Electronic Journal of Differential Equations, Vol. **2001**(2001), No. 44, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

An elementary proof of the Harnack inequality for non-negative infinity-superharmonic functions *

Tilak Bhattacharya

Abstract

We present an elementary proof of the Harnack inequality for non-negative viscosity supersolutions of $\Delta_\infty u = 0$. This was originally proven by Lindqvist and Manfredi using sequences of solutions of the p-Laplacian. We work directly with the Δ_∞ operator using the distance function as a test function. We also provide simple proofs of the Liouville property, Hopf boundary point lemma and Lipschitz continuity.

1 Introduction

Our effort in this note will be to provide an elementary proof of the Harnack inequality for nonnegative ∞-superharmonic functions. The ∞-harmonic operator, in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is defined as

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}. \tag{1.1}$$

A function $u = u(x_1, x_2, \ldots, x_n)$ is said to be ∞-harmonic if u is a solution of $\Delta_\infty u = 0$. In this work, by a solution u we will mean a viscosity solution. For definitions and background for such equations see [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13] and the references therein, in particular we mention the remarkable work of Jensen [6]. These references also discuss the relevance and the importance of the notion of viscosity solutions in the context of such nonlinear equations. We may also define ∞-superharmonicity: u is said to be ∞-superharmonic if it is a viscosity supersolution of (1.1) i.e., u is lower semicontinuous and satisfies

$$-\Delta_\infty u \geq 0,$$

in the viscosity sense. For our work we will take $u \geq 0$.

Let $\Omega \subset \mathbb{R}^n$, be a bounded domain and $\partial\Omega$ be its boundary; also let $B_r(P)$ denote the open ball in $\mathbb{R}^n$, of radius r and center P. The main result of this work is

* *Mathematics Subject Classifications:* 35J70, 26A16.
Key words: Viscosity solutions, Harnack inequality, infinite harmonic operator, distance function.

Submitted January 15, 2001. Revised May 17, 2001. Published June 14, 2001.

Theorem 1 *Let $u \geq 0$ be a viscosity supersolution of (1.1) in Ω. Also let $P \in \Omega$, $0 < r \leq \operatorname{dist}(P, \partial\Omega)$ and $B_r = B_r(P)$. If $M = sup_{B_{r/2}} u(x)$ and $m = inf_{B_{r/2}} u(x)$, then*

$$m \geq M/8\,.$$

The result we present here is not new; see, for instance, [5] for the case when solutions are C^2 and [11] for viscosity solutions of (1.1). The proof in [11] makes use of approximating sequences involving the p-Laplacian and captures the ∞-harmonic operator as the limiting operator when $p \to \infty$. This work also provides sharp local Lipschitz bounds for viscosity solutions of (1.1). However, we work directly with (1.1). We now briefly discuss the basic idea of our proof. For a ball $B_r(P)$ in Ω, define $d(x) = \operatorname{dist}(x, \partial B_r(P))$ for $x \in B_r(P)$. Our observation is that this distance function d acts as a universal barrier for the ∞-harmonic operator Δ_∞. This follows from rather elementary calculations. See Lemmas 1 and 2 in Section 2. More precisely, we show that if u is ∞-superharmonic and $u \geq 0$, but not identically 0, then

$$u(x) \geq u(P)\frac{d(x)}{d(P)}.$$

This estimate is the main contribution of our work and this in turn leads to an elementary proof of the Harnack inequality. The proof of this result appears in Lemma 2. As a matter of fact this observation also leads to straightforward proofs of the Hopf boundary point lemma, the well known Liouville property and local Lipschitz regularity. These appear in the Appendix. It has been pointed out to us by Juan Manfredi that some of the ideas presented in this work may be applicable to other situations such as the Heisenberg group. In a more recent work, we have been able to adapt the ideas of this work to prove similar results in the case of nonnegative viscosity supersolutions of the p-Laplacian, when $p > n$. We thank Peter Lindqvist and Juan Manfredi for having read an earlier version of this manuscript and for their comments. We also thank the referee for comments that have clarified the presentation greatly. For more information on the Harnack inequality, in this context, see [5, 11].

2 Preliminary results and the proof of the main theorem

We start with the following rather elementary result. It will set the stage for showing that the function d acts as a barrier for the ∞-harmonic operator.

Lemma 1 *Let $B_t(0)$ be the open ball in $\mathbb{R}^n$ centered at 0 and radius t and let $d(x) = \operatorname{dist}(x, \partial B_t(0)) = t - |x|$, $\forall\, x \in \mathbb{R}^n$. Then for $x \neq 0$ and $|x| \neq t$,*

$$\Delta_\infty d^\alpha(x) = \alpha^3(\alpha - 1)d(x)^{3\alpha-4}. \tag{2.1}$$

Proof. We observe that

$$\begin{aligned} D_i d^\alpha &= \alpha d^{\alpha-1}\left(\frac{-x_i}{|x|}\right) \quad \text{and} \\ D_{ij}(d^\alpha) &= \alpha d^{\alpha-1}\left(\frac{x_i x_j}{|x|^3} - \frac{\delta_{ij}}{|x|}\right) + \alpha(\alpha-1)d^{\alpha-2}\frac{x_i x_j}{|x|^2}. \end{aligned}$$

Thus we see that

$$\begin{aligned} \Delta_\infty d^\alpha &= (\alpha d^{\alpha-1})^2 \frac{x_i x_j}{|x|^2}\left[\alpha(\alpha-1)d^{\alpha-2}\frac{x_i x_j}{|x|^2} + \alpha d^{\alpha-1}\left(\frac{x_i x_j}{|x|^3} - \frac{\delta_{ij}}{|x|}\right)\right] \\ &= \left(\alpha d^{\alpha-1}\right)^2\left[\alpha(\alpha-1)d^{\alpha-2} + \alpha d^{\alpha-1}\left(\frac{1}{|x|} - \frac{1}{|x|}\right)\right] \\ &= \alpha^3(\alpha-1)d^{3\alpha-4}. \quad \diamondsuit \end{aligned}$$

We now prove the main estimate mentioned in the introduction. It appears as part (i) of Lemma 2. In what follows, we will take u to be lower semicontinuous. Note that the estimates are stated in terms of distances of points from the boundary of a certain ball they lie in. The basic observation is that the function $ku(x) - d(x)$ attains its infimum at the center of the ball which is being used for defining d. Here k is a suitable scaling constant. This fact leads to the estimate pointed out in Section 1.

Lemma 2 *Let $P \in \Omega$, $r \le \operatorname{dist}(P, \partial\Omega)$ and $B_r = B_r(P)$ be the open ball of radius r and center P. Set $d(x) = r - |x - P| = \operatorname{dist}(x, \partial B_r)$ for all $x \in \Omega$. Let $u(x) \ge 0$ solve $-\Delta_\infty u \ge 0$ in the viscosity sense. Assume $u(P) > 0$ and if $k > 0$ is such that $d(P) = ku(P) = r$, then $\forall\, x \in B_r$,*

(i) $u(x) \ge u(P)\frac{d(x)}{d(P)}$;

(ii) $u(x) - u(P) \ge -|x - P|/k$ *or* $u(x) + \frac{|x-P|}{k} \ge u(P)$.

Proof. We will scale the functions u and d as follows. We define $u_c(x) = cu(x)/r$ and $v(x) = d(x)/r$ where $0 < c < k$. By the definition of k,

$$u_c(P) = \frac{cu(P)}{r} < v(P) = \frac{ku(P)}{r} = 1\,.$$

Note that $v(x) = 0$ whenever $x \in \partial B_r$ and also note $d(P) = r$; fix c and set

$$w = u_c - v = \frac{cu(x)}{r} - \frac{d(x)}{r},$$

then $w(P) < 0$ and $w \ge 0$ on $\partial B_r(P)$. Clearly there is a negative infimum of w in $B_r(P)$. Our intention is to show that this infimum occurs at P. We proceed by contradiction. Suppose there is a point $x_c \ne P$ with the property that

$$\inf_{B_r} w(x) = w(x_c) < w(P) < 0\,.$$

Consider the function $v(x)^\alpha = (d(x)/r)^\alpha$, $\alpha > 1$ and we look at $w_\alpha(x) = u_c(x) - v^\alpha(x)$. Clearly,

$$w_\alpha(P) = u_c(P) - 1 < 0 \text{ and } w_\alpha(x) \geq 0 \quad \text{on } \partial B_r.$$

Choose α sufficiently close to 1 so that the point of infimum of w_α, denoted by $x_{c,\alpha}$, is different from P and

$$w_\alpha(x_{c,\alpha}) < w_\alpha(P) = u_c(P) - 1 < 0\,.$$

Note $x_{c,\alpha}$ is not in ∂B_r. Unscaling w_α, this implies that the function

$$\frac{rw_\alpha(x)}{c} = u(x) - \left[\frac{rv(x)^\alpha}{c}\right] = u(x) - \left[\frac{d(x)^\alpha}{cr^{\alpha-1}}\right]$$

has a negative infimum at $x_{c,\alpha} \neq P$. Now $v(x)^\alpha$ is C^2 near $x_{c,\alpha}$ and as u is a viscosity supersolution of (1.1), we have

$$-\Delta_\infty \left[\frac{d^\alpha(x_{c,\alpha})}{cr^{\alpha-1}}\right] \geq 0.$$

By Lemma 1,

$$\Delta_\infty \left[\frac{d^\alpha(x_{c,\alpha})}{cr^{\alpha-1}}\right] = \frac{\alpha^3(\alpha-1)d^{3\alpha-4}(x_{c,\alpha})}{(cr^{\alpha-1})^3} > 0,$$

since $\alpha > 1$, which results in a contradiction. Thus the infimum of w occurs at P. Hence, $u_c(x) - v(x) \geq u_c(P) - 1$, i. e.,

$$\frac{cu(x)}{r} - \frac{d(x)}{r} \geq \frac{cu(P)}{r} - 1, \ \forall\, x \in\ B_r \text{ and } \forall\, c < k.$$

Letting $c \to k$, we obtain

$$ku(x) - d(x) \geq ku(P) - d(P) = 0\,.$$

The inequality $ku(x) \geq d(x)$ together with the fact $k = r/u(P) = d(P)/u(P)$ yields (i). Rearranging the above inequality now yields

$$ku(x) - ku(P) \geq d(x) - d(P) = -|x - P|.$$

This clearly implies (ii). $\diamondsuit$

Before proceeding to the proof of the main result, we make a few observations below. These follow from the results of Lemma 2 and will be used quite frequently in what follows. Remark 1 is a straightforward observation, while Remark 2 shows that if u is positive somewhere then it is positive everywhere, a fact necessary for our proof.

Remark 1. Observe from (i) of Lemma 2, $u(x) \geq u(P)/2, \quad \forall\, x \in\ B_{r/2}(P)$.

Remark 2. We show that if u is positive somewhere in Ω then it is positive everywhere in Ω. Clearly S, the set of points where $u > 0$, is open. Suppose y is a limit point of S. There are two possiblities: either y is in $\partial\Omega$ or it is in the interior of Ω. If the latter happens then there is a $B_\delta(y)$ that lies completely in Ω. Clearly there is a point $z \in S$ that lies in $B_{\delta/4}(y)$. Now $u(z) > 0$ and $B_{\delta/2}(z) \subset B_\delta(y)$ with $y \in B_{\delta/2}(z)$. Part (i) of Lemma 2 implies $u(y) \geq u(z)/2 > 0$. Thus S is both open and closed and Ω being connected we have $S = \Omega$.

We now present the proof of the main result.

Proof of the main Theorem. By Remark 2, $u > 0$ in Ω. Let Q be the point of infimum of u on $B_{r/2}(P)$. By Remark 1, $u(Q) \geq u(P)/2$. Let x be in $B_{r/2}(P)$. Let R be the midpoint of the segment joining x to P. Let $l = |x - P|$, then by applying Remark 1 to the $B_l(x)$, we see that $u(R) \geq u(x)/2$. Clearly, $l \leq r/2$ and $|R - P| \leq r/4$. Finally, by applying part (i) of Lemma 2 to the ball $B_{r/2}(R)$ (now P lies in this ball with distance from P to the boundary of this ball is at least $r/2 - r/4 = r/4$), we get $u(P) \geq u(R)/2$. Putting these inequalities together we obtain

$$u(Q) \geq u(P)/2 \geq u(R)/4 \geq u(x)/8, \quad \forall\, x \in B_{r/2}(x).$$

3 Appendix

We now present the proofs of the Hopf boundary lemma, the Liouville property and the local Lipschitz regularity. They follow from the basic estimates proved in Lemma 2.

Remark 3 (The Liouville Property). If $u \geq 0$ is a viscosity supersolution of (1.1) defined on all of $\mathbb{R}^n$ then it is a constant function. To see this, we take two distinct points x and z in $\mathbb{R}^n$. Consider the ball $B_R(z)$ with $R > |x - z|$. By part (i) of Lemma 2,

$$u(z) \leq u(x)\frac{d(z)}{d(x)}, \text{ and } d(z) = d(x) + |x - z| = R.$$

Letting $R \to \infty$ we get $u(z) \leq u(x)$. Switching the roles of x and z we get the reverse inequality.

Remark 4 (The Hopf Boundary Point Lemma) . We drop the requirement that $u \geq 0$. Let $Q \in \partial\Omega$ be such that there is a ball $B_r(P) \subset \Omega$ with $Q \in \partial\Omega \cap \partial B_r$. Assume that $u(Q) = inf_\Omega\, u$ and $u(P) > u(Q)$. We apply part (i) of Lemma 2 to the function $v(x) = u(x) - u(Q) \geq 0$ in the ball $B_r(P)$. Then

$$v(x) \geq v(P)\frac{d(x)}{d(P)} = v(P)\frac{d(x)}{r}.$$

Clearly, we obtain

$$\frac{u(x)-u(Q)}{d(x)} \geq \frac{u(P)-u(Q)}{d(P)}.$$

Implying then

$$liminf_{x\to Q}\frac{u(x)-u(Q)}{d(x)} \geq \frac{u(P)-u(Q)}{d(P)} > 0.$$

Also see the work in [13] in this regard. $\diamondsuit$

The estimates in Lemma 2 also imply local Lipschitz continuity of u. See [11] in this regard. We first prove this for $u \geq 0$ which are supersolutions of (1.1). A somewhat modified estimate continues to hold if the assumption of nonnegativity is dropped. We do this in Remark 5.

Lemma 3 (Lipschitz Continuity) *Let y be in Ω and $\delta = \mathrm{dist}(y, \partial\Omega)$. Then for all x in $B_{\delta/4}(y)$, we have*

$$|u(x)-u(y)| \leq \frac{4u(y)|x-y|}{\delta} \leq \frac{4M|x-y|}{\mathrm{dist}(y,\partial\Omega)},$$

where $M = supu$.

Proof. We apply part (ii) of Lemma 2. Let x and y be as in the statement of the lemma. Clearly,

$$u(x)-u(y) \geq -\frac{|x-y|}{k} = -\frac{u(y)|x-y|}{r}.$$

The ball $B_{\delta/2}(x)$ lies in $B_\delta(y)$ and contains y. Another application of part (ii) of Lemma 2 to $B_{\delta/2}(x)$ implies

$$u(y)-u(x) \geq -\frac{u(x)|x-y|}{\delta/2}.$$

Putting together these two inequalities, we obtain

$$-\frac{u(y)|x-y|}{\delta} \leq u(x)-u(y) \leq \frac{2u(x)|x-y|}{\delta}.$$

The conclusion is now obtained by observing that $u(y) \geq u(x)/2$ (apply part (i) of Lemma 2 to $B_{\delta/2}(x)$ with the observation that $|y-x| \leq \delta/4$). $\diamondsuit$

Remark 5. In order to prove Lemma 3 for a more general u, we proceed as follows. First redefine $\delta = \mathrm{dist}(y, \partial\Omega)/2$. Let $m = inf_{B_\delta(y)}\ u$; clearly,

$v(x) = u(x) - m \geq 0$ in $B_\delta(y)$ and is a supersolution of (1.1). Going through the proof of Lemma 4, we find that

$$\begin{aligned} |u(x) - u(y)| &= |v(x) - v(y)| \leq \frac{4v(y)|x-y|}{\delta} \\ &\leq \frac{(u(y) - m)|x-y|}{\delta} \\ &\leq \frac{8 sup|u| \; |x-y|}{\operatorname{dist}(y, \partial\Omega)}. \end{aligned}$$

Finally, we state a somewhat more precise version of part (i) of Lemma 2.

Remark 6. Let $B_r(z)$ be in Ω. We show that for $x \in B_r(z)$, the function $u(x)/d(x)$ is increasing along radial lines emanating from z. To state this from precisely, let e be a unit vector in $\mathbb{R}^n$ and $0 < t < r$, we claim that $u(z+te)/d(z+te)$ is increasing as a function of t. Set $x = z+te$ and $y = z+se$, where $t < s < r$. Fix t and s. Note that $d(x) = \operatorname{dist}(x, \partial B_r(z))$ and $d(y) = \operatorname{dist}(y, \partial B_r(z))$. Note that the ball $B_{d(x)}(x)$ contains y. Applying Lemma 2 to this ball and observing that $d(y) = \operatorname{dist}(y, \partial B_{d(x)}(x))$, we deduce that $u(x)/d(x) \leq u(y)/d(y)$.

References

[1] Aronsson, G., On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$, Arkiv fur Matematik 7 (1968), 395-425.

[2] Bhattacharya, T., DiBenedetto, E. and Manfredi, J., Limits of extremal problems, Classe Sc. Math. Fis. Nat., Rendiconti del Sem. Mat. Fascicolo Speciale Non Linear PDE's, Univ. de Torino, 1989, 15-68.

[3] Crandall, M. G., Ishii, H. and Lions, P. L., User's guide to viscosity solutions of second order partial differential equations, Bulletin of AMS, Vol 27, no 1, 1992.

[4] DiBenedetto, E. and Trudinger, N., Harnack inequalities for quasiminima of variational integrals, Analyse Nonlineaire, Ann. Inst. Henri Poincare' 1(1984), 295-308.

[5] Evans, L., Estimates for smooth absolutely minimizing Lipschitz extensions, Electronic Journal of Differential Equations Vol. 1993(1993), No. 3, 1-10.

[6] Jensen, R., Uniqueness of Lipschitz extensions: Minimizing the sup-norm of the gradient, Arch. for Rational Mechanics and Analysis 123(1993), 51-74.

[7] Juutinen, P., Lindqvist, P. and Manfredi, J., The ∞- Eigenvalue Problem, Arch. Rat. Mech and Anal, 148(1999), pp 89-105.

[8] Juutinen, P., Lindqvist, P. and Manfredi, J., On the Equivalence of Viscosity Solutions and Weak Solutions for a Quasilinear Equation. Preprint.

[9] Juutinen, P. and Manfredi, J., Viscosity Solutions of the p-Laplace equation. Preprint.

[10] Lindqvist, P., On the definition and the properties of p-superharmonic functions, J. Reine Angew. Math. 365(1986), 67-79.

[11] Lindqvist, P. and Manfredi, J., The Harnack Inequality for ∞-harmonic functions, Electronic J. Diff. Equations, 1995(1995), No. 4, 1-5.

[12] Lindqvist, P. and Manfredi, J., Note on ∞-superharmonic functions, Revista Matematica de la Univ. Complutense de Madrid 10(2), 1997, 1-9.

[13] Rosset Edi, A Lower Bound for the Gradient of Infinity-Harmonic Functions, Electronic Journal of Differential Equations, Vol 1996(1996), No. 2, 1-12.

[14] Serrin, J., Local behaviour of solutions of quasilinear equations, Acta Math. 111(1964), 247-302.

Tilak Bhattacharya
Indian Statistical Institute
7, S.J.S. Sansanwal Marg
New Delhi 110 016 India
e-mail: tlk@isid.isid.ac.in

Current address:
Mathematics Department, Central Michigan University
Mount Pleasant, MI 48859 USA
e-mail: tilakb@hotmail.com