follows immediately by taking limit $\epsilon \to 0$ in (3.8). ☐

When the mixed boundary condition in (3.1) is replaced by the Dirichlet boundary condition, results similar to Lemma 3.1, Theorem 3.2 and Theorem 3.3 follows with no additional complication. We only state the following theorem.

Theorem 3.4. *Let δ be any small positive constant and let (λ_1, ϕ_1) be the first eigenpair of the problem*

$$-\Delta\phi = \lambda\phi \quad \text{in } D_\delta$$
$$\phi = 0 \quad \text{on } \partial D_\delta,$$

where $D_\delta = \{(x,y) : x^2 + y^2 < 1 + \delta\}$. Let $K = \max_{0\le r\le 1} \left|\frac{T_0 - T_s}{\phi_1(r)}\right|$. Then,

$$|T(r,t) - T_s(r)| \le Ke^{-\lambda_1 t}\phi_1(r) \quad \forall (r,t) \in D \times (0,\infty),$$

where T is the solution of (3.1) with Dirichlet boundary condition and T_s the steady state solution given by (2.2).

4. The elliptic model

If heat transfer by conduction in the flow direction is not neglected, then we need to add $L\frac{\partial^2 T}{\partial t^2}$ to the right hand side of (3.1). Replacing the variable t by $\sqrt{L}t'$, we then consider the following elliptic boundary value problem

$$\begin{gathered} -\Delta T + \frac{(1-r^\nu)}{\sqrt{L}}\frac{\partial T}{\partial t'} = ce^{-nBT}r^\nu \quad \text{in } D \times (0,+\infty), \\ T = T_0 \quad \text{in } D \times \{0\}, \\ \frac{\partial T}{\partial n} = h(T_w - T) \quad \text{on } \partial D \times [0,+\infty), \\ |T(t,r)| < \infty \quad \text{in } D \times (0,+\infty), \end{gathered} \tag{4.1}$$

where $\Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial t'^2}$, D is the unit disc in the xy-plane as defined in the previous section. Similar problems were studied in [4], [5], and [13] for existence and regularity of solutions. Again, for simplicity in notation, we denote t' by t in the rest of the section.

Theorem 4.1. *There exists a unique positive solution to the boundary value problem (4.1) which is everywhere regular except at the boundary $\partial D \times \{0\}$.*

Proof. Let $\Omega = D \times (0, \infty)$. Consider the following nonlinear operator

$$a(u, v) = (\nabla u, \nabla v) + (\eta \frac{\partial u}{\partial t}, v) + [u, v]_1, -(g(u), v)$$

and the linear operator

$$l(u, v) = [f, v]_2,$$

where $\eta = \frac{(1-r^\nu)}{\sqrt{L}}$, $(\cdot, \cdot)$ is the inner product defined on $L^2(\Omega)$,

$$[u, v]_1 = \int_{\partial D \times (0,\infty)} huvds,$$
$$[f, v]_2 = \int_{\partial D \times (0,\infty)} hT_w vds,$$

and $g(u) = ce^{-nB|u|}$.
Let $V = \{u \in H^2(\Omega) : u = T_0 \text{ on } D \times \{0\}\}$ and $V_0 = \{u \in H^2(\Omega) : u = 0 \text{ on } D \times \{0\}\}$. First, we look for a solution of the following problem: Find $u \in V$ such that

$$a(u, v) = l(u, v), \forall v \in V_0. \tag{4.2}$$

We now use the theory of Fredholm alternative for nonlinear operators to show that the above abstract problem is well-posed. In particular, we use Theorem 33.5 of [10] which is also valid for general boundary conditions [11]. Therefore, we have a unique weak solution u of (4.2) in V. Since the boundary conditions are smooth, and the boundary is everywhere smooth except at part of the boundary $\partial D \times \{0\}$. The standard regularity, see, e.g. pp 317-326 of [7], can be applied to show that T is everywhere smooth except at $D \times \{0\}$ which is the flow inlet boundary.

Then we show by maximum principle that the solution is positive and therefore, it is a solution of (4.1). The uniqueness of solution follows similarly as that of the parabolic problem. □

In the following, we derive rate of convergence of the solution of the elliptic model (4.1) to the steady state solution.

Theorem 4.2. *Let T be the solution of (4.1) and T_s the solution of (2.1). Let (λ_1, ϕ_1) be the first eigenpair of the problem*

$$-\Delta\phi = \lambda\phi \qquad \text{in } D$$
$$\frac{\partial \phi}{\partial n} + h\phi = 0 \quad \text{on } \partial D,$$

where $D = \{(x, y) : x^2 + y^2 < 1\}$. Then, there exits a $K > 0$ such that

$$|T(r, t) - T_s(r)| \le Ke^{-\sqrt{\lambda_1}t}\phi_1(r) \quad \forall (r, t) \in \bar{D} \times (0, \infty).$$

Proof. Let $T \in C(D \times [0,\infty)) \cap C^2(D \times (0,\infty))$ be the solution of (4.1) such that $\frac{\partial T}{\partial t}|_{t=\infty} = 0$, then by the maximum principle, $T \geq min\{T_w, T_0\}$. Let $W = T_s + Ke^{-\beta t}\phi_1$ where ϕ_1 is the first eigenfunction of (3.3), T_s is the solution of (2.1) given by (2.3), $K, \beta > 0$ are to be determined. Then

$$\begin{aligned} &-\Delta W + \frac{(1-r^\nu)}{\sqrt{L}}\frac{\partial W}{\partial t} - ce^{-nBW}r^\nu \\ &\qquad = (\lambda_1 - \beta^2 - \beta\frac{(1-r^\nu)}{\sqrt{L}})Ke^{-\beta t}\phi_1 + ce^{-nBT_s}r^\nu(1-e^{-nBKe^{-\beta t}\phi_1}). \end{aligned} \tag{4.3}$$

Since the second term is nonnegative, we only need to choose $\beta \in (0, \min_{0\leq r\leq 1}\hat{\beta})$, where $\hat{\beta} = \frac{1}{2\sqrt{L}}(\sqrt{(1-r^\nu)^2 + 4L\lambda_1} - (1-r^\nu))$ is the positive root of $\lambda_1 - \beta^2 - \beta\frac{(1-r^\nu)}{\sqrt{L}} = 0$ so that

$$-\Delta W + \frac{(1-r^\nu)}{\sqrt{L}}\frac{\partial W}{\partial t} - ce^{-nBW}r^\nu \geq 0.$$

For this purpose, we choose $\beta = \sqrt{\lambda_1}$. Let $V = T - W$. Then

$$-\Delta V + (1-r^\nu)\frac{\partial V}{\partial t} \leq ce^{-nBT}r^\nu - ce^{-nBW}r^\nu.$$

Therefore,

$$-\Delta V + (1-r^\nu)\frac{\partial V}{\partial t} + cnBr^\nu e^\theta V \leq 0$$

for some θ satisfying $T \leq \theta \leq W$. By maximum principle, V attains its maximum value on the boundary $\partial\Omega = D \times [0,\infty)$. On $t = 0$, $V = T_0 - T_s - K\phi_1 < 0$ for large K and $V|_{r=1} = -Ke^{-\beta t}\phi_1 < 0$. Therefore, the maximum value of V will occur only on $\partial D \times (0,\infty)$. Since T is bounded by Lemma 3.1, $V = T - T_s - Ke^{-\beta L}\phi_1 \leq 0$ on $D \times (0,\infty)$ for large enough K. Therefore $V \leq 0$, e.g., $T \leq T_s + Ke^{-\beta t}\phi_1$ in $D \times (0,\infty)$. Similarly, for large enough K, we can show that $T_s - Ke^{-\beta t}\phi_1 \leq T$ in $D \times (0,\infty)$. We conclude that

$$|T - T_s| \leq Ke^{-\beta t}\phi_1,$$

which is the result of our theorem. □

5. Comparing the two models

In this section, we will estimate the difference between the solutions of the parabolic model and the elliptic model. We will use the standard norm in the space $L^2(0,\hat{t}; H_0^1(D))$ where $\hat{t} > 0$ and D is the cross section of the pipe. We denote the norm by $|||u|||$ for $u \in L^2(0,\hat{t}; H_0^1(D))$, and have $|||u|||_{\hat{t}} = (\int_0^{\hat{t}} ||u||^2 dt)^{\frac{1}{2}}$, where $||u|| = (\int_D (|\nabla u|^2 + u^2)dxdy)^{\frac{1}{2}}$.

Theorem 5.1. *Let T_e be the solution of (4.1) and T_p the solution of (3.1). Then, there exits a constant $C > 0$ which depends only on D such that*

$$|||T_p - T_e|||_{\hat{t}} \leq C|||\frac{\partial^2 T_e}{\partial t^2}|||_{\hat{t}} L,$$

where $L = \left[\frac{\nu k}{(\nu+2)\rho c_p u_{av} R^2}\right]^2$.

Proof. Take the difference of (4.1) and (3.1) and let $E = T_e - T_p$, we have

$$(1 - r^\nu)\frac{\partial E}{\partial t} = \Delta E + cr^\nu(e^{-nBT_e} - e^{-nBT_e}) + L\frac{\partial T_e}{\partial t}.$$

Multiply both sides of the above equation by E, use the mean value theorem, and integrate over $D \times (0, \hat{t})$, we get

$$\begin{aligned}\int_D (1 - r^\nu)E^2 dxdy + \int_0^{\hat{t}}\int_D |\nabla E|^2 dxdyd\tau + \\ \int_0^{\hat{t}}\int_D cnBe^{-cnB\theta}E^2 dxdyd\tau = L\int_0^{\hat{t}}\int_D \frac{\partial^2 T_e}{\partial t^2} Edxdyd\tau,\end{aligned} \tag{5.1}$$

where $\theta = \alpha T_p + (1 - \alpha)T_e$ for some $0 \le \alpha \le 1$. The result then follows from Poinceré's inequality. □

6. Discussion of the results

To our knowledge the steady state solution (2.2) or (2.3) are not available in the literature. It can be used to verify if a numerical simulation of the unsteady state is reasonable. Here is an example taken out of [1] for a temperature-dependent high-density polyethylene melt flowing inside a tube held at a constant temperature. The following is the list of data: $T_0 = 403.15°K$, $T_w = 433.15°K, u_{av} = 15$ cm/s, $n = 0.453$, $\rho = 0.000794$ kg/cm^3, $c_p = 2510$J/(kg°K), $k = 0.00255$W/(cm°K), $R = 0.125$ cm, $A = 2.82$Ns^n/cm^2, $B = .0240K^{-1}$, $T_m = 399.5°K$. We provide the solution profile in Figure 1, where the temperature is given in Celsius and the radial distance is $0.125cm$.

One of the quantities that is important to engineers is the bulk temperature, which is given by

$$T_{\text{bulk}}(t) = \frac{\int_0^1 T(r,t)u(r,t)rdr}{\int_0^1 u(r,t)rdr}.$$

It can be calculated by using the steady state solution and obtain

$$\lim_{t\to\infty} T_{bulk}(t) = 452.39°C$$

for this particular example. Galerkin's method with linear axisymmetric triangular finite elements is used to approximate the solution of the elliptic problem (4.1). The resulting system of nonlinear algebraic equations is then solved iteratively by using Newton's method. The initial guess for Newton's method is taken to be the solution of the problem for $n = 1.0$ and $A = 0.0$. Numerical steady state is reached at approximately $z = 740cm$. The analytic steady state solution can be used to check the reliability of the numerical results.

The results given by Theorem 3.3, Theorem 3.4, Theorem 4.2, and Theorem 5.1 partially validate the use of these two different models for the same engineering problem mathematically. In this particular example, the quantity L in Theorem 5.1 has a values of 1.13×10^{-5} and therfore both models should produce similar numerical results. The error estimates given by the theorems, although computable, are not sharp away from the boundary of the pipe wall. The analysis provided

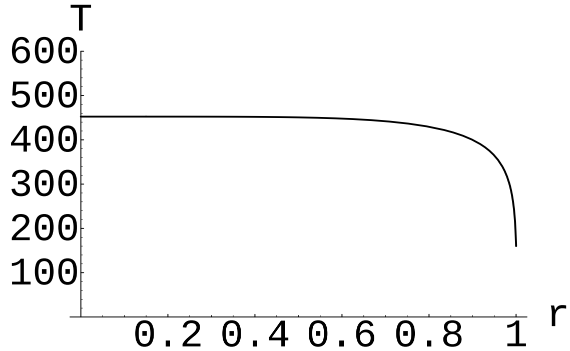

FIGURE 1. Steaty State Temperature Profile

help build confidence in using these two models for calculating the temperature distributions in the pipes.

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