Global bifurcation result for the p-biharmonic operator *

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Abstract

We prove that the nonlinear eigenvalue problem for the p-biharmonic operator with $p > 1$, and $\Omega$ a bounded domain in $\mathbb{R}^N$ with smooth boundary, has principal positive eigenvalue $\lambda_1$ which is simple and isolated. The corresponding eigenfunction is positive in $\Omega$ and satisfies $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$, $\Delta u_1 < 0$ in $\Omega$. We also prove that $(\lambda_1, 0)$ is the point of global bifurcation for associated nonhomogeneous problem. In the case $N = 1$ we give a description of all eigenvalues and associated eigenfunctions. Every such an eigenvalue is then the point of global bifurcation.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. For $p \in (1, +\infty)$ consider the nonlinear eigenvalue problem

\begin{equation}
\Delta(|\Delta u|^{p-2}\Delta u) = \lambda|u|^{p-2}u \quad \text{in } \Omega \\
u = \Delta u = 0 \quad \text{on } \partial\Omega
\end{equation}

In this paper we prove that (1.1) has a principal positive eigenvalue $\lambda_1 = \lambda_1(p)$ which is simple and isolated. Moreover, we prove that there exists strictly positive eigenfunction $u_1 = u_1(p)$ in $\Omega$ associated with $\lambda_1(p)$ and satisfying $\frac{\partial u_1}{\partial n} < 0$ on $\partial\Omega$. We also study the dependence of $\lambda_1(p)$ on $p$ and show that $p \mapsto \lambda_1(p)$ is a continuous function in $(1, +\infty)$. Making use of this result we prove that $\lambda_1(p)$ is a bifurcation point of

\begin{equation}
\Delta(|\Delta u|^{p-2}\Delta u) = \lambda|u|^{p-2}u + g(x, \lambda, u) \quad \text{in } \Omega \\
u = \Delta u = 0 \quad \text{on } \partial\Omega,
\end{equation}

from which a global continuum of nontrivial solutions emanates.

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In one dimensional case \((N = 1, \Omega = (0, 1))\) we obtain a complete characterization of the spectrum of the eigenvalue problem

\[
\begin{aligned}
(|u''|^p - 2 u'')'' &= \lambda |u|^p - 2 u \quad \text{in} \ (0, 1) \\
u(0) = u''(0) = u(1) = u''(1) = 0.
\end{aligned}
\]  

(1.3)

We prove that the spectrum of (1.3) consists of a sequence of simple eigenvalues \(0 < \lambda_1 < \ldots < \lambda_n < \ldots \rightarrow +\infty\). The eigenfunction \(u_n\) associated with \(\lambda_n(n \geq 2)\) has precisely \(n\) bumps in \((0, 1)\) and it is reproduced from \(u_1\) by using the symmetry of (1.3). As a simple consequence we then obtain that any \(\lambda_n\) is a global bifurcation point of the symmetry of (1.3). As a simple consequence we then obtain that any \(\lambda_n\) is a global bifurcation point of the symmetry of (1.3).

\[
\begin{aligned}
(|u''|^p - 2 u'')'' &= \lambda |u|^p - 2 u + g(t, \lambda, u) \quad \text{in} \ (0, 1) \\
u(0) = u''(0) = u(1) = u''(1) = 0.
\end{aligned}
\]  

(1.4)

Our main results are stated in the following theorems.

**Theorem 1.1** The problem (1.1) has the least positive eigenvalue \(\lambda_1(p)\) which is simple and isolated in the sense that the set of all solutions of (1.1) with \(\lambda = \lambda_1(p)\) forms a one dimensional linear space spanned by a positive eigenfunction \(u_1\) associated with \(\lambda_1(p)\) such that \(\Delta u_1(p) < 0\) in \(\Omega\) and \(\frac{\partial u_1(p)}{\partial n} < 0\) on \(\partial \Omega\) and that there exists a positive number \(\delta\) so that \((\lambda_1(p), \lambda_1(p) + \delta)\) does not contain any eigenvalues of \((E_N)_p\). Moreover, (1.1) has a positive solution if and only if \(\lambda = \lambda_1\) and the function \(p \mapsto \lambda_1(p)\) is continuous.

**Theorem 1.2** Let \(p > 1\) be fixed and the function \(g = g(x, \lambda, s), g(x, \lambda, 0) = 0\), represents higher order terms in (1.2) (see Section 4 for precise assumptions). Then there exists a continuum of nontrivial solutions \((\lambda, u)\) of (1.2) bifurcating from \((\lambda_1(p), 0)\) which is either unbounded or meets the point \((\lambda_n(p), 0)\), where \(\lambda_n(p) > \lambda_1(p)\) is some eigenvalue of (1.1).

**Theorem 1.3** The set of all eigenvalues of (1.3) is formed by a sequence

\[0 < \lambda_1(p) < \lambda_2(p) < \ldots < \lambda_n(p) < \ldots \rightarrow +\infty.\]

For any \(n = 1, 2, \ldots\), the function \(p \mapsto \lambda_n(p)\) is continuous. Every \(\lambda_n(p)\) is simple and the corresponding one dimensional space of solutions of (1.3) with \(\lambda = \lambda_n(p)\) is spanned by a function having precisely \(n\) bumps in \((0, 1)\).

Each \(n\)-bump solution is constructed by the reflection and compression of the eigenfunction \(u_1\) associated with \(\lambda_1(p)\).

**Theorem 1.4** Let \(p > 1\) be fixed and \(g = g(t, \lambda, s), g(t, \lambda, 0) = 0\), represents higher order terms in (1.4) (see Section 5 for precise assumptions). Then for every \(n = 1, 2, \ldots\), there exists a continuum of nontrivial solutions \((\lambda, u)\) of (1.4) bifurcating from \((\lambda_n(p), 0)\) which is either unbounded or meets the point \((\lambda_k(p), 0)\), with \(k \neq n\).
The paper is organized as follows. In Section 2 we define the notion of the solution, and prepare some auxiliary results. Section 3 contains the proof of Theorem 1.1. The essential part of it relies on the abstract result of Idogawa and Ôtani [7] and the verification of its assumptions. In Section 4 we prove the bifurcation result stated in Theorem 1.2 using the degree argument and the well-known result of Rabinowitz [R]. The last Section 5 is devoted to the one dimensional case and Theorems 1.3, 1.4 are proved there.

2 Auxiliaries

For \( p > 1 \) we define the function \( \psi_p : \mathbb{R} \to \mathbb{R} \) by \( \psi_p(s) = |s|^{p-2}s, s \neq 0 \) and \( \psi_p(0) = 0 \). Denoting \( p' = \frac{p}{p-1} \), we immediately obtain that \( z = \psi_p'(z) \) if and only if \( s = \psi_p'(z) \). The eigenvalue problem (1.1) can be thus written in the form

\[
\Delta \psi_p(\Delta u) = \lambda \psi_p(u) \quad \text{in } \Omega \\
u = \Delta u = 0 \quad \text{on } \partial \Omega. \tag{2.1}
\]

Before we define the weak solution to (2.1) we recall some properties of the Dirichlet problem for Poisson equation:

\[
-\Delta w = f \quad \text{in } \Omega \\
w = 0 \quad \text{on } \partial \Omega. \tag{2.2}
\]

It is well known that (2.2) is uniquely solvable in \( L^p(\Omega) \) for any \( p \in (1, \infty) \) and that the linear solution operator \( \Lambda : L^p(\Omega) \to W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), \lambda f = w, \) has the properties stated in the following lemma, (see, e.g., [6]).

Lemma 2.1  
(i) (Continuity) There exists a constant \( c_p > 0 \) such that

\[ \| \Lambda f \|_{W^{2,p}} \leq c_p \| f \|_{L^p} \]

holds for all \( p \in (1, \infty) \) and \( f \in L^p(\Omega) \).

(ii) (Continuity) Given \( k \geq 1, k \in \mathbb{N} \), there exists a constant \( c_{p,k} > 0 \) such that

\[ \| \Lambda f \|_{W^{k+2,p}} \leq c_{p,k} \| f \|_{W^{k,p}} \]

holds for all \( p \in (1, \infty) \) and \( f \in W^{k,p}(\Omega) \).

(iii) (Symmetry) The following identity

\[ \int_\Omega \Lambda u \cdot vdx = \int_\Omega u \cdot \Lambda vdx \]

holds for all \( u \in L^p(\Omega) \) and \( v \in L^{p'}(\Omega) \) with \( p \in (1, \infty) \).

(iv) (Regularity) Given \( f \in L^\infty(\Omega) \), we have \( \Lambda f \in C^{1,\alpha}(\overline{\Omega}) \) for all \( \alpha \in (0,1) \); moreover, there exist \( c_\alpha > 0 \) such that

\[ \| \Lambda f \|_{C^{1,\alpha}} \leq c_\alpha \| f \|_{L^\infty}. \]
(v) (Regularity and Hopf-type maximum principle) Let $f \in C(\Omega)$ and $f \geq 0$, then $w = \Lambda f \in C^{1,\alpha}(\Omega)$, for all $\alpha \in (0,1)$ and $w$ satisfies: $w > 0$ in $\Omega, \frac{\partial w}{\partial n} < 0$ on $\partial \Omega$.

(vi) (Order preserving property) Given $f, g \in L^p(\Omega), f \leq g$ in $\Omega$, we have $\Delta f \leq \Delta g$ in $\Omega$.

Let us denote $v := -\Delta u$ in (1.1). Then the problem (1.1) can be restated as an operator equation

$$
\psi_p(v) = \lambda \psi_p(\Lambda v) \quad \text{in } \Omega \tag{2.3}
$$

or as

$$
v = \lambda \psi_p(\Lambda v) \quad \text{in } \Omega. \tag{2.4}
$$

Indeed, let us assume that $v \in L^p(\Omega)$ solves (2.3). Then from Lemma 2.1 (i) and the properties of the Nemytskii operator induced by $\psi_p$, we obtain:

$$
\begin{align*}
&u = \Lambda v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \Rightarrow \psi_p(\Lambda v) \in L^{p'}(\Omega) \Rightarrow \\
&\Rightarrow \Lambda \psi_p(\Lambda v) \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega) \Rightarrow \\
&\Rightarrow \psi_p(v) \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega) \Rightarrow \\
&\Rightarrow -\Delta \psi_p(-\Delta u) = \lambda \psi_p(u) \text{ holds in } L^{p'}(\Omega).
\end{align*}
$$

This enables us to give the following definition of the solution of (1.1).

**Definition 2.2** The function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is called a solution of (1.1) if $v = -\Delta u$ solves (2.3) in $L^{p'}(\Omega)$. The parameter $\lambda_e$ is called an eigenvalue of (1.1) if there is a nonzero solution $u_e$ of (1.1) with $\lambda = \lambda_e$. The function $u_e$ is then called the eigenfunction associated with the eigenvalue $\lambda_e$.

**Lemma 2.3** (Duality). Let $\lambda_e = \lambda_e(p) \neq 0$ be the eigenvalue of $(E_N)_p$ and $u_e(p)$ be the eigenfunction associated with $\lambda_e$. Define $\lambda_e(p')$ and $u_e(p')$ by $\lambda_e(p') = \lambda_e^{-1}(p')e$ and $u_e(p') = \lambda_e^{-1}(p')\psi_p(\Delta u_e(p))$. Then $\lambda_e(p')$ becomes an eigenvalue of $(E_N)_{p'}$ with $p' = \frac{p}{p-1}$ and $u_e(p')$ gives the eigenfunction associated with $\lambda_e(p')$.

**Proof.** We have

$$
\begin{align*}
\Delta \psi_p(\Delta u_e(p)) &= \lambda_e(p)\psi_p(u_e(p)) \quad \text{in } \Omega \\
u_e(p) &= \Delta u_e(p) = 0 \quad \text{on } \partial \Omega. \tag{2.5}
\end{align*}
$$

Let $w_p := \psi_p(\Delta u_e(p))$, then $w_p \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$. It is easy to see that to solve (2.5) is nothing but to find $(u_e(p), w_p)$ satisfying the system

$$
\begin{align*}
\Delta w_p &= \lambda_e(p)\psi_p(u_e(p)) \\
\Delta u_e(p) &= \psi_p'(w_p). \tag{2.6}
\end{align*}
$$
Since \( u_e(p') = \frac{1}{\lambda_e(p')} w_p \in W^{2,p'}(\Omega) \cap W^{1,p'}_0(\Omega) \) satisfies \( \psi_{p'}(u_e(p')) = \lambda_e(p')^{1-p'} \psi_{p'}(w_p) = \lambda_e(p')^{-1} \psi_{p'}(w_p) \), we easily find that \((u_e(p'), w_p)\) with \( w_p = u_e(p) \) solves (2.6) with \( p = p' \).

**Remark 2.4** The duality proved in the previous lemma enables us to deduce several properties of (1.1) for \( p > 2 \) from those for \( p \in (1, 2) \) and vice versa.

The following technical lemma will be useful for the verification of certain abstract assumptions in the next section.

**Lemma 2.5** Let \( A, B, C \) and \( p \) be real numbers satisfying \( A \geq 0, B \geq 0, C \geq \max\{B - A, 0\} \) and \( p > 1 \). Then
\[
|A + C|^p + |B - C|^p \geq A^p + B^p. \tag{2.7}
\]

**Proof.** If \( C = 0 \) (i.e., \( B \leq A \)), then (2.7) is trivial. So it suffices to show (2.7) when \( B \geq A \). Due to the strict convexity of the function \( s \mapsto s^p \) in \((0, +\infty)\) we have
\[
|A + C|^p \geq B^p + pB^{p-1}[C - (B - A)],
\]
\[
|B - C|^p \geq A^p - pA^{p-1}[C - (B - A)].
\]
Adding these inequalities, we derive the assertion. \( \Box \)

### 3 Eigenvalue problem

Let us define convex functionals \( f^1_p, f^2_p : L^p(\Omega) \to \mathbb{R} \) as follows:
\[
f^1_p(v) = \frac{1}{p} \int_\Omega |v|^p dx, \quad f^2_p(v) = \frac{1}{p} \int_\Omega |\Lambda v|^p dx.
\]
Then it is clear that \( f^1_p \) and \( f^2_p \) are Fréchet differentiable in \( L^p(\Omega) \). Since for every Fréchet differentiable convex functional \( f \), its subdifferential \( \partial f \) coincides with its Fréchet derivative \( f' \), we get that (2.3) is equivalent to
\[
\partial f^1_p(v) = \lambda \partial f^2_p(v) \quad \text{in} \quad L^p(\Omega), \tag{3.1}
\]
where \( \partial f^i_p \) are the subdifferentials of \( f^i_p \), \( i = 1, 2 \). We are going to verify the hypotheses (A0), (A0)', (6.1) - (6.10) of [7] with \( A = \partial f^1_p, B = \partial f^2_p \) and \( V = L^p(\Omega) \). The assumptions (6.1) (i)--(iii), (6.2) (i)--(iii), (6.3), (6.4) (i) and (6.5) are clearly satisfied. Concerning (6.4) (ii) we should verify that
\[
f^2_p(\max\{u, w\}) + f^2_p(\min\{u, w\}) \geq f^2_p(u) + f^2_p(w) \tag{3.2}
\]
for any \( u, w \in L^p(\Omega) \) satisfying \( u \geq 0 \) and \( w \geq 0 \) a.e. in \( \Omega \). We have \( \max\{u, w\} = u + (w - u)^+ \) and \( \min\{u, w\} = w - (w - u)^+ \). By Lemma 2.1
by Sobolev’s embedding theorem and the property of the Nemytskii operator, we obtain
\begin{equation}
\int_{\Omega} |Au + \Lambda (w - u)^+|^p dx + \int_{\Omega} |Aw - \Lambda (w - u)^+|^p dx \geq \int_{\Omega} |Au|^p dx + \int_{\Omega} |Aw|^p dx.
\end{equation}
(3.3)
Then (3.3) implies (3.2). The assumption (6.10) is a consequence of Lemma 2.1 (vi). Hence it remains to verify (A0) and \((A0)^*\).

**Lemma 3.1** Let \(v \in L^p(\Omega)\) solve (2.3) in \(L^{p'}(\Omega)\). Then \(v \in C(\Omega)\).

**Proof.** The main part of the proof is to show the following fact:

Suppose, that \(v \in L^{p_0}(\Omega)\), then we find that

(i) \(v \in L^{p_1}(\Omega)\) with \(\frac{1}{p_1} = \frac{1}{p_0} - \frac{N}{2p'}\) if \(p_0 < \frac{N}{2p'}\).

(ii) \(v \in C(\Omega)\) if \(p_0 > \frac{N}{2p'}\), \(p' = \frac{p}{p-1}\).

Let \(v \in L^{p_0}(\Omega)\), and \(p_0 < \frac{N}{2p'}\), then \(Av \in W^{2,p_0}(\Omega)\) by Lemma 2.1(i). Then, by Sobolev’s embedding theorem and the property of the Nemytskii operator: \(r \mapsto \psi_p(r)\), we get \(Av \in L^{r_0}(\Omega)\) and \(\psi_p(Av) \in L^{\frac{2r_0}{N}}\) with \(r_0 = \frac{Np_0}{2N}\). Again, by Sobolev’s embedding theorem and the property of the Nemytskii operator, we obtain

\[ \Lambda \psi_p(Av) \in W^{2,\frac{2r_0}{N}}(\Omega) \hookrightarrow L^{r_1}(\Omega) \]

and

\[ \psi_p'(\Lambda \psi_p(Av)) \in L^{\frac{2r_1}{N}}(\Omega) = L^{r_1(p-1)}(\Omega) \]

with \(r_1 = \frac{N^{r_0}}{N(p-1) - 2r_0}\). Consequently, (2.4) implies that \(v \in L^{p_1}(\Omega)\) with \(p_1 = r_1(p-1)\), i.e., \(\frac{1}{p_1} = \frac{1}{p_0} - \frac{2p'}{N}\), whence follows assertion (i). If \(\frac{N}{2p'} < p_0 < \frac{N}{2}\) or \(p_0 = \frac{N}{2p'}\) (or \(p_0 = \frac{N}{2}\)), noting that \(W^{2,\frac{2r_0}{N}}(\Omega) \hookrightarrow C(\Omega)\) (or \(W^{2,\frac{2r_0}{N}}(\Omega) \hookrightarrow C(\Omega)\) for sufficiently large \(r\)) we easily see that \(v \in C(\Omega)\). Then assertion (ii) is verified. Now take suitable \(p_0 \in (1, p]\) and \(k \in \mathbb{N}\) such that

\[ p_{k-1} < \frac{N}{2p'} < p_k \text{ with } \frac{1}{p_k} = \frac{1}{p_0} - \frac{2p'}{N}k. \]

Then applying assertion (i) with \(p_0 = p_0, p_1, \ldots, p_{k-1}\), we deduce \(v \in L^{p_k}(\Omega)\). Hence from assertion (ii), \(v \in C(\Omega)\) follows. \(\square\)

**Remark 3.2** In particular, it follows from above proof that given bounded sequences \(\{p_n\} \subset (1, \infty)\) and \(\{\lambda_n\} \subset (0, \infty)\), the sequence of elements \(v_n\) solving (2.3) with \(\lambda = \lambda_n\) and \(p = p_n\) which are normalized by \(\|v_n\|_{L^q} = 1, q \in (1, \infty)\), we find a constant \(c > 0\) (independent of \(n\)) such that

\[ \|v_n\|_{L^\infty} \leq c. \]
By the same reason, if \( \lambda_n \to \lambda_0 \) and \( v_0 \) solves (2.3) with \( \lambda = \lambda_0, \| v_0 \|_{L^p} = 1 \), the proof of Lemma 3.1 implies that
\[
\lim_{n \to \infty} \| v_n - v_0 \|_{L^\infty} = 0.
\]

**Lemma 3.3** Let \( p \geq 2 \) and \( v \in L^p(\Omega), v \geq 0 \) a.e. in \( \Omega \), and let \( v \) solve (2.3) in \( L^p(\Omega) \). Then \( v \in C^1(\Omega), v > 0 \) everywhere in \( \Omega \) and \( \frac{\partial v}{\partial n} = -\infty \) on \( \partial \Omega \).

**Proof.** It follows from Lemma 2.1 (v), Lemma 3.1 and (2.3) that \( w := \psi_p(v) \) satisfies \( w \in C^{1,\alpha}(\bar{\Omega}), \alpha \in (0,1), w > 0 \) in \( \Omega \) and \( \frac{\partial w}{\partial n} < 0 \) on \( \partial \Omega \). This fact assures that \( v > 0 \) in \( \Omega \) and \( (p-1)|v|^{p-2} \frac{\partial v}{\partial n} < 0 \) on \( \partial \Omega \). Then \( \frac{\partial v}{\partial n} = -\infty \) follows from the fact that \( v = 0 \) on \( \partial \Omega \). \( \square \)

For \( p \geq 2 \) the assumption \( (A0)' \) now follows from Lemma 3.3 while instead of \( (A0)' \) we obtain the following property - \( (A0)'^* \): Every positive solution \( v \) of (3.1) satisfies \( v \in C^1(\Omega), v = 0 \) on \( \partial \Omega \) and \( \frac{\partial v}{\partial n} = -\infty \) on \( \partial \Omega \).

It is easy to see that the results of [7] remain true even if \( (A0)' \) is substituted by \( (A0)'^* \). Applying now the results of [7] we deduce that, for \( p \geq 2 \),
\[
0 < \lambda_1(p) := \left( \sup_{v \in L^p(\Omega), v \neq 0} \frac{\int_\Omega \rho^2(v)}{\int_\Omega \rho_1(v)} \right)^{-1},
\]
is the least simple eigenvalue of (3.1) with associated positive eigenfunction \( v_1(p), \| v_1(p) \|_{L^p} = 1 \) and (3.1) has a positive solution if and only if \( \lambda = \lambda_1(p) \). The assertion for \( p \in (1,2) \) now follows from Lemma 2.3 and Remark 2.4.

As a consequence of this fact we find that \( u_1(p) = \Lambda v_1(p) \) is the corresponding first eigenfunction of (E_N)p satisfying \( u_1(p) > 0 \) in \( \Omega, \Delta u_1(p) < 0 \) in \( \Omega \) and \( \frac{\partial u_1(p)}{\partial n} < 0 \) on \( \partial \Omega \) due to Lemma 2.1 (vi). Moreover, if \( u \) is another positive solution of (E_N)p then \( v = -\Delta u > 0 \) solves (2.3) in \( L^p(\Omega) \). Therefore (2.4) holds with \( \Lambda v = u \). Hence according to the above mentioned argument, it holds that \( \lambda = \lambda_1(p) \) and \( v = v_1(p) \), i.e. \( u = u_1(p) \).

**Lemma 3.4** \( \lambda_1(p) \) is isolated, i.e. there is \( \delta > 0 \) such that the interval \( (\lambda_1(p), \lambda_1(p) + \delta) \) does not contain any eigenvalue of (3.1).

**Proof.** Assume the contrary, i.e., there are sequences \( \{\lambda_n\}, \{v_n\} \) such that \( \lambda_n \to \lambda_1(p), \| v_n \|_{L^p} = 1 \) and that \( v_n \) solves (3.1) with \( \lambda = \lambda_n \). Then both \( v_n \) and \( \Lambda v_n \) must change sign in \( \Omega \) and
\[
\lim_{n \to \infty} \| v_n - v_1(p) \|_{L^\infty} = 0
\]
according to Remark 3.2. But Lemma 2.1 (iv) implies that \( \Lambda v_n \to \Lambda v_1(p) \) in \( C^{1,\alpha}(\bar{\Omega}) \) for some \( \alpha \in (0,1) \) which leads to a contradiction with the fact that \( \Lambda v_1(p) > 0 \) in \( \Omega \) and \( \frac{\partial \Lambda v_1(p)}{\partial n} < 0 \) on \( \partial \Omega \). \( \square \)
It remains to show the continuity of \( p \mapsto \lambda_1(p) \). Let us note first that

\[
\lambda_1(p) = \inf \frac{1}{\int_\Omega f(v)}
\]

where the infimum is taken over all \( v \in L^p(\Omega) \), \( \|v\|_{L^p} = p \). It follows from Lemma 2.1 (i) that \( \lambda_1(p) \) is bounded uniformly away from zero and infinity for any \( p \) belonging to a compact subinterval of \((1, \infty)\). Let \( p_n \to p \in (1, \infty) \). Then \( \{\lambda_1(p_n)\} \) is a bounded sequence. Denote by \( v_1(p_n) \) the positive eigenfunction associated with \( \lambda_1(p_n) \) and normalized by

\[
\|v_1(p_n)\|_{L^p} = p. 
\]  

(3.4)

Extracting a suitable subsequence we can assume that

\[
\lambda_1(p_n) \to \lambda_0, v_1(p_n) \to v_0 \in L^p(\Omega).
\]  

(3.5)

In particular, we derive from (3.5) that \( v_0 \geq 0 \, \text{a.e.} \) in \( \Omega \), and the compactness of \( \Lambda \) (cf. Lemma 2.1 (i)) yields \( \Lambda v_1(p_n) \to \Lambda v_0 \) in \( L^p(\Omega) \). Extracting again to a subsequence we get

\[
\Lambda v_1(p_n) \to \Lambda v_0 \, \text{a.e. in } \Omega. 
\]  

(3.6)

It follows from Remark 3.2 and Lemma 2.1 (iv) that there is a constant \( c > 0 \) independent of \( n \) such that

\[
|\Lambda v_1(p_n)| \leq c. 
\]  

(3.7)

Hence it follows from (3.6), (3.7) and Lemma 2.1 (iv) that

\[
\Lambda \psi_{p_n}(\Lambda v_1(p_n)) \to \Lambda \psi_p(\Lambda v_0) \, \text{a.e. in } \Omega, \quad \text{i.e.,}
\]

\[
\psi_{p_n}(\Lambda \psi_{p_n}(\Lambda v_1(p_n))) \to \psi_p(\Lambda(\psi_p(\Lambda v_0))) \, \text{a.e. in } \Omega. 
\]  

(3.8)

Now taking arbitrary \( \varphi \in L^p(\Omega) \), it follows from (3.4), (3.5), (3.7), (3.8), Lemma 2.1 (iv) and the Lebesgue dominated convergence theorem that

\[
\int_{\Omega} \psi_{p_n}(\Lambda \psi_{p_n}(\Lambda v_1(p_n))) \varphi dx \to \int_{\Omega} \psi_p(\Lambda(\psi_p(\Lambda v_0))) \varphi dx. 
\]  

(3.9)

It also follows from (3.5) that

\[
\int_{\Omega} v_1(p_n) \varphi dx \to \int_{\Omega} v_0 \varphi dx.
\]  

(3.10)

So it follows from (2.4), (3.9) and (3.10) that

\[
v_0 = \lambda_0^{\frac{1}{p-1}} \psi_p(\Lambda(\psi_p(\Lambda v_0))). 
\]  

(3.11)

On the other hand (3.6), (3.7) the definition of \( \lambda_1 \) and the Lebesgue dominated convergence theorem imply

\[
1 = \lim_{n \to \infty} \lambda_1(p_n) \int_{\Omega} |\Lambda v_1(p_n)|^{p_n} dx = \lambda_0 \int_{\Omega} |\Lambda v_0|^p dx, 
\]
i.e. \( v_0 \neq 0 \). It follows from here and (3.11) that \( v_0 \) is a positive solution of (2.3) with \( \lambda = \lambda_0 \). According to the first part of Theorem 1.1 (cf. [7]) it must be \( \lambda_0 = \lambda_1(p) \), \( v_0 = v_1(p) \). Since the above argument does not depend on the choice of subsequences, the continuity of the function

\[ p \mapsto \lambda_1(p) \]

is proved. This also completes the proof of Theorem 1.1

4 Global bifurcation result

For \( p > 1 \) set \( X = L^p(\Omega) \). Then \( X^* = L^{p'}(\Omega) \) and the Nemytskii operator

\[ \Psi_p : v \mapsto \psi_p(v) \]

is one to one mapping between \( X \) and \( X^* \).

Lemma 4.1 \( \Psi_p \) satisfies condition \((S_+)\), i.e.

\[ v_n \rightharpoonup v_0 \text{ weakly in } X. \tag{4.1} \]

and

\[ \limsup_{n \to \infty} \int_{\Omega} \psi_p(v_n)(v_n - v_0)dx \leq 0 \tag{4.2} \]

imply \( v_n \to v_0 \) strongly in \( X \).

Proof. The monotonicity of \( \psi_p \), (4.1) and (4.2) imply

\[
0 \geq \limsup_{n \to \infty} \int_{\Omega} \psi_p(v_n)(v_n - v_0)dx = \\
= \limsup_{n \to \infty} \int_{\Omega} (\psi_p(v_n) - \psi_p(v_0))(v_n - v_0)dx \\
\geq \limsup_{n \to \infty} \left[ \left( \int_{\Omega} |v_n|^p dx \right)^{1/p'} - \left( \int_{\Omega} |v_0|^p dx \right)^{1/p'} \right]^* \\
\times \left[ \left( \int_{\Omega} |v_n|^p dx \right)^{1/p} - \left( \int_{\Omega} |v_0|^p dx \right)^{1/p} \right] \geq 0
\]

Hence \( \|v_n\|_X \to \|v_0\|_X \), which together with (4.1) yields the desired strong convergence.

Let the function \( g : \Omega \times \mathbb{R}^2 \to \mathbb{R} \) be a Carathéodory function, i.e. \( g(x, \cdot, \cdot) \) is continuous for a.e. \( x \in \Omega \) and \( g(\cdot, \lambda, s) \) is measurable for all \( (\lambda, s) \in \mathbb{R}^2 \). Moreover, let \( g(x, \lambda, 0) = 0 \) for any \( (x, \lambda) \in \Omega \times \mathbb{R} \) and given any bounded interval \( J \subset \mathbb{R} \) we assume that there exists an exponent \( q \in (p, p^{**}) \) with
Note that (1.2) can be written in the equivalent form
\[ |g(x, \lambda, s)| \leq \varepsilon |s|^{p-1} + C_\varepsilon |s|^{q-1} \quad \text{for a.e. } x \in \Omega \text{ and all } \lambda \in J, s \in \mathbb{R}. \tag{4.3} \]
Due to (4.3) the right hand side of (4.4) defines an operator
\[ T_{\lambda, g} : v \mapsto \lambda \Lambda \psi_p(\Lambda v) + \Lambda g(x, \lambda, \Lambda v). \tag{4.4} \]
Due to (4.3) the right hand side of (4.4) defines an operator
\[ T_{\lambda, g} : v \mapsto \lambda \Lambda \psi_p(\Lambda v) + \Lambda g(x, \lambda, \Lambda v) \]
from \( X \) into \( X^* \) which is compact. Indeed, by Lemma 2.1 (i) we get \( \Lambda v \in W^{2,p}(\Omega) \) and \( \Lambda \psi_p(\Lambda v) \in W^{2,p'}(\Omega) \). Furthermore by using (4.3) and the fact that \( W^{2,p}(\Omega) \subset L^q(\Omega) \), we find that \( \Lambda g(x, \lambda, \Lambda v) \in W^{2,q'}(\Omega) \). Thus \( T_{\lambda, g} \) maps any bounded set of \( X \) onto a bounded set of \( W^{2,q'}(\Omega) \), which is compactly embedded in \( X^* \), since \( q < p^{**} \). Then this fact and Lemma 4.1 imply that \( \Psi_p - T_{\lambda, g} \) satisfies condition \((S_+)\). So, given an open and bounded set \( D \subset X \) such that \( \Psi_p(v) - T_{\lambda, g}(v) \neq 0 \) for any \( v \in \partial D \), the generalized degree of Browder and Petryshin
\[ \text{Deg}[\Psi_p - T_{\lambda, g}; D, 0] \]
is well defined.

**Lemma 4.2** \( \|\Lambda g(x, \lambda, \Lambda v)\|_{X^*} = o(\|v\|_{X}^{p-1}) \) as \( \|v\|_{X} \to 0 \).

**Proof.** Since \( \Lambda \) is symmetric, we have
\[ \|\Lambda g(x, \lambda, \Lambda v)\|_{X^*} = \sup_{\|\varphi\|_{X} \leq 1} \int_{\Omega} \Lambda g(x, \lambda, \Lambda v) \varphi dx = \sup_{\|\varphi\|_{X} \leq 1} \int_{\Omega} g(x, \lambda, \Lambda v) \Lambda \varphi dx. \tag{4.5} \]
Then, for any \( \varepsilon > 0 \), by virtue of (4.3) and Lemma 2.1 (i), we find
\[ \left| \int_{\Omega} g(x, \lambda, \Lambda v) \Lambda \varphi dx \right| \leq \int_{\Omega} \varepsilon |\Lambda v|^{p-1} |\Lambda \varphi| dx + \int_{\Omega} C_\varepsilon |\Lambda v|^{q-1} |\Lambda \varphi| dx \]
\[ \leq \varepsilon \|\Lambda v\|_{L^p}^{p-1} \|\Lambda \varphi\|_{L^p} + C_\varepsilon \|\Lambda v\|_{L^q}^{q-1} \|\Lambda \varphi\|_{L^q} \]
\[ \leq \varepsilon c_p^p \|v\|_{X}^{p-1} \|\varphi\|_{X} + C_\varepsilon c_q^q \|\Lambda v\|_{W^{2,p}}^{q-1} \|\Lambda \varphi\|_{W^{2,p}} \]
\[ \leq \varepsilon c_p^p \|v\|_{X}^{p-1} + C_\varepsilon c_q^q \|\varphi\|_{X}^{q-1}, \tag{4.6} \]
where \( c_p \) is the constant appearing in Lemma 2.1 (i) and \( c > 0 \) is the embedding constant for \( W^{2,p}(\Omega) \hookrightarrow L^q(\Omega) \). Thus the assertion follows from (4.5) and (4.6), since \( p < q \). \( \square \)

Let \( \delta > 0 \) be as in Lemma 3.4 and consider \( \lambda < \lambda_1(p) + \delta, \lambda \neq \lambda_1(p) \). Then Lemma 4.2 and simple homotopy argument yields
\[ \text{Deg}[\Psi_p - T_{\lambda, g}; B_r(0), 0] = \text{Deg}[\Psi_p - T_{\lambda, 0}; B_\lambda(0), 0] \tag{4.7} \]
if \( r > 0 \) is chosen sufficiently small (cf. [4], [5], [2], [3] or [R]). Here \( B_r(0) \) is the ball centred at the origin and with radius \( r > 0 \).
Lemma 4.3 Deg[Ψ_p - T_{λ,0}; B_r(0), 0] = ±1 for λ < λ_1(p) + δ, λ ≠ λ_1(p) and sgn(λ_1(p) − λ) = ±1.

Proof. To prove the “jump” of the degree we adopt the method developed in [5] (see also [4]). Thus we just sketch the proof and refer to [DKN, Theorem 3.7] or [D, Theorem 14.18] for the details. Consider the functional

\[ F_λ(v) = \frac{1}{p} \int_Ω |v|^p dx - \frac{λ}{p} \int_Ω |∇v|^p dx. \]

It follows from the variational characterization of λ_1(p) (see Section 3) that for λ < λ_1(p) we have

\[ \langle F'_λ(v), v \rangle_X > 0 \]

for v ∈ ∂B_r(0) and v = 0 is the only critical point of F_λ (here \( \langle \cdot, \cdot \rangle_X \) denotes the duality between \( X^* \) and \( X \)) and hence

\[ \text{Deg}[Ψ_p - T_{λ,0}; B_r(0), 0] = \pm 1 \] (4.8)

by the properties of the degree (cf.[9]). Let now λ ∈ (λ_1(p), λ_1(p) + δ). As in (DKN, Theorem 3.7) we define a function η : \( \mathbb{R} \to \mathbb{R} \) by

\[ η(t) = \begin{cases} 0, & \text{for } t < K, \\ \frac{2δ}{λ_1(p)}(t - 2K), & \text{for } t \geq 3K, \end{cases} \]

The function η(t) is continuously differentiable, positive and strictly convex in \( (K, 3K), K > 0 \). Let us modify \( F_λ \) as follows

\[ \tilde{F}_λ(v) := F_λ(v) + η(\frac{1}{p} \int_Ω |v|^p dx). \]

The properties of λ_1(p) stated in Theorem 1.1 now imply the following properties of \( \tilde{F}_λ \):

- \( \tilde{F}_λ \) is continuously Fréchet differentiable and its critical point \( v_0 ∈ X \) corresponds to a solution of the equation

\[ \psi_p(v_0) - \frac{λ}{1 + η'(\frac{1}{p} \int_Ω |v_0|^p dx)} Λψ_p(Λv_0) = 0. \]

- For λ ∈ (λ_1(p), λ_1(p) + δ) the only nontrivial critical points of \( \tilde{F}_λ \) occur if

\[ η'\left(\frac{1}{p} \int_Ω |v_0|^p dx\right) = \frac{λ}{λ_1(p)} - 1. \]

- Due to the definition of η we then have

\[ \frac{1}{p} \int_Ω |v_0|^p dx ∈ (K, 3K) \]

and due to the simplicity of λ_1(p), either \( v_0 = -tv_1(p) \) or \( v_0 = tv_1(p) \), for some \( t ∈ ((pK)^{1/p}, (3pK)^{1/p}) \), \( v_1(p) \) as in the Section 3.
Global bifurcation result for the $p$-biharmonic operator

$\tilde{F}_\lambda$ has precisely three isolated critical points $-tv_1(p), 0, tv_1(p)$.

$\tilde{F}_\lambda$ is weakly lower semicontinuous and even.

$\tilde{F}_\lambda$ is coercive, i.e.

$$\lim_{\|v\|_X \to \infty} \tilde{F}_\lambda(v) = \infty$$

$-tv_1(p), tv_1(p)$ are the points of the global minimum of $\tilde{F}_\lambda$; $0$ is an isolated critical point of “saddle type”.

$\langle \tilde{F}_\lambda'(v), v \rangle_X > 0$ for $\|v\|_X = R$ if $R > 0$ is large enough.

The properties of the degree now imply that for small $\rho > 0$ and large $R > 0$ we have

$$\text{Deg}[\tilde{F}_\lambda'; B_\rho(\pm tv_1(p)), 0] = \text{Deg}[\tilde{F}_\lambda'; B_R(0), 0] = 1.$$  

The additivity property of the degree then yields for $0 < r < (pK)^{1/p}$,

$$\text{Deg}[\Psi_p - T_{\lambda,0}; B_r(0), 0] = \text{Deg}[\tilde{F}_\lambda'; B_r(0), 0] = -1. \quad (4.9)$$

The assertion of Lemma 4.3 follows now from (4.8) and (4.9). □

If we combine (4.7) with Lemma 4.3 we come to the following conclusion: for $r > 0$ sufficiently small

$$\text{Deg}[\Psi_p - T_{\lambda,0}; B_r(0), 0] = \pm 1$$

for $\text{sgn}(\lambda_1(p) - \lambda) = \pm 1$. Following the proof of [R, Theorem 1.3] we prove that continuum of nontrivial solutions $(\lambda, v) \in \mathbb{R} \times X$ of (4.4) bifurcates from $(\lambda_1(p), 0)$ and it is either unbounded in $\mathbb{R} \times X$ or meets the point $(\lambda_e(p), 0)$, where $\lambda_e(p) > \lambda_1(p)$ is an eigenvalue of (3.1). The assertion of Theorem 1.2 now follows from the fact that $(\lambda, u)$ solves $(\text{BP}_N)_p$ if and only if $(\lambda, -\Delta u)$ solves (4.4).

5 One-dimensional problem

Let $N = 1$ and $\Omega = (0, 1)$. Then $(E_N)_p$ reduces to (1.3) and obviously the assertions of Theorems 1.1, 1.2 remain true. We point out that $W^{2,p}(0, 1) \hookrightarrow C^1([0, 1])$ in the case $N = 1$, and so $\psi_p(v) \in C^1([0, 1])$, $v(0) = v(1) = 0$ for any solution $v$ of (2.3). Hence we do not need Lemmas 3.1 and 3.3 in this case. For the sake of brevity we shall write $\lambda_1 := \lambda_1(p), u_1 := u_1(p)$. It follows from the symmetry of (1.3) and Theorem 1.1 (simplicity of $\lambda_1$) that $u_1(t) = u_1(1 - t)$ for $t \in [0, 1]$, i.e. $u_1$ is even with respect to $\frac{1}{2}$. Making use of this observation, we give a precise description of all eigenvalues and eigenfunctions of $(E_1)_p$. Indeed,
Then $u_n(t), t \in [0, 1]$, is an eigenfunction of (1.3) associated with the eigenvalue $\lambda_n = n^{2p}\lambda_1$. On the other hand, let $u = u(t)$ be an eigenfunction of $(E_1)_p$ associated with some eigenvalue $\lambda_e$. According to Theorem 1.1 it must be $\lambda_e > \lambda_1$ and $u$ changes sign in $(0, 1)$. By Lemma A.4 the number of nodes of $u$ in $(0, 1)$ is finite. Assume first that $\lambda_e = \lambda_n$, for some $n > 1$. Let us normalize $u$ as follows: $u'(0) = u_n'(0) > 0$. Note that since $u$ and $u_n$ are oscillatory, we must have, according to Lemma A.3, that

$$(\psi_p(u''(t)))'|_{t=0} < 0 \quad \text{and} \quad (\psi_p(u''_n(t)))'|_{t=0} < 0,$$

respectively. Let $(\psi_p(u''(t)))'|_{t=0} = (\psi_p(u''_n(t)))'|_{t=0}$. Then Lemma A.1 implies that $u(t) = u_n(t), t \in [0, 1]$. Let $(\psi_p(u''(t)))'|_{t=0} \neq (\psi_p(u''_n(t)))'|_{t=0}$. Then Lemma A.2 implies that $u(1) \neq 0$, a contradiction. Let $\lambda_e \neq \lambda_k$ for any $k \geq 2$.

Define

$$\tilde{u}(t) = u_1 \left( \left( \frac{\lambda_e}{\lambda_1} \right)^{1/(2p)} t \right), \quad t \in \left[ 0, \left( \frac{\lambda_1}{\lambda_e} \right)^{1/(2p)} \right],$$

$$\tilde{u}(t) = -u_1 \left( \left( \frac{\lambda_e}{\lambda_1} \right)^{1/(2p)} t - 1 \right), \quad t \in \left[ \left( \frac{\lambda_1}{\lambda_e} \right)^{1/(2p)}, \left( \frac{\lambda_1}{\lambda_e} \right)^{1/(2p)} \right], \quad \text{etc.}$$

Then $\tilde{u}(1)\tilde{u}'(1) < 0$. Let us normalize $u$ as $u'(0) = \tilde{u}'(0) > 0$. Then it follows from Lemma A.2 that $u(1) = u''(1) = 0$ cannot hold at the same time. Thus Theorem 1.3 is proved.

Let $X = C([0, 1])$. Let $g : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying $g(t, \lambda, 0) = 0$ for any $(t, \lambda) \in (0, 1) \times \mathbb{R}$ and given any bounded interval $J \subset \mathbb{R}$ we assume that

$$|g(t, \lambda, s)| = o(|s|^{p-1}) \quad (5.1)$$

holds near $s = 0$ uniformly for all $(t, \lambda) \in [0, 1] \times J$. Note that $(BP_1)_p$ can be written in the equivalent form

$$v = \psi_p' (\Lambda \psi_p(Av) + Ag(t, \lambda, Av)). \quad (5.2)$$

Due to Lemma 2.1 (i), the right hand side of (5.2) defines an operator

$$R_{p, \lambda, g} : (p, \lambda, v) \mapsto \psi_p' (\Lambda \psi_p(Av) + Ag(t, \lambda, Av))$$

which is compact from $(1, \infty) \times \mathbb{R} \times X$ into $X$. If $I : X \to X$ denotes the identity mapping, the Leray-Schauder degree

$$\text{deg}[I - R_{p, \lambda, g}; D, 0]$$
Lemma 5.1 Let \( \lambda \neq \lambda_n \). Then there is \( r > 0 \) (sufficiently small) such that
\[
\deg[I - R_{p,\lambda,g}; B_r(0), 0] = \deg[I - R_{p,\lambda,0}; B_r(0), 0].
\] (5.3)

Proof. Standard argument based on (5.1) yields that the homotopy
\[
H(\tau, v) = v - \psi'_{\lambda}(\lambda \Lambda \psi_{\lambda}(\Lambda v) + \tau \Lambda g(t, \lambda, \Lambda v))
\]
satisfies \( H(\tau, v) \neq 0 \) for all \( \tau \in [0,1] \) and \( v \in \partial B_r(0) \) if \( r > 0 \) is small enough. So (5.3) follows from the homotopy invariance property of the Leray-Schauder degree. \( \square \)

Let \( \lambda \in (\lambda_n(p), \lambda_{n+1}(p)), n = 0, 1, 2, \ldots \), where we set \( \lambda_0(p) = -\infty \) and \( \lambda_1(p), \lambda_2(p), \ldots \) are as above, then we have.

Lemma 5.2 \( \deg[I - R_{p,\lambda,0}; B_r(0), 0] = (-1)^n. \)

Proof. We follow the idea in [2]. Note that it follows from Theorems 1.1, 1.3 that
\[
\lambda_n : p \mapsto \lambda_n(p), n = 1, 2, \ldots ,
\]
are continuous functions on \((1, \infty)\). Assume that \( p < 2 \). Define \( \lambda(q), q \in [p, 2], \)
by the following way
\[
\lambda(q) := \frac{\lambda - \lambda_n(p)}{\lambda_{n+1}(p) - \lambda_n(p)} \cdot (\lambda_{n+1}(q) - \lambda_n(q)) + \lambda_n(q), \quad n \geq 1,
\]
\[
\lambda(q) := \lambda_1(q) - (\lambda_1(p) - \lambda), n = 0.
\]
Then
\[
H(q, v) := v - R_{q,\lambda(q),0}(v) = v - \psi'_{\lambda}(\lambda(q) \Lambda \psi_{\lambda}(\Lambda v))
\]
satisfies \( H(q, v) \neq 0 \) for all \( q \in [p, 2] \) and \( v \in \partial B_r(0) \). It follows from the homotopy invariance property of the Leray-Schauder degree that
\[
\deg[I - R_{p,\lambda,0}; B_r(0), 0] = \deg[I - R_{2,\lambda(2),0}; B_r(0), 0].
\] (5.4)
The same approach but in the interval \([2, p]\) yields to the same conclusion also for \( p > 2 \). Since \( \lambda_n(2) < \lambda(2) < \lambda_{n+1}(2) \), the classical Leray-Schauder index formula implies that
\[
\deg[I - R_{2,\lambda(2),0}; B_r(0), 0] = (-1)^n.
\] (5.5)
The assertion of lemma follows now from (5.4) and (5.5). \( \square \)

With Lemmas 5.1 and 5.2 in hand we can follow the proof of [R, Theorem 1.3] to prove that continua of nontrivial solutions \((\lambda, v) \in \mathbb{R} \times X\) of (5.2) bifurcate from \((\lambda_n(p), 0), n = 1, 2, \ldots \), and they are either unbounded in \( \mathbb{R} \times X \) or meet the point \((\lambda_m(p), 0)\) with \( m \neq n \). The assertion of Theorem 1.4 follows from the fact that \((\lambda, u)\) solves (1.4) if and only if \((\lambda, -\Delta u)\) solves (5.2).
6 Appendix

To justify some statements in Section 5 we present here a brief study of the initial value problem associated with the equation in $(E_1)_p$ with $\lambda > 0$:

$$u'' = \psi_p(w), \quad u(t_0) = \alpha, \quad u'(t_0) = \beta,$$

$$w'' = \lambda \psi_p(u), \quad w(t_0) = \gamma, \quad w'(t_0) = \delta. \quad (6.1)$$

By a solution of (6.1) we understand a couple of functions $(u, w)$ which are of class $C^2$ and fulfil the equations and initial conditions in (6.1).

**Lemma 6.1** The solution to (6.1) is locally unique.

**Proof.** Without loss of generality we can restrict ourselves to $t_0 = 0$ and $p \in (1, 2)$ (the case $p > 2$ is treated similarly). Local existence is a consequence of the Schauder fixed point theorem. For its uniqueness we have to distinguish among several cases:

(I) $\alpha \neq 0$ implies that both functions $\psi_p(u(t))$ and $\psi_p(w(t))$ are of class $C^1$ in the neighbourhood of $t = 0$ and so the assertion follows from the classical theory.

(II) $\alpha = 0$, in this case $\psi_p(u(t))$ is not $C^1$ in $t = 0$.

(II)(i) $\alpha = 0, \beta \neq 0$. Let $(u, w_1), (v, w_2)$ be two solutions of (6.1) in $(0, \varepsilon)$ with some $\varepsilon > 0$. Then

$$\psi_p(u''(t)) - \psi_p(v''(t)) = \lambda \int_0^t (t - \tau) (\psi_p(u(\tau)) - \psi_p(v(\tau))) d\tau. \quad (6.2)$$

By the assumption, $u(\tau), v(\tau)$ lie in the neighbourhood of $\beta \neq 0$ for $\tau \in (0, \varepsilon)$ with $\varepsilon$ small enough. We thus have $K_1 > 0$ such that

$$\left| \psi_p\left(\frac{u(\tau)}{\tau}\right) - \psi_p\left(\frac{v(\tau)}{\tau}\right) \right| \leq K_1 \left| \frac{u(\tau)}{\tau} - \frac{v(\tau)}{\tau} \right|, \quad (6.3)$$

$\tau \in (0, \varepsilon), K_1$ independent of $\varepsilon << 1$. On the other hand there is $K_2 > 0$ such that

$$|\psi_p(u''(t)) - \psi_p(v''(t))| \geq K_2 |u''(t) - v''(t)|, \quad (6.4)$$

t $\in (0, \varepsilon)$. Now, it follows from (6.2)–(6.4)

$$K_2 |u''(t) - v''(t)| \leq \lambda \int_0^t (t - \tau) \tau^{p-1} K_1 \left| \frac{u(\tau)}{\tau} - \frac{v(\tau)}{\tau} \right| d\tau.$$

Taking into account

$$u(\tau) - v(\tau) = \int_0^\tau (\tau - \sigma) (u''(\sigma) - v''(\sigma)) d\sigma,$$
we arrive at
\[ \|u'' - v''\|_\varepsilon \leq \lambda \frac{K_1}{K_2} \varepsilon^{p+2}\|u'' - v''\|_\varepsilon, \]  
(6.5)
where \( \| \cdot \|_\varepsilon \) is the sup norm on \([0, \varepsilon]\). This implies \( u = v \) (and thus \( w_1 = w_2 \)) for \( \varepsilon \) small enough.

(II) (ii) \( \alpha = \beta = 0, \gamma \neq 0 \) and (iii) \( \alpha = \beta = \gamma = 0, \delta \neq 0 \). Instead of (6.2) we use the following fact

\[ \psi_{p'}(w_1''(t)) - \psi_{p'}(w_2''(t)) = \psi_{p'}(\lambda) \int_0^t (t - \tau)(\psi_{p'}(w_1(\tau)) - \psi_{p'}(w_2(\tau)))d\tau. \]

(6.6)
Since \( p' > 2 \), we have

\[ |\psi_{p'}(w_1(\tau)) - \psi_{p'}(w_2(\tau))| \leq K_1|w_1(\tau) - w_2(\tau)|, \]

\( \tau \in (0, \varepsilon) \). Hence

\[ \left| \int_0^t (t - \tau)(\psi_{p'}(w_1(\tau)) - \psi_{p'}(w_2(\tau)))d\tau \right| \leq K_1 \varepsilon^2\|w_1 - w_2\|_\varepsilon. \]  
(6.7)
It follows from the initial conditions that \( \frac{w_i''(t)}{t^{2(p-1)}} = \lambda \varepsilon \lambda, i = 1, 2, \) lie near \( \lambda \gamma \psi_p(\frac{1}{t^2}), i = 1, 2, \) lie near \( \lambda \delta \psi_p \left( \frac{1}{(p' + 1)} \right) \neq 0 \) in the case (iii). Hence there exists \( K_2 > 0 \) such that

\[ \left| \psi_{p'} \left( \frac{w_1''(t)}{t^{2(p-1)}} \right) - \psi_{p'} \left( \frac{w_2''(t)}{t^{2(p-1)}} \right) \right| \geq K_2 \left| \frac{w_1''(t)}{t^{2(p-1)}} - \frac{w_2''(t)}{t^{2(p-1)}} \right| \]  
(6.8)
in the case (ii) and

\[ \left| \psi_{p'} \left( \frac{w_1''(t)}{t^{2(p-1)}} \right) - \psi_{p'} \left( \frac{w_2''(t)}{t^{2(p-1)}} \right) \right| \geq K_2 \left| \frac{w_1''(t)}{t^{2(p-1)}} - \frac{w_2''(t)}{t^{2(p-1)}} \right| \]  
(6.9)
in the case (iii). Taking into account

\[ w_1(t) - w_2(t) = \int_0^t (t - \tau)(w_1''(\tau) - w_2''(\tau))d\tau \]
we derive from (6.6), (6.7), (6.8) and (6.9) that

\[ \|w_1 - w_2\|_\varepsilon \leq \frac{K_1}{K_2} \psi_{p'}(\lambda) \varepsilon^{2p+2}\|w_1 - w_2\|_\varepsilon \]
in the case (ii) and

\[ \|w_1 - w_2\|_\varepsilon \leq \frac{K_1}{K_2} \psi_{p'}(\lambda) \varepsilon^{2p+3}\|w_1 - w_2\|_\varepsilon \]
in the case (iii).
(II)(iv) \( \alpha = \beta = \gamma = \delta = 0 \). In this case (6.1) has always the trivial solution \( u_0 = w_0 = 0 \). Let \((u, w)\) be a nontrivial solution. Then

\[
|\psi_p(u''(t))| \leq \lambda \int_0^t (t - \tau)\psi_p(|u(\tau)|)d\tau \leq \lambda \varepsilon^2 ||u||_{L^p}^{p-1}, \quad t \in (0, \varepsilon),
\]

which yields

\[
||u''||_{L^p}^{p-1} \leq \lambda \varepsilon^2 ||u''||_{L^p}^{p-1},
\]

i.e. \( u = w = 0 \). This completes the proof. \( \Box \)

**Lemma 6.2** Let \((u, w)\) and \((\tilde{u}, \tilde{w})\) be solutions of (6.1) defined on \([0, 1]\), respectively, \( u(0) = w(0) = \tilde{u}(0) = \tilde{w}(0) = 0 \), \( u'(0) = \tilde{u}'(0) > 0 \), \( w'(0) < \tilde{w}'(0) \). Then \( u(t) < \tilde{u}(t) \) and \( w(t) < \tilde{w}(t) \) for any \( t \in (0, 1) \).

**Proof.** Assume that the assertion is not true. Then it follows from Lemma A.1 that there is \( t_1 > 0 \) such that \( u(t_1) = \tilde{u}(t_1) \) and \( u(t) < \tilde{u}(t), t \in (0, t_1) \). Simultaneously, the fact that both \( u \) and \( \tilde{u} \) solve (E1) imply that

\[
\int_0^{t_1} (t_1 - \tau)\psi_p' \left( \lambda \int_0^\tau (\tau - \sigma)\psi_p(u(\sigma))d\sigma + w'(0)\tau \right)d\tau
= \int_0^{t_1} (t_1 - \tau)\psi_p' \left( \lambda \int_0^\tau (\tau - \sigma)\psi_p(\tilde{u}(\sigma))d\sigma + \tilde{w}'(0)\tau \right)d\tau
\]

which contradicts the monotone character of the functions \( \psi_p \) and \( \psi_p' \). The same argument applies for \( w \) and \( \tilde{w} \). \( \Box \)

**Lemma 6.3** Let \((u, w)\) be a nonzero solution of (6.1) defined on \([0, 1]\) and satisfying \( u(0) = w(0) = u(1) = w(1) = 0 \). Then \( u'(0)w'(0) < 0 \).

**Proof.** Multiply the first (second) equation in (6.1) by \( w'(x) \) and add to get

\[
u'(x)w'(x) = \frac{|w(x)|^{p'}}{p'} + \lambda \frac{|u(x)|^p}{p} - C \text{ for all } x \in [0, 1]. \tag{6.10}
\]

Let \( x_0 \in (0, 1) \) be the point satisfying

\[
|u(x_0)| = \max_{x \in [0, 1]} |u(x)| > 0.
\]

Then (6.10) implies

\[
0 = \frac{|w(x_0)|^{p'}}{p'} + \lambda \frac{|u(x_0)|^p}{p} - C,
\]

i.e. \( C > 0 \). Hence \( u'(0)w'(0) < 0 \) by (6.10). \( \Box \)

**Lemma 6.4** Let us assume the same as in the previous lemma. Then \( u \) (and also \( w \)) changes sign in \((0, 1)\) at most finitely many times.
Proof. Let $u$ have an infinite number of bumps in $(0, 1)$. Then there exist sequences $x_n, y_n$ such that $u(x_n) = u'(y_n) = 0, x_n \to x_0, y_n \to x_0, x_n, y_n, x_0 \in [0, 1]$. Then $u(x_0) = u'(x_0) = 0$, hence (6.10) gives

$$0 = \frac{|w(x_0)|^p}{p'} - C.$$

Since $C > 0$, we have

$$w(x_0) > 0 \; \text{or} \; w(x_0) < 0.$$

Due to

$$u'(x) = \int_{x_0}^{x} \psi_p(w(y))dy + \psi_p(w(x_0)),$$

the function $u'(x)$ should be of definite sign in a neighbourhood of $x = x_0$, which contradicts the observation that $u'(y_n) = 0, y_n \to x_0$. □

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