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PARABOLIC EQUATIONS WITH VMO COEFFICIENTS IN MORREY SPACES

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Abstract. Global regularity in Morrey spaces is derived for the regular oblique derivative for linear uniformly parabolic operators. The principal coefficients of the operator are supposed to be discontinuous, belonging to Sarason's class of functions with vanishing mean oscillation (VMO).

1. Introduction

Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, $n \geq 1$, and denote by $Q_T = \Omega \times (0,T)$ a cylinder in $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$. Set $S_T = \partial\Omega \times (0,T)$ for the lateral boundary of Q_T and denote by $x = (x', t) = (x_1, \dots, x_n, t)$ a point in $\mathbb{R}^{n+1}$. We consider the following regular oblique derivative problem for the uniformly parabolic operator $\mathcal{P}$ with discontinuous coefficients

$$
\begin{aligned}
\mathcal{P}u = u_t - \sum_{i,j=1}^n a^{ij}(x) D_{ij} u = f(x) \quad \text{in } Q_T,\\
\mathcal{I}u = u(x',0) = 0 \quad \text{on } \Omega,\\
\mathcal{B}u = \sum_{i=1}^n \ell^i(x) D_i u = 0 \quad \text{on } S_T.
\end{aligned}
\tag{1.1}
$$

There is a vast number of existence results in Hölder and Sobolev spaces for initial-boundary value problems for linear elliptic and parabolic operators with Hölder continuous coefficients a^{ij} (see [16], [21], [22]). In our considerations, we suppose the coefficients a^{ij} belong to the Sarason class of functions with *vanishing mean oscillation* VMO (cf. [28]).

Recall that the class VMO consists of functions with *bounded mean oscillation* BMO (cf. [20]) whose integral oscillation over balls shrinking to a point converges uniformly to zero. The interest to the space VMO is due mainly to the fact that it contains as a proper subspace the bounded uniformly continuous functions. This ensures the extension of the L^p-theory for operators with *continuous* coefficients to the case of *discontinuous* ones. Moreover, the Sobolev spaces $W^{1,n}(\Omega)$ and $W^{\theta,n/\theta}(\Omega)$, $0<\theta<1$ are also contained in VMO, which makes the discontinuity of

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the coefficients $a^{ij} \in VMO$ more general than those studied before either for elliptic ($a^{ij} \in W^{1,n}(\Omega)$, [24]) or parabolic operators ($D_x a^{ij} \in L^{n+2}$, $D_t a^{ij} \in L^{(n+2)/2}$, [17], [18]).

In two innovative articles, Chiarenza, Frasca and Longo ([10], [11]) modify the classical methods for obtaining L^p-estimates of solutions to Dirichlet boundary problem for linear elliptic equations. This allow them to move from $a^{ij}(x) \in C^0(\bar{\Omega})$ into $a^{ij}(x) \in VMO$. Roughly speaking, the approach goes back to Calderón and Zygmund (see [5], [6]) and makes use of an explicit representation formula for D^2u in terms of singular integrals and their commutators with variable Calderón-Zygmund type kernel. We refer the reader to the survey [8], where an excellent presentation of the state-of-art and relations with another similar results for linear and quasilinear (cf. [25]) elliptic operators can be found.

Later on, the articles [4] and [29] consider unique solvability in the Sobolev spaces $W^{2,1}_p$, $p \in (1,\infty)$ of the Cauchy-Dirichlet and oblique derivative problem for the operator $\mathcal{P}$, while [30] presents existence results for initial-boundary value problems for quasilinear parabolic equations of the type $u_t - a^{ij}(x,t,u)D_{ij}u = f(x,t,u,Du)$. An up-to-date overview of the classical and the modern results regarding elliptic and parabolic equations with discontinuous data can be found in the monograph [23].

In the present work we are interested of the Morrey regularity of solutions to (1.1). Let us note that the parabolic Morrey spaces $L^{p,\lambda}$ are subspaces of L^p for every $p \in (1,\infty)$ and $\lambda \in (0,n+2)$ ($\lambda \in (0,n)$ in the elliptic case). Whence the existence results in the Sobolev spaces $W^{2,1}_p$ for elliptic and parabolic operators with right-hand side $f \in L^p$ still hold if $f \in L^{p,\lambda}$. A natural question which arises is whether $\mathcal{P}u \in L^{p,\lambda}$ implies $u \in W^{2,1}_{p,\lambda}$. It is true for elliptic operators, as it is shown in [27] (see also [14], [13]). For parabolic operators, in a difference, there is no results concerning the regularizing properties of the operator $\mathcal{P}$ in the Morrey spaces.

The main goal of the present work is to show that the solution of (1.1) belongs to $W^{2,1}_{p,\lambda}(Q_T)$ assuming the coefficients of the uniformly parabolic operator $\mathcal{P}$ to be bounded VMO functions and $f \in L^{p,\lambda}(Q_T)$, $p \in (1,\infty)$, $\lambda \in (0,n+2)$. The crucial point of our investigations is the establishment of suitable integral estimates of singular integral operators and their commutators with variable *parabolic Calderón–Zygmund (PCZ) kernel.* The expansion of the kernel into spherical harmonics allow us to reduce our considerations over integral operators with constant PCZ kernel which possesses good enough regularity. Constructing a diadic partition of the space subordinated to the utilized parabolic metric (the standard one or that defined in [15]) we derive the desired Morrey estimates (see Section 3). In Section 4, there are established analogous $L^{p,\lambda}$ estimates for nonsingular integrals and commutators making use of a *generalized reflection* similar to the one used for constructing of half space Green function. These results are applied later in Section 5 to derive $L^{p,\lambda}(Q_T)$ estimates for the second spatial derivatives of the solution of (1.1). The $W^{2,1}_{p,\lambda}(Q_T)$ a priori estimate of the solution is established analogously as the $W^{2,1}_p(Q_T)$ estimate obtained in [29]. Finally, the Morrey regularity of solution implies Hölder regularity of its gradient (see Corollary 5.2), which is finer than the one known in the case $\mathcal{P}u \in L^p$ (cf. [29, Corollary 1]).

2. Definitions and preliminary results

Consider the regular oblique derivative problem

$$\begin{aligned} \mathcal{P}u &\equiv u_t - a^{ij}(x)D_{ij}u = f(x) \quad \text{in } Q_T,\\ \mathcal{I}u &\equiv u(x',0) = 0 \quad \text{on } \Omega,\\ \mathcal{B}u &\equiv \ell^i(x)D_iu = 0 \quad \text{on } S_T. \end{aligned} \tag{2.1}$$

where $Q_T = \Omega \times (0,T)$ is a cylinder in $\mathbb{R}^n \times \mathbb{R}_+$, $n \geq 1$, the base Ω is bounded $C^{1,1}$ domain in $\mathbb{R}^n$ and $T > 0$. Set $S_T = \partial\Omega \times (0,T)$ for the lateral boundary of Q_T.

Throughout the paper the standard summation convention on repeated upper and lower indices is adopted. Moreover, we set $D_iu = \partial u/\partial x_i$, $D_{ij}u = \partial^2 u/\partial x_i \partial x_j$, $u_t = D_tu = \partial u/\partial t$, $Du = (D_1u, \dots, D_nu)$ means the spatial gradient of u, $D^2u = \{D_{ij}\}_{ij=1}^n$ stands for its Hessian matrix and $x = (x',t) = (x_1,\dots,x_n,t) \in \mathbb{R}^{n+1}$. In our further considerations we shall use the notations $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$, and $\mathbb{D}^{n+1}_+ = \mathbb{R}^n_+ \times \mathbb{R}_+$.

We suppose that $\mathcal{P}$ is a *uniformly parabolic operator,* i.e., there exists a positive constant Λ such that

$$\Lambda^{-1}|\xi|^2 \leq a^{ij}(x,t)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \text{a.a. } x \in Q_T,\ \forall \xi \in \mathbb{R}^n. \tag{2.2}$$

Besides that, the requirement the coefficients matrix $\mathbf{a} = \{a^{ij}\}_{ij=1}^n$ to be symmetric, leads to essential boundedness of a^{ij}'s (cf. [29]).

The boundary operator $\mathcal{B}$ is prescribed in terms of a directional derivative with respect to the unit vector field $\ell(x) = (\ell^1(x),\dots,\ell^n(x),0)$ defined on S_T. We suppose that $\mathcal{B}$ is a *regular oblique derivative operator* (cf. [26]), i.e., the field ℓ is never tangential to the boundary S_T:

$$\ell(x)\cdot\nu(x') = \ell^i(x)\nu_i(x') > 0 \quad \text{on } S_T\,,\ \ell^i \in \mathrm{Lip}(\bar{S}_T). \tag{2.3}$$

Here $\mathrm{Lip}(\bar{S}_T)$ is the class of functions which are uniformly Lipschitz continuous on $\bar{S}_T$ and $\nu(x') = (\nu_1(x'),\dots,\nu_n(x'))$ stands for the unit inner normal to $\partial\Omega$.

Denote by $\mathcal{P}_0$ a linear parabolic operator with constant coefficients a_0^{ij} that satisfy (2.2). It is well known from the linear theory (cf. [21]) that the fundamental solution of the operator $\mathcal{P}_0$ is given by the formula

$$\Gamma^0(y) = \begin{cases} \frac{1}{(4\pi\tau)^{n/2}\sqrt{\det \mathbf{a}_0}} \exp\Big\{-\frac{A_0^{ij}y_iy_j}{4\tau}\Big\} & \text{for} \quad \tau > 0,\\ 0 & \text{for} \quad \tau < 0, \end{cases} \tag{2.4}$$

where $\mathbf{a}_0 = \{a_0^{ij}\}$ is the matrix of the coefficients of $\mathcal{P}_0$ and $\mathbf{A}_0 = \{A_0^{ij}\} = \mathbf{a}_0^{-1}$ is its inverse matrix. Hereafter we denote by Γ_i^0 and Γ_{ij}^0 the derivatives $\partial\Gamma^0/\partial y_i$ and $\partial^2\Gamma^0/\partial y_i\partial y_j$. In the problem under consideration (2.1), the coefficients of the operator $\mathcal{P}$ depend on x. To express this dependence in the fundamental solution we define

$$\Gamma(x;y) = \begin{cases} \frac{1}{(4\pi\tau)^{n/2}\sqrt{\det \mathbf{a}(x)}} \exp\Big\{-\frac{A^{ij}(x)y_iy_j}{4\tau}\Big\} & \text{for} \quad \tau > 0,\\ 0 & \text{for} \quad \tau < 0, \end{cases} \tag{2.5}$$

where $\mathbf{a}(x) = \{a^{ij}(x)\}$ is the matrix of the coefficients of $\mathcal{P}$ and $\mathbf{A}(x) = \{A^{ij}(x)\} = \mathbf{a}(x)^{-1}$ is its inverse matrix. The derivatives Γ_i and Γ_{ij} are taken with respect to the second variable y.

For the goal of our further considerations, besides the standard parabolic metric $\widetilde{\rho}(x) = \max(|x'|, |t|^{1/2})$, $|x'| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ we are going to use the one introduced by Fabes and Riviére in [15]

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}}, \quad d(x,y) = \rho(x-y). \tag{2.6}$$

A ball with respect to the metric d centered at zero and of radius r is simply an ellipsoid

$$\mathcal{E}_r(0) = \left\{ x \in \mathbb{R}^{n+1} \colon \quad \frac{|x'|^2}{r^2} + \frac{t^2}{r^4} < 1 \right\}.$$

Obviously, the unit sphere with respect to this metric coincides with the unit sphere in $\mathbb{R}^{n+1}$, i.e.

$$\partial\mathcal{E}_1(0) \equiv \Sigma_{n+1} = \Big\{ x \in \mathbb{R}^{n+1} \colon |x| = \Big(\sum_{i=1}^n x_i^2 + t^2 \Big)^{1/2} = 1 \Big\}.$$

Let I be a *parabolic cylinder* centered at some point x and with radius r, that is $I = I_r(x) = \{y \in \mathbb{R}^{n+1} \colon |x'-y'| < r, |t-\tau| < r^2\}$. It is easy to see that for any ellipsoid $\mathcal{E}_r$ there exist cylinders $\underline{I}$ and $\overline{I}$ with measures comparable to r^{n+2} and such that $\underline{I} \subset \mathcal{E}_r \subset \overline{I}$. Obviously, that relation gives an equivalence of the both metrics and the introduced by them topologies. Later we shall use this equivalence without making reference to, except if it is necessary.

It is worth noting that (2.6) has been employed in the study of singular integral operators with Calderón-Zygmund kernels of mixed homogeneity (see [15]).

Definition 2.1. *A function $k(x)$ is said to be a parabolic Calderón-Zygmund (PCZ) kernel in the space $\mathbb{R}^{n+1}$ if*

i) k is smooth on $\mathbb{R}^{n+1} \setminus \{0\}$;

ii) $k(rx', r^2t) = r^{-(n+2)}k(x',t)$ for each $r > 0$;

iii) $\int_{\rho(x)=r} k(x)d\sigma_x = 0$ for each $r > 0$.

Definition 2.2. *We say that a function $k(x;y)$, $x \in \mathbb{R}^{n+1}$, $y \in \mathbb{R}^{n+1} \setminus \{0\}$ is a variable PCZ kernel, if:*

i) $k(x;\cdot)$ is a PCZ kernel (in the sense of Definition 2.1) for a.a. $x \in \mathbb{R}^{n+1}$;

ii) $\sup_{\rho(y)=1} \left| \left(\frac{\partial}{\partial y}\right)^\beta k(x;y) \right| \le C(\beta)$ for every multiindex β, independently of x.

For the sake of the completeness we shall recall here the definitions and some properties of the spaces we are going to use.

Definition 2.3. *We say that the measurable and locally integrable function f belongs to BMO if the seminorm*

$$\|f\|_* = \sup_I \frac{1}{|I|} \int_I |f(y) - f_I| dy \tag{2.7}$$

is finite. Here I ranges over all parabolic cylinders in $\mathbb{R}^{n+1}$ with radius r, and centered at some point x and $f_I = \frac{1}{|I|}\int_I f(y)dy$. Then $\|f\|_*$ is a norm in BMO modulo constant functions under which BMO is a Banach space.

Definition 2.4. *Let $f \in BMO$, $r_0 > 0$ and denote*

$$\gamma_f(r_0) = \sup_{I_r} \frac{1}{|I_r|} \int_{I_r} |f(y) - f_{I_r}| dy. \tag{2.8}$$

We say that $f \in VMO$ if $\gamma_f(r_0) \to 0$ as $r_0 \to 0$ where the supremum is taken over all parabolic cylinders I_r centered at some point x with radius $r \le r_0$. The quantity $\gamma_f(r_0)$ is referred to as VMO modulus of f.

The spaces $BMO(Q_T)$ and $VMO(Q_T)$ can be defined by taking $I \cap Q_T$ and $I_r \cap Q_T$ instead of I and I_r in the definitions of $\|f\|_*$ and $\gamma_f(r_0)$.

Having a function f defined in Q_T and belonging to $BMO(Q_T)$, it is possible to extend it to the whole $\mathbb{R}^{n+1}$ and the BMO norm of the extension could be estimated by the BMO norm of the original function. If in addition $f \in VMO(Q_T)$, then we may extend it preserving its VMO-modulus, as it follows by the results of Jones [19] and Acquistapace [1, Proposition 1.3]. The next theorem offers several alternative descriptions of VMO.

Theorem 2.5. ([28, Theorem 1]) *For $f \in BMO$, the following conditions are equivalent:*

(i) *f is in VMO;*
(ii) *f is in the BMO-closure of the space of bounded uniformly continuous functions;*
(iii) $\lim_{y\to 0} \|f(x-y) - f(x)\|_* = 0$;

If f is a uniformly continuous function, then its VMO-modulus $\gamma_f(r)$ coincides with the modulus of continuity $\omega_f(r)$. Moreover, for a given $f \in VMO$ we can find a sequence $\{f_k\} \in L^\infty \cap C^\infty(\mathbb{R}^{n+1})$ of functions with $\gamma_{f_k}(r) \equiv \omega_{f_k}(r)$, such that $f_k \to f$ in VMO as $k \to \infty$ and $\gamma_{f_k}(r) \le \gamma_f(r)$ for all integer numbers k. In what follows we use these results without explicit reference.

The problem (2.1) has been already studied in the framework of the Sobolev spaces $W^{2,1}_p(Q_T)$, $p \in (1,\infty)$ (cf. [29], [23]). Precisely, assuming (2.2), (2.3) and $a^{ij} \in VMO(Q_T)$, it is proved that for any $f \in L^p(Q_T)$, $p \in (1,\infty)$, there exists a unique weakly differentiable function u belonging to $L^p(Q_T)$ with all its derivatives $D^r_t D^s_x u$, $0 \le 2r+s \le 2$, such that u satisfies the equation in (2.1) almost everywhere in Q_T and the boundary conditions holds in trace sense.

Our goal here is to obtain finer regularity of that solution supposing that $\mathcal{P}u$ belongs to the Morrey space $L^{p,\lambda}$.

Definition 2.6. *We say that a measurable function $f \in L^p_{\rm loc}$ belongs to the parabolic Morrey space $L^{p,\lambda}$ if for any $p \in (1,+\infty)$ and $\lambda \in (0, n+2)$ the following norm is finite*

$$\|f\|_{p,\lambda} = \Big(\sup_{r>0} \frac{1}{r^\lambda} \int_I |f(y)|^p dy \Big)^{1/p} \tag{2.9}$$

where I ranges over all parabolic cylinders in $\mathbb{R}^{n+1}$ of radius r.

To define the space $L^{p,\lambda}(Q_T)$, we insist the norm

$$\|f\|_{p,\lambda;Q_T} = \Big(\sup_{r>0} \frac{1}{r^\lambda} \int_{Q_T \cap I} |f(y)|^p dy \Big)^{1/p} \tag{2.10}$$

to be finite.

Definition 2.7. *We say that the function u lies in the Morrey space $W^{2,1}_{p,\lambda}(Q_T)$, $1 < p < \infty$, $0 < \lambda < n+2$, if it is weakly differentiable and belongs to $L^{p,\lambda}(Q_T)$, along with all its derivatives $D^r_t D^s_x u$, $0 \le 2r + s \le 2$. Then the following norm is finite*

$$\|u\|_{W^{2,1}_{p,\lambda}(Q_T)} = \|u\|_{p,\lambda;Q_T} + \|D^2 u\|_{p,\lambda;Q_T} + \|D_t u\|_{p,\lambda;Q_T}.$$

For a given measurable function $f \in L^1_{loc}$ we define the *Hardy-Littlewood maximal operator*

$$Mf(x) = \sup_I \frac{1}{|I|} \int_I |f(y)| dy \quad \text{for a.a. } x \in \mathbb{R}^{n+1}, \tag{2.11}$$

where the supremum is taken over all parabolic cylinders I centered at the point x.

A variant of it is the *sharp maximal operator*

$$f^{\#}(x) = \sup_I \frac{1}{|I|} \int_I |f(y) - f_I| dy \quad \text{for a.a. } x \in \mathbb{R}^{n+1}. \tag{2.12}$$

The following lemmas give $L^{p,\lambda}$ estimates for f, Mf and $f^{\#}$. Their L^p variants are proved in [2]. Analogous $L^{p,\lambda}$ estimates, but in the space $\mathbb{R}^n$ endowed with the Euclidean metric can be found in [9] and [13]. To prove the $L^{p,\lambda}$ estimates below, we follow the same lines of reasoning as in the paper cited above, making use of the parabolic metrics $\widetilde{\rho}$ or ρ and corresponding to them diadic partition of the space:

$$\mathbb{R}^{n+1} = 2I + \bigcup_{k=1}^{\infty} 2^{k+1} I \setminus 2^k I$$

where I is either parabolic cylinder or ellipsoid centered at some point $x \in \mathbb{R}^{n+1}$ with radius r. We note that $2^k I$ means parabolic cylinder (ellipsoid) with the same center and radius $2^k r$.

Lemma 2.8. (Maximal inequality) *Let $f \in L^{p,\lambda}$, $p \in (1, \infty)$, $\lambda \in (0, n+2)$. Then there exists a constant C independent of f such that*

$$||Mf||_{p,\lambda} \leq C\|f\|_{p,\lambda}. \tag{2.13}$$

Lemma 2.9. (Sharp inequality) *Let f be the same as in Lemma 2.8. Then there exists a constant C independent of f such that*

$$\|f\|_{p,\lambda} \leq C\|f^{\#}\|_{p,\lambda} \tag{2.14}$$

Analogous estimates are valid also in $\mathbb{D}^{n+1}_+$ where the corresponding diadic partition of the space has the form

$$\mathbb{D}^{n+1}_+ = 2I_+ + \bigcup_{k=1}^{\infty} 2^{k+1} I_+ \setminus 2^k I_+$$

where $I_+ = I \cap \{x_n > 0, t > 0\}$ and I is a parabolic cylinder. Then

$$\|Mf\|_{p,\lambda;\mathbb{D}^{n+1}_+} \leq C\|f\|_{p,\lambda;\mathbb{D}^{n+1}_+}, \quad \|f\|_{p,\lambda;\mathbb{D}^{n+1}_+} \leq C\|f^{\#}\|_{p,\lambda;\mathbb{D}^{n+1}_+}.$$

Lemma 2.10. (John-Nirenberg type lemma) *Let $1 < p < \infty$, $a \in BMO$ and I be a parabolic cylinder. Then*

$$\left(\frac{1}{|I|} \int_I |a(y) - a_I|^p \, dy\right)^{1/p} \leq C(p)\|a\|_*.$$

Lemma 2.11. [4, Lemma 2.10] *Let $a \in BMO$. Then, for any positive integer j and parabolic cylinder I*

$$|a_{2^j I} - a_I| \leq C(n) j \|a\|_*.$$

The next lemma gives an important property of the Calderón-Zygmund kernels.

Lemma 2.12. [4, Pointwise Hörmander condition] *Let k be a PCZ kernel. Then for any parabolic cylinder I_0 of center x_0 one has*

$$|k(x-y)-k(x_0-y)| \le C(k)\frac{\rho(x_0-x)}{\rho(x_0-y)^{n+3}}$$

for any $x \in I_0$ and $y \notin 2I_0$.

3. Singular integral estimates in Morrey spaces

Let $k(x;y)$ be a variable PCZ kernel. For any functions $f \in L^{p,\lambda}$, $p \in (1,\infty)$, $\lambda \in (0,n+2)$ and $a \in L^\infty$ define a singular integral operator $\mathcal{K}f$ and its commutator $\mathcal{C}[a,f]$ by

$$\mathcal{K}f(x) = \lim_{\varepsilon\to 0}\int_{\rho(x-y)>\varepsilon} k(x;x-y)f(y)dy = \lim_{\varepsilon\to 0}\mathcal{K}_\varepsilon f(x) \tag{3.1}$$

$$\begin{aligned}\mathcal{C}[a,f](x) &= \lim_{\varepsilon\to 0}\int_{\rho(x-y)>\varepsilon} k(x;x-y)[a(y)-a(x)]f(y)dy \\ &= \lim_{\varepsilon\to 0}\mathcal{C}_\varepsilon f(x) = \mathcal{K}(af)(x) - a(x)\mathcal{K}f(x).\end{aligned} \tag{3.2}$$

The aim of this section is to derive $L^{p,\lambda}$ a priori estimates for the singular operators $\mathcal{K}f$ and $\mathcal{C}[a,f]$. For this goal we are going to exploit the well known technique, based on an expansion into spherical harmonics (cf. [5], [6], [10], [4]).

Any homogeneous polynomial $p(x)$, $x \in \mathbb{R}^N$ of degree m, solution of Laplaces equation $\Delta u = 0$, is called *N-dimensional solid harmonic* of degree m. Its restriction to the unit sphere Σ_N is called *N-dimensional spherical harmonic* of degree m.

Denote by Y_m the space of $(n+1)$-dimensional spherical harmonics of degree m. It is a finite-dimensional space with $\dim Y_m = g_m$ where

$$g_m = \binom{m+n}{n} - \binom{m+n-2}{n} \le C(n)m^{n-1} \tag{3.3}$$

and the second binomial coefficient is equal to 0 when $m = 0,1$, i.e., $g_0 = 1$, $g_1 = n+1$. Further, let $\{Y_{sm}(x)\}_{s=1}^{g_m}$ be an *orthonormal base* of Y_m. Then $\{Y_{sm}(x)\}_{s=1,m=0}^{g_m,\ \infty}$ is a *complete orthonormal base* in $L^2(\Sigma_{n+1})$ and

$$\sup_{x\in\Sigma_{n+1}}\left|\left(\frac{\partial}{\partial x}\right)^\beta Y_{sm}(x)\right| \le C(n)m^{|\beta|+(n-1)/2}, \quad m = 1,2,\dots. \tag{3.4}$$

If, for instance, $\phi \in C^\infty(\Sigma_{n+1})$ then $\phi(x) \sim \sum_{s,m} b_{sm}Y_{sm}(x)$ is the Fourier series expansion of $\phi(x)$ with respect to $\{Y_{sm}(x)\}$, where

$$b_{sm} = \int_{\Sigma_{n+1}}\phi(y)Y_{sm}(y)d\sigma, \quad |b_{sm}| \le C(l)m^{-2l}\sup_{\substack{|\beta|=2l\\ y\in\Sigma_{n+1}}}\left|\left(\frac{\partial}{\partial y}\right)^\beta\phi(y)\right| \tag{3.5}$$

for every $l > 1$ and the notation $\sum_{s,m}$ stands for $\sum_{m=0}^\infty\sum_{s=1}^{g_m}$.

We are in a position now to formulate our result concerning singular operators.

Theorem 3.1. *Let $k(x;y)$ be a variable PCZ kernel. Then for any $f \in L^{p,\lambda}$, $p \in (1,\infty)$, $\lambda \in (0,n+2)$ and $a \in L^\infty$ the integrals $\mathcal{K}f$ and $\mathcal{C}[a,f]$ there exist and*

$$\lim_{\varepsilon\to 0}\|\mathcal{K}_\varepsilon f - \mathcal{K}f\|_{p,\lambda} = \lim_{\varepsilon\to 0}\|\mathcal{C}_\varepsilon[a,f] - \mathcal{C}[a,f]\|_{p,\lambda} = 0. \tag{3.6}$$

Furthermore, there exists a constant $C = C(n, p, k)$, independent of f, such that

$$\|\mathcal{K}f\|_{p,\lambda} \le C\|f\|_{p,\lambda}, \quad \|\mathcal{C}[a, f]\|_{p,\lambda} \le C\|a\|_*\|f\|_{p,\lambda}. \tag{3.7}$$

Proof. By density arguments it is enough to prove the theorem for $f \in C_0^\infty(\mathbb{R}^{n+1})$.

Let $x, y \in \mathbb{R}^{n+1}$ and $\bar{y} = \frac{y}{\rho(y)} \in \Sigma_{n+1}$. Having in mind the homogeneity properties of the variable PCZ kernel, we can write

$$\rho(y)^{n+2}k(x; y) = k(x; \bar{y}) = \sum_{s,m} b_{sm}(x)Y_{sm}(\bar{y}).$$

Hence $k(x; y) = \rho(y)^{-(n+2)} \sum_{s,m} b_{sm}(x)Y_{sm}(\bar{y})$.

¿From the Definition 2.2 *ii)* and the estimate (3.5) it follows

$$\|b_{sm}\|_\infty \le C(l, \beta)m^{-2l}. \tag{3.8}$$

Let we note that the function $k(x, \bar{y})$ is C^∞ with respect to $\bar{y}$ and hence it is equal to its series expansion. So we consider the integrals

$$\mathcal{K}_\varepsilon f(x) = \int_{\rho(x-y)>\varepsilon} \sum_{s,m} b_{sm}(x)\mathcal{H}_{sm}(x - y)f(y)dy, \tag{3.9}$$

$$\mathcal{C}_\varepsilon[a, f](x) = \int_{\rho(x-y)>\varepsilon} \sum_{s,m} b_{sm}(x)\mathcal{H}_{sm}(x - y)[a(y) - a(x)]f(y)dy. \tag{3.10}$$

where $\mathcal{H}_{sm}(x - y)$ stands for the kernel $Y_{sm}(\overline{x - y})\rho(x - y)^{-(n+2)}$. We note that the series

$$\begin{aligned}\big|\sum_{s,m} b_{sm}(x)\mathcal{H}_{sm}(x - y)f(y)\big| &\le |f(y)|\varepsilon^{-(n+2)} \sum_{s,m} \|b_{sm}\|_\infty\|Y_{sm}\|_\infty \\ &\le C\varepsilon^{-(n+2)}|f(y)| \sum_{m=1}^{\infty} m^{-2l}m^{n-1}m^{(n-1)/2}\end{aligned}$$

converges for $l > (3n - 1)/4$. Hence, by the dominated convergence theorem, we can write

$$\mathcal{K}_\varepsilon f(x) = \sum_{s,m} b_{sm}(x) \int_{\rho(x-y)>\varepsilon} \mathcal{H}_{sm}(x - y)f(y)dy. \tag{3.11}$$

Identical arguments are valid also for the commutator, so

$$\mathcal{C}_\varepsilon[a, f](x) = \sum_{s,m} b_{sm}(x) \int_{\rho(x-y)>\varepsilon} \mathcal{H}_{sm}(x - y)[a(x) - a(y)]f(y)dy.$$

It is easy to check that $\mathcal{H}_{sm}(x)$ is PCZ kernel in the sense of Definition 2.1. Moreover,

$$\sup_{x\in\Sigma_{n+1}} |\nabla_{x'}\mathcal{H}_{sm}(x)| \le C(n)m^{(n+1)/2}$$

according to (3.4). Later on, for all $x \in \mathbb{R}^{n+1}$ we have estimates also for the derivatives of $\mathcal{H}_{sm}(x)$, that is

$$\begin{aligned}D_i\mathcal{H}_{sm}(x) &= D_i\big(Y_{sm}\left(\frac{x}{\rho(x)}\right)\rho(x)^{-(n+2)}\big) \\ &= D_iY_{sm}\big(\frac{x}{\rho(x)}\big)\frac{1}{\rho(x)}\rho(x)^{-(n+2)} - (n + 2)\rho(x)^{-(n+3)}D_i\rho(x)Y_{sm}\big(\frac{x}{\rho(x)}\big).\end{aligned}$$

Hence

$$|D_i\mathcal{H}_{sm}(x)| \le C(n)m^{(n+1)/2}\rho(x)^{-(n+3)}. \tag{3.12}$$

The derivative with respect to t is calculated analogously

$$|D_t \mathcal{H}_{sm}(x)| \le C(n) m^{(n+1)/2} \rho(x)^{-(n+4)}. \tag{3.13}$$

Now it is easy to see that $\mathcal{H}_{sm}(x)$ satisfies Hörmander type condition.

Lemma 3.2. *Let I_0 be a cylinder centered at x_0 with radius r. Consider $x \in I_0$ and $y \notin 2I_0$. Then the PCZ kernel $\mathcal{H}_{sm}(x)$ satisfies*

$$|\mathcal{H}_{sm}(x-y) - \mathcal{H}_{sm}(x_0-y)| \le C(n) m^{(n+1)/2} \frac{\rho(x_0-x)}{\rho(x_0-y)^{n+3}}. \tag{3.14}$$

The proof is analogous to that of Lemma 2.12 making use of (3.12) and (3.13).

Lemma 3.3. *Let f and a be the same as above. Then the singular integrals*

$$\begin{aligned}
\mathcal{K}_{sm} f(x) &= P.V. \int_{\mathbb{R}^{n+1}} \mathcal{H}_{sm}(x-y) f(y) dy, \\
\mathcal{C}_{sm}[a,f](x) &= P.V. \int_{\mathbb{R}^{n+1}} \mathcal{H}_{sm}(x-y) [a(y)-a(x)] f(y) dy
\end{aligned}$$

satisfy the estimates

$$(\mathcal{K}_{sm} f)^{\#}(x) \le C m^{\frac{n+1}{2}} \big(M(|f|^p)(x)\big)^{\frac{1}{p}}, \tag{3.15}$$

$$(\mathcal{C}_{sm}[a,f])^{\#}(x) \le C \|a\|_* \Big\{ \big(M(|\mathcal{K}_{sm} f|^p)(x)\big)^{\frac{1}{p}} + m^{\frac{n+1}{2}} \big(M(|f|^p)(x)\big)^{\frac{1}{p}} \Big\}, \tag{3.16}$$

where the constants depend on n, p, λ but not on f.

Proof. For arbitrary $x_0 \in \mathbb{R}^{n+1}$ and cylinder I centered at x_0 with radius r, we construct the operator

$$\begin{aligned}
J &= \frac{1}{|I|} \int_I |\mathcal{K}_{sm} f(y) - (\mathcal{K}_{sm} f)_I| \, dx \\
&= \frac{1}{|I|} \int_I |\mathcal{K}_{sm} f(y) - \mathcal{K}_{sm2r} f(x_0) + \mathcal{K}_{sm2r} f(x_0) - (\mathcal{K}_{sm} f)_I| \, dx \\
&\le \frac{2}{|I|} \int_I |\mathcal{K}_{sm} f(y) - \mathcal{K}_{sm2r} f(x_0)| \, dx,
\end{aligned}$$

where $\mathcal{K}_{sm2r} f(x_0) = \int_{\rho(y-x_0)>2r} \mathcal{H}_{sm}(x_0-y) f(y) dy$. We define $(2I)^c = \mathbb{R}^{n+1} \setminus 2I$ and write $f(x) = f(x)\chi_{2I}(x) + f(x)\chi_{(2I)^c}(x) = f_1(x) + f_2(x)$. Hence

$$\begin{aligned}
(\mathcal{K}_{sm} f)^{\#}(x_0) \le & \frac{C}{|I|} \int_I |\mathcal{K}_{sm} f_1(y)| dy \\
& + \frac{C}{|I|} \int_I |\mathcal{K}_{sm} f_2(y) - \mathcal{K}_{sm2r} f(x_0)| dy = J_1 + J_2.
\end{aligned}$$

Thus,

$$\begin{aligned}
J_1 &\le \frac{C}{|I|} \Big(\int_I 1 dy \Big)^{1/p'} \Big(\int_I |\mathcal{K}_{sm} f_1(y)|^p dy \Big)^{1/p} = \frac{C}{|I|^{1/p}} \|\mathcal{K}_{sm} f_1\|_p \\
&\le \frac{C}{|I|^{1/p}} \|f_1\|_p \le C \big(M(|f|^p)(x_0)\big)^{1/p}
\end{aligned}$$

after applying [15, Theorem 1] and taking the supremum with respect to I. The second integral gives

$$\begin{aligned}
J_2 &\le \frac{C}{|I|}\int_I \left(\int_{(2I)^c} |\mathcal{H}_{sm}(y-\xi) - \mathcal{H}_{sm}(x_0-\xi)|\,|f(\xi)|d\xi \right) dy \\
&\le Cm^{(n+1)/2}\frac{1}{|I|}\int_I \left(\int_{(2I)^c} \frac{\rho(x_0-y)}{\rho(x_0-\xi)^{n+3}}|f(\xi)|d\xi \right) dy \\
&\le Cm^{(n+1)/2} r \sum_{k=1}^{\infty} \int_{2^{k+1}I\setminus 2^k I} \frac{|f(\xi)|}{\rho(x_0-\xi)^{n+3}} d\xi \\
&\le Cm^{(n+1)/2} r \sum_{k=1}^{\infty} \frac{1}{(2^k r)^{n+3}} \left(\int_{2^{k+1}I} 1 d\xi \right)^{1/p'} \left(\int_{2^{k+1}I} |f(\xi)|^p d\xi \right)^{1/p} \\
&\le Cm^{(n+1)/2} \frac{1}{r^{n+2}} \sum_{k=1}^{\infty} \frac{1}{(2^{k(n+3)}} |2^{k+1}I| \left(\frac{1}{|2^{k+1}I|} \int_{2^{k+1}I} |f(\xi)|^p d\xi \right)^{1/p} \\
&\le Cm^{(n+1)/2} \big(M(|f|^p)(x_0) \big)^{1/p},
\end{aligned}$$

where we have applied Lemma 3.2 for the cylinder I centered at x_0 and containing y while $\xi \in (2I)^c$. The final inequality has been reached after taking the supremum with respect to I. Since x_0 was chosen arbitrary, the estimate (3.15) holds true for any $x \in \mathbb{R}^{n+1}$.

As it concerns the commutator we shall employ the idea of Stromberg (see [31]) which consists of expressing $\mathcal{C}_{sm}[a,f]$ as a sum of integral operators and estimating their sharp functions. Precisely

$$\begin{aligned}
\mathcal{C}_{sm}[a,f](x) &= \mathcal{K}_{sm}(a-a_I)f(x) - (a(x)-a_I)\mathcal{K}_{sm}f(x) \\
&= \mathcal{K}_{sm}(a-a_I)f\chi_{2I}(x) + \mathcal{K}_{sm}(a-a_I)f\chi_{2I^c}(x) - (a(x)-a_I)\mathcal{K}_{sm}f(x) \\
&= A(x) + B(x) + C(x).
\end{aligned}$$

Now for any $p > 1$ and $q \in (1,p)$ we have

$$\begin{aligned}
G_1 =& \frac{1}{|I|}\int_I |A(x) - A_I|dx \le \frac{2}{|I|}\int_I |\mathcal{K}_{sm}(a-a_I)f\chi_{2I}(x)|dx \\
&\le \frac{C}{|I|} \left(\int_I |\mathcal{K}_{sm}(a-a_I)f\chi_{2I}(x)|^q dx \right)^{1/q} \left(\int_I 1^{q'} dx \right)^{1/q'} \\
&\le |I|^{-1/q} \left(\int_{2I} |f(y)|^p dy \right)^{1/p} \left(\int_{2I} |a(y)-a_I|^{pq/(p-q)} dy \right)^{(p-q)/pq},
\end{aligned}$$

as follows from [15, Theorem 1]. Further the John-Nirenberg type lemma and Lemma 2.11 applied to the second integral yield

$$\begin{aligned}
\int_{2I} |a(y)-a_I|^{pq/(p-q)} dy &\le \int_{2I} |a(y)-a_{2I}|^{pq/(p-q)} dy + \int_{2I} |a_{2I}-a_I|^{pq/(p-q)} dy \\
&\le C(p,q)\|a\|_*^{pq/(p-q)}
\end{aligned}$$

and hence

$$G_1 \le C\|a\|_* \left(\frac{1}{|2I|} \int_{2I} |f(y)|^p dy \right)^{1/p} \le C\|a\|_* \big(M(|f|^p)(x_0) \big)^{1/p}.$$

To estimate the sharp function of $B(x)$ we proceed analogously as we did for $\mathcal{K}_{sm}f$

$$G_2 = \frac{1}{|I|}\int_I |B(x) - B_I|dx \le \frac{2}{|I|}\int_I |B(x) - B(x_0)|dx.$$

The integrand above satisfies

$$\begin{aligned}
&|B(x) - B(x_0)| = |\mathcal{K}_{sm}(a - a_I)f\chi_{(2I)^c}(x) - \mathcal{K}_{sm}(a - a_I)f\chi_{(2I)^c}(x_0)| \\
&\le \int_{(2I)^c} |\mathcal{H}_{sm}(x - y) - \mathcal{H}_{sm}(x_0 - y)|\, |a(y) - a_I||f(y)|dy \\
&\le C(n)m^{(n+1)/2}\rho(x_0 - x)\int_{(2I)^c} \frac{|a(y) - a_I||f(y)|}{\rho(x_0 - y)^{n+3}}dy \\
&\le C(n)m^{(n+1)/2}r\left(\int_{(2I)^c} \frac{|f(y)|^p}{\rho(x_0 - y)^{n+3}}dy\right)^{1/p}\left(\int_{(2I)^c} \frac{|a(y) - a_I|^{p'}}{\rho(x_0 - y)^{n+3}}dy\right)^{1/p'},
\end{aligned}$$

$\frac{1}{p} + \frac{1}{p'} = 1$, as consequence from the Hörmander pointwise estimate since $x \in I$ and $y \in (2I)^c$. The first integral above is estimated directly

$$\begin{aligned}
\int_{(2I)^c} \frac{|f(y)|^p}{\rho(x_0 - y)^{n+3}}dy &= \sum_{k=1}^{\infty}\int_{2^{k+1}I\setminus 2^kI} \frac{|f(y)|^p}{\rho(x_0 - y)^{n+3}}dy \\
&\le \sum_{k=1}^{\infty} \frac{(2^{k+1}r)^{n+2}}{(2^kr)^{n+3}}\frac{1}{|2^{k+1}I|}\int_{2^{k+1}I} |f(y)|^p dy \le \frac{2^{n+2}}{r}M(|f|^p)(x_0),
\end{aligned}$$

while the second one is estimated by the help of Lemmas 2.10 and 2.11

$$\begin{aligned}
&\int_{(2I)^c} \frac{|a(y) - a_I|^{p'}}{\rho(x_0 - y)^{n+3}}dy = \sum_{k=1}^{\infty}\int_{2^{k+1}I\setminus 2^kI} \frac{|a(y) - a_I|^{p'}}{\rho(x_0 - y)^{n+3}}dy \\
&\le \sum_{k=1}^{\infty} \frac{1}{(2^kr)^{n+3}}\int_{2^{k+1}I} |a(y) - a_I|^{p'}dy \\
&\le \sum_{k=1}^{\infty} \frac{1}{(2^kr)^{n+3}}\left(\int_{2^{k+1}I} |a(y) - a_{2^{k+1}I}|^{p'}dy + \int_{2^{k+1}I} |a_{2^{k+1}I} - a_I|^{p'}dy\right) \\
&\le \frac{C(n,p')}{r}\|a\|_*^{p'}\sum_{k=1}^{\infty}\frac{(k+1)^{p'}}{2^k} \le \frac{C(n,p')}{r}\|a\|_*^{p'}.
\end{aligned}$$

Hence

$$G_2 \le C(n,p)m^{(n+1)/2}\|a\|_*\big(M(|f|^p)(x_0)\big)^{1/p}.$$

Finally,

$$\begin{aligned}
G_3 &= \frac{1}{|I|}\int_I |C(x) - C_I|dx \le \frac{2}{|I|}\int_I |a(x) - a_I||\mathcal{K}_{sm}f(x)|dx \\
&\le 2\left(\frac{1}{|I|}\int_I |a(x) - a_I|^{p'}dx\right)^{1/p'}\left(\frac{1}{|I|}\int_I |\mathcal{K}_{sm}f(x)|^p dx\right)^{1/p} \\
&\le C(p)\|a\|_*\big(M(|\mathcal{K}_{sm}f|^p)(x_0)\big)^{1/p}.
\end{aligned}$$

Combining G_1, G_2, G_3, taking the supremum with respect to I and having in mind that the point was chosen arbitrary we get (3.16). $\square$

The above lemma and the sharp inequality yield $L^{p,\lambda}$ estimates for the integral operator $\mathcal{K}_{sm}f$ and its commutator.

Lemma 3.4. *Let f, a, $\mathcal{K}_{sm}f$ and $\mathcal{C}_{sm}[a,f]$ be as above. Then*

$$\|\mathcal{K}_{sm}f\|_{p,\lambda} \leq Cm^{(n+1)/2}\|f\|_{p,\lambda} \tag{3.17}$$

$$\|\mathcal{C}_{sm}[a,f]\|_{p,\lambda} \leq Cm^{(n+1)/2}\|a\|_*\|f\|_{p,\lambda} \tag{3.18}$$

where the constants depend on n, p, λ but not on f.

Proof. We are going to estimate the $L^{p,\lambda}$ norms of the sharp functions of the corresponding operators in order to employ the sharp inequality (Lemma 2.9). Let we note that (3.15) holds true for any $q \in (1,p)$. Therefore, the maximal inequality (Lemma 2.8) asserts

$$\begin{aligned}\int_I |(\mathcal{K}_{sm}f)^\#(x)|^p dx &\leq Cm^{(n+1)p/2}\int_I |M(|f|^p)|^{p/q}(x)dx \\ &\leq Cm^{(n+1)p/2}r^\lambda\|M(|f|^q)\|^{p/q}_{p/q,\lambda}, \leq Cm^{(n+1)p/2}r^\lambda\||f|^q\|^{p/q}_{p/q,\lambda} \\ &\leq Cm^{(n+1)p/2}r^\lambda\|f\|^p_{p,\lambda}.\end{aligned}$$

Dividing of r^λ and taking the supremum with respect to r we get

$$\|(\mathcal{K}_{sm}f)^\#\|_{p,\lambda} \leq C(n,p)m^{(n+1)/2}\|f\|_{p,\lambda}$$

and the assertion follows from Lemma 2.9.

The $L^{p,\lambda}$ estimate for the commutator follows analogously. After using (3.16) for $q \in (1,p)$ and applying the maximal inequality we arrive to

$$\int_I |(\mathcal{C}_{sm}[a,f])^\#(x)|^p dx \leq C(n,p)\|a\|^p_* r^\lambda\Big\{\|\mathcal{K}_{sm}f\|^p_{p,\lambda} + m^{(n+1)p/2}\|f\|^p_{p,\lambda}\Big\}$$

and by the help of (3.17) we get

$$\|(\mathcal{C}_{sm}[a,f])^\#\|_{p,\lambda} \leq C(n,p)\|a\|_* m^{(n+1)/2}\|f\|_{p,\lambda}$$

which leads to the assertion after applying Lemma 2.9. □

Lemma 3.5. *Denote by $\mathcal{K}_{sm\varepsilon}f$ and $\mathcal{C}_{sm\varepsilon}[a,f]$ the nonsingular integral operators with constant PCZ kernel*

$$\mathcal{H}_{sm\varepsilon}(x-y) = \begin{cases}\mathcal{H}_{sm}(x-y) & \textit{for } \rho(x-y) \geq \varepsilon \\ 0 & \textit{for } \rho(x-y) < \varepsilon.\end{cases}$$

Then for any functions a and f as above, we have

$$\|\mathcal{K}_{sm\varepsilon}f\|_{p,\lambda} \leq C(n,p)m^{(n+1)/2}\|f\|_{p,\lambda} \tag{3.19}$$

$$\|\mathcal{C}_{sm\varepsilon}[a,f]\|_{p,\lambda} \leq C(n,p)m^{(n+1)/2}\|a\|_*\|f\|_{p,\lambda}. \tag{3.20}$$

Moreover

$$\lim_{\varepsilon\to 0}\|\mathcal{K}_{sm\varepsilon}f - \mathcal{K}_{sm}f\|_{p,\lambda} = \lim_{\varepsilon\to 0}\|\mathcal{C}_{sm\varepsilon}[a,f] - \mathcal{C}_{sm}[a,f]\|_{p,\lambda} = 0. \tag{3.21}$$

Proof. Fix an ellipsoid $\mathcal{E}_\varepsilon \equiv \mathcal{E}_\varepsilon(x_0) = \{y \in \mathbb{R}^{n+1} : \rho(x_0 - y) < \varepsilon\}$ and set for $\mathcal{E}_{\varepsilon/2}$ the ellipsoid centered at the same point with radius $\varepsilon/2$. Hence

$$\begin{aligned}\mathcal{K}_{sm\varepsilon} f(x_0) &= \frac{C}{\varepsilon^{n+2}} \int_{\mathcal{E}_{\varepsilon/2}} |\mathcal{K}_{sm\varepsilon} f(x_0)| dy \\ &\le \frac{C}{\varepsilon^{n+2}} \int_{\mathcal{E}_{\varepsilon/2}} |\mathcal{K}_{sm} f(y)| dy + \frac{C}{\varepsilon^{n+2}} \int_{\mathcal{E}_{\varepsilon/2}} |\mathcal{K}_{sm\varepsilon} f(x_0) - \mathcal{K}_{sm} f(y)| dy.\end{aligned}$$

The density f could be written as a sum of the kind

$$f(x) = f(x)\chi_{\mathcal{E}_\varepsilon}(x) + f(x)\chi_{\mathcal{E}_\varepsilon^c}(x) = f_1(x) + f_2(x)$$

which allows us to write $\mathcal{K}_{sm} f = \mathcal{K}_{sm} f_1 + \mathcal{K}_{sm} f_2$ and hence

$$\begin{aligned}\mathcal{K}_{sm\varepsilon} f(x_0) =& \frac{C}{\varepsilon^{n+2}} \int_{\mathcal{E}_{\varepsilon/2}} \big|\mathcal{K}_{sm} f_1(y)\big| dy \\ &+ \frac{C}{\varepsilon^{n+2}} \int_{\mathcal{E}_{\varepsilon/2}} \big|\mathcal{K}_{sm} f_2(y) - \mathcal{K}_{sm\varepsilon} f(x_0)\big| dy = E_1 + E_2.\end{aligned}$$

The first integral is analogous to J_1 from Lemma 3.3 and hence

$$E_1 \le C\big(M(|f|^q)(x_0)\big)^{1/q}$$

for any $q \in (1, p)$.

The second integral is analogous to J_2 and hence

$$E_2 \le C m^{(n+2)/2}\big(M(|f|^q)(x_0)\big)^{1/q}$$

for any $q \in (1, p)$.

Since the point x_0 was chosen arbitrary, the estimates hold true for any $x \in \mathbb{R}^{n+1}$. Using the same arguments as in the proof of Lemma 3.4 we get the desired estimates (3.19) and (3.20).

It is known from [15] and [4] that the limits

$$\lim_{\varepsilon\to 0} \mathcal{K}_{sm\varepsilon} f(x) = \mathcal{K}_{sm} f(x), \quad \lim_{\varepsilon\to 0} \mathcal{C}_{sm\varepsilon}[a, f](x) = \mathcal{C}_{sm}[a, f](x),$$

there exist in L^p. This allows us to assert that taking $\varepsilon \to 0$ in (3.19) and (3.20) we get exactly (3.17) and (3.18), respectively, and the assertion (3.21) follows. □

Now, after giving the proofs of several helpful results, we shall turn back to the proof of Theorem 3.1.

First of all, the spherical expansion of the kernel leads to expansions of the nonsingular integrals $\mathcal{K}_\varepsilon f$ and $\mathcal{C}_\varepsilon[a, f]$, that is

$$\mathcal{K}_\varepsilon f(x) = \sum_{s,m} b_{sm}(x)\mathcal{K}_{sm\varepsilon} f(x), \tag{3.22}$$

$$\mathcal{C}_\varepsilon[a, f](x) = \sum_{s,m} b_{sm}(x)\mathcal{C}_{sm\varepsilon}[a, f](x) \tag{3.23}$$

In Lemma 3.5 we show that $\mathcal{K}_{sm\varepsilon} f$ and $\mathcal{C}_{sm\varepsilon}[a, f]$ are bounded in $L^{p,\lambda}$ uniformly with respect to ε. Moreover, the series

$$\sum_{s,m} \|b_{sm}\mathcal{K}_{sm\varepsilon} f\|_{p,\lambda} \le C\|f\|_{p,\lambda} \sum_{m=1}^{\infty} m^{-2l+(n+1)/2+(n-1)},$$

$$\sum_{s,m} \|b_{sm}\mathcal{C}_{sm\varepsilon}[a,f]\|_{p,\lambda} \le C\|a\|_*\|f\|_{p,\lambda} \sum_{m=1}^{\infty} m^{-2l+(n+1)/2+(n-1)}$$

are totally convergent in $L^{p,\lambda}$, uniformly in ε for $l > (3n+1)/4$. Whence

$$\|\mathcal{K}_\varepsilon f\|_{p,\lambda} \le \|f\|_{p,\lambda}, \quad \|\mathcal{C}_\varepsilon[a,f]\|_{p,\lambda} \le C\|a\|_*\|f\|_{p,\lambda}$$

with $C = C(n,p,\lambda,k)$. Setting

$$\mathcal{K}f(x) = \sum_{s,m} b_{sm}(x)\mathcal{K}_{sm}f(x), \quad \mathcal{C}[a,f](x) = \sum_{s,m} b_{sm}(x)\mathcal{C}_{sm}[a,f](x),$$

we obtain through Lemma 3.4

$$\|\mathcal{K}f\|_{p,\lambda} \le C\|f\|_{p,\lambda}, \quad \|\mathcal{C}[a,f]\|_{p,\lambda} \le C\|a\|_*\|f\|_{p,\lambda}.$$

Finally, the dominated convergence theorem, applied in $L^{p,\lambda}$ to the series expansions (3.22) and (3.23) gives

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathcal{K}_\varepsilon f(x) &= \lim_{\varepsilon\to 0} \sum_{s,m} b_{sm}(x)\mathcal{K}_{sm\varepsilon}f(x) = \sum_{s,m} b_{sm}(x) \lim_{\varepsilon\to 0} \mathcal{K}_{sm\varepsilon}f(x)\\ &= \sum_{s,m} b_{sm}(x)\mathcal{K}_{sm}f(x) = \mathcal{K}f(x),\end{aligned}$$

$$\lim_{\varepsilon\to 0} \mathcal{C}_\varepsilon[a,f](x) = \sum_{s,m} b_{sm}(x) \lim_{\varepsilon\to 0} \mathcal{C}_{sm\varepsilon}[a,f](x) = \mathcal{C}[a,f](x).$$

This completes the proof of Theorem 3.1. □

Theorem 3.6. *Let Q be a cylinder in $\mathbb{R}^{n+1}$ and $k(x;y)$ be a variable PCZ kernel defined in Q. Let $f \in L^{p,\lambda}(Q)$, $1 < p < \infty$, $0 < \lambda < n+2$ and $a \in BMO(Q)$. Then $\mathcal{K}f$ and $\mathcal{C}[a,f]$ belong to $L^{p,\lambda}(Q)$ and*

$$\lim_{\varepsilon\to 0} \|\mathcal{K}_\varepsilon f - \mathcal{K}f\|_{p,\lambda;Q} = 0 \tag{3.24}$$

$$\lim_{\varepsilon\to 0} \|\mathcal{C}_\varepsilon[a,f] - \mathcal{C}[a,f]\|_{p,\lambda;Q} = 0 \tag{3.25}$$

uniformly with respect to ε. Moreover, the following estimates hold true

$$\|\mathcal{K}f\|_{p,\lambda;Q} \le C\|f\|_{p,\lambda;Q} \tag{3.26}$$

$$\|\mathcal{C}[a,f]\|_{p,\lambda;Q} \le C\|a\|_*\|f\|_{p,\lambda;Q} \tag{3.27}$$

where the constants depend on n, p, λ and the kernel k.

Proof. Define the functions

$$\bar{k}(x;y) = \begin{cases} k(x;y) & \text{for } x \in Q,\ y \in \mathbb{R}^{n+1} \setminus \{0\},\\ 0 & \text{elsewhere,}\end{cases}$$

$$\bar{f}(x) = \begin{cases} f(x) & \text{for } x \in Q,\\ 0 & \text{for } x \notin Q.\end{cases}$$

Define the operator $\bar{\mathcal{K}}\bar{f}(x) = \int_{\mathbb{R}^{n+1}} \bar{k}(x,x-y)\bar{f}(y)dy$. So we can consider $\mathcal{K}f$ as a restriction of $\bar{\mathcal{K}}\bar{f}$ on Q which means

$$\|\mathcal{K}_\varepsilon f - \mathcal{K}f\|_{p,\lambda;Q} \le \|\bar{\mathcal{K}}_\varepsilon \bar{f} - \bar{\mathcal{K}}\bar{f}\|_{p,\lambda}.$$

Then (3.24) follows from (3.6).

The extension theorems for BMO functions (cf. [19], [1]) allow to define $\bar{a} \in BMO(\mathbb{R}^{n+1})$ such that $\bar{a}|_Q = a$. Arguments similar to those already applied for $\mathcal{K}f$ lead to (3.25). Finally

$$\|\mathcal{K}f\|_{p,\lambda;Q} \le \|\bar{\mathcal{K}}\bar{f}\|_{p,\lambda} \le C\|\bar{f}\|_{p,\lambda} = C\|f\|_{p,\lambda;Q}.$$

The estimate (3.27) follows in the same manner. □

Theorem 3.7. *Let $k(x;y)$ be a variable PCZ kernel and $a \in VMO \cap L^\infty(Q)$ with VMO modulus $\gamma_a(r)$ defined by (2.8). Then for any $\varepsilon > 0$ there exists a positive number $r_0 = r_0(\varepsilon, \gamma_a)$ such that for any $r \in (0, r_0)$ and any parabolic cylinder $I_r \subset Q$ one has*

$$\|\mathcal{C}[a, f]\|_{p,\lambda;I_r} \le C\varepsilon\|f\|_{p,\lambda;I_r} \tag{3.28}$$

for any function $f \in L^{p,\lambda}(I_r)$.

Proof. As it was pointed out above, by the equivalence of the topologies induced by the standard parabolic metric and the metric (2.6), we may prove the theorem in ellipsoids instead of parabolic cylinders. For this goal we consider $\mathcal{E}_r$ centered at x_0 and of radius r

$$\mathcal{E}_r = \Big\{x \in \mathbb{R}^{n+1} : \frac{(x' - x_0')^2}{r^2} + \frac{(t-t_0)^2}{r^4} < 1\Big\} \text{ and } \quad \mathcal{E}_r^c = \mathbb{R}^{n+1} \setminus \mathcal{E}_r.$$

From the properties of BMO and VMO functions (see Theorem 2.5), it follows that for any $\varepsilon > 0$ there exists a number $r_0(\varepsilon, \gamma_a)$ and a continuous and uniformly bounded function g with modulus of continuity $\omega_g(r_0) < \varepsilon/2$ and $\|a - g\|_* < \varepsilon/2$. Fix an ellipsoid $\mathcal{E}_r \subset Q$, such that $r \in (0, r_0)$ and construct a function

$$h(x) = \begin{cases} g(x) & \text{for } x \in \mathcal{E}_r, \\ g\Big(x_0' + r\frac{x'-x_0'}{\rho(x-x_0)}, t_0 + r^2\frac{t-t_0}{\rho(x_0-x)^2}\Big) & \text{for } x \in \mathcal{E}_r^c \end{cases}$$

which is uniformly continuous in $\mathbb{R}^{n+1}$ and its oscillation in $\mathcal{E}_r^c$ equals the oscillation of g over the surface $\partial\mathcal{E}_r$. Then the oscillation of h in $\mathbb{R}^{n+1}$ is no greater than the oscillation of g in $\mathcal{E}_{r_0}$. Then

$$\|\mathcal{C}[a, f]\|_{p,\lambda;\mathcal{E}_r} \le \|\mathcal{C}[a - g, f]\|_{p,\lambda;\mathcal{E}_r} + \|\mathcal{C}[g, f]\|_{p,\lambda;\mathcal{E}_r}.$$

The first norm could be estimated according to (3.27). For the second one we employ the constructed above function. Whence

$$\begin{aligned} \|\mathcal{C}[a, f]\|_{p,\lambda;\mathcal{E}_r} &\le C\left(\|a - g\|_* + \|h\|_*\right)\|f\|_{p,\lambda;\mathcal{E}_r} \\ &\le C\left(\|a - g\|_* + \omega_g(r_0)\right)\|f\|_{p,\lambda;\mathcal{E}_r} \le C\varepsilon\|f\|_{p,\lambda;\mathcal{E}_r}. \end{aligned}$$

4. Nonsingular integral estimates in Morrey spaces

Suppose now that the coefficients of the operator $\mathcal{P}$ are defined in $\mathbb{D}_+^{n+1}$ and construct a *generalized reflection* T in the next manner. Denote by $\mathbf{a}^n(y)$ the last row of the matrix $\mathbf{a} = \{a^{ij}\}$ and define

$$T(x', t; y', t) = x' - 2x_n\frac{\mathbf{a}^n(y', t)}{a^{nn}(y', t)}, \quad T(x) = T(x', t; x', t), \tag{4.1}$$

for any $x', y' \in \mathbb{R}_+^n$ and any fixed $t \in \mathbb{R}_+$. Obviously, T maps $\mathbb{R}_+^n$ into $\mathbb{R}_-^n$ and $k(x, T(x) - y)$ turns out to be nonsingular kernel for any $x, y \in \mathbb{D}_+^{n+1}$. The following $L^{p,\lambda}$ estimates concerns integrals with kernels like that.

Theorem 4.1. *Let $f \in L^{p,\lambda}(\mathbb{D}_+^{n+1})$, $a \in L^\infty(\mathbb{D}_+^{n+1})$ and*

$$\widetilde{\mathcal{K}}f(x) = \int_{\mathbb{D}_+^{n+1}} k(x, T(x)-y) f(y)dy$$

$$\widetilde{\mathcal{C}}[a,f](x) = \int_{\mathbb{D}_+^{n+1}} k(x, T(x)-y)[a(y)-a(x)]f(y)dy$$

be integral operators with nonsingular kernels. Then there exist constants depending on n, p and λ, such that

$$\|\widetilde{\mathcal{K}}f\|_{p,\lambda;\mathbb{D}_+^{n+1}} \le C\|f\|_{p,\lambda;\mathbb{D}_+^{n+1}} \tag{4.2}$$

$$\|\widetilde{\mathcal{C}}[a,f]\|_{p,\lambda;\mathbb{D}_+^{n+1}} \le C\|a\|_*\|f\|_{p,\lambda;\mathbb{D}_+^{n+1}}. \tag{4.3}$$

Proof. For all $x = (x_1, \dots, x_n, t), x_n > 0$ we define $\tilde{x} = (x_1, \dots, -x_n, t)$ for any $t \in \mathbb{R}_+$. Then there exist two positive constants C_1 and C_2 depending on n and Λ, such that

$$C_1\rho(\tilde{x}-y) \le \rho(T(x)-y) \le C_2\rho(\tilde{x}-y)$$

for every $x, y \in \mathbb{D}_+^{n+1}$ (cf. [10], [4]). We consider again the expansion in spherical harmonics of $k(x; T(x)-y)$. For the numbers s and m as in (3.3) and (3.4), we have

$$k(x, T(x)-y) = \sum_{s,m} b_{sm}(x)\frac{Y_{sm}(\overline{T(x)-y})}{\rho(T(x)-y)^{n+2}} = \sum_{s,m} b_{sm}(x)\mathcal{H}_{sm}(T(x)-y).$$

Hence

$$\widetilde{\mathcal{K}}f(x) = \sum_{s,m} b_{sm}(x)\int_{\mathbb{D}_+^{n+1}} \mathcal{H}_{sm}(T(x)-y)f(y)dy = \sum_{s,m} b_{sm}(x)\widetilde{\mathcal{K}}_{sm}f(x).$$

¿From the properties of the spherical harmonics (3.3) and (3.4), it follows

$$|\mathcal{H}_{sm}(T(x)-y)| \le C\frac{m^{(n-1)/2}}{\rho(T(x)-y)^{n+2}} \le C\frac{m^{(n-1)/2}}{\rho(\tilde{x}-y)^{n+2}}.$$

Consider the operator

$$Rf(x) = \int_{\mathbb{D}_+^{n+1}} \frac{f(y)}{\rho(\tilde{x}-y)^{n+2}}dy \tag{4.4}$$

for which $|\widetilde{\mathcal{K}}_{sm}f| \le C(n,\Lambda)m^{(n-1)/2}|Rf|$. We choose $x_0 = (x'', 0, t), x'' \in \mathbb{R}^{n-1}, t \in \mathbb{R}_+$ and consider a cylinder $I = I_r(x_0)$ centered at x_0 with radius r. As usual 2^kI means $I_{2^kr}(x_0)$ for any integer k, $I_+ = I \cap \{x_n > 0, t > 0\}$ and $I_- = I \cap \{x_n < 0, t > 0\}$. Then we can write $f(x)$ as

$$f(x) = f(x)\chi_{2I_+}(x) + \sum_{k=1}^{\infty} f(x)\chi_{2^{k+1}I_+\setminus 2^kI_+}(x) = \sum_{k=0}^{\infty} f_k(x).$$

It follows from [4, Lemma 3.3] that,

$$\begin{aligned}\|Rf_0\|_{p,I_+}^p &\le \|Rf_0\|_{p,\mathbb{D}_+^{n+1}}^p \le C(n,p)\|f_0\|_{p,\mathbb{D}_+^{n+1}}^p \\ &= C\int_{2I_+} |f(y)|^p dy \le C(n,p,\lambda)r^\lambda\|f\|_{p,\lambda;\mathbb{D}_+^{n+1}}^p.\end{aligned}$$

Later on,

$$|Rf_k(x)| \le \int_{\mathbb{D}_+^{n+1}} \frac{|f_k(y)|}{\rho(\tilde{x}-y)^{n+2}} dy$$

where $\rho(\tilde{x}-y) \ge \rho(x-y) \ge r(2^k-1) \ge 2^{k-1}r$ since $x \in I_+$, $\tilde{x} \in I_-$ and $y \in 2^{k+1}I_+ \setminus 2^k I_+$. Hence

$$\begin{aligned} |Rf_k(x)|^p &\le \left(\int_{\mathbb{D}_+^{n+1}} \frac{|f(y)|\chi_{2^{k+1}I_+\setminus 2^k I_+}(y)}{\rho(\tilde{x}-y)^{n+2}} dy \right)^p \\ &\le \frac{1}{(2^{k-1}r)^{p(n+2)}} \left(\int_{2^{k+1}I_+} |f(y)| dy \right)^p \\ &\le \frac{1}{(2^{k-1}r)^{p(n+2)}} \left(\int_{2^{k+1}I_+} 1 dy \right)^{p/p'} \left(\int_{2^{k+1}I_+} |f(y)|^p dy \right) \\ &\le C(n,p,\lambda) 2^{k(\lambda-(n+2))} r^{\lambda-(n+2)} \|f\|^p_{p,\lambda;\mathbb{D}_+^{n+1}} \end{aligned}$$

and

$$\int_{I_+} |Rf(y)|^p dy \le C r^\lambda \sum_{k=0}^{\infty} 2^{k(\lambda-(n+2))} \|f\|^p_{p,\lambda;\mathbb{D}_+^{n+1}}.$$

The series above is convergent since $\lambda < n+2$ and therefore

$$\|Rf\|_{p,\lambda;\mathbb{D}_+^{n+1}} \le C(n,p,\lambda) \|f\|_{p,\lambda;\mathbb{D}_+^{n+1}}. \tag{4.5}$$

¿From the expansion of $\widetilde{\mathcal{K}}f$ it follows

$$\begin{aligned} \|\widetilde{\mathcal{K}}f\|_{p,\lambda;\mathbb{D}_+^{n+1}} &\le \sum_{s,m} \|b_{sm}\|_\infty \|\widetilde{\mathcal{K}}_{sm} f\|_{p,\lambda;\mathbb{D}_+^{n+1}} \\ &\le C(n,p,\lambda,\Lambda) \|f\|_{p,\lambda;\mathbb{D}_+^{n+1}} \sum_{m=1}^{\infty} m^{-2l+(n-1)/2+(n-1)} \end{aligned}$$

for any $l > 0$. So the series converges for a suitable choices of l which proves (4.2).

The commutator could be written as a sum of integral operators analogously as it was already done for $\widetilde{\mathcal{K}}f$

$$\widetilde{\mathcal{C}}[a,f](x) = \sum_{s,m} b_{sm}(x) \widetilde{\mathcal{C}}_{sm}[a,f](x).$$

Moreover, using similar arguments as those for $\widetilde{\mathcal{K}}f$, we get

$$|\widetilde{\mathcal{C}}_{sm}[a,f]| \le C m^{(n-1)/2} \int_{\mathbb{D}_+^{n+1}} \frac{|a(y)-a(x)|}{\rho(\widetilde{x}-y)^{n+2}} |f(y)| dy.$$

Denote by

$$R_a f(x) = \int_{\mathbb{D}_+^{n+1}} \frac{|a(y)-a(x)|}{\rho(\tilde{x}-y)^{n+2}} |f(y)| dy.$$

From [3, Theorem 2.1] we have that for any $q \in (1,p)$ there exists a constant $C(q)$ such that

$$|(R_a f)^\#(x_0)| \le C \|a\|_* \left\{ \left(M(R|f|)^q(x_0) \right)^{1/q} + \left(M(|f|^q)(x_0) \right)^{1/q} \right\}$$

for every $x_0 \in \mathbb{D}_+^{n+1}$. Let $x_0 = (x'', 0, 0)$, $r > 0$, $f \in L^{p,\lambda}$. Hence

$$\begin{aligned}\int_{I_+} |(R_a f)^\#(y)|^p dy \le & C\|a\|_*^p \Big\{ \int_{I_+} \big(M(R|f|)^q(y)\big)^{p/q} dy \\ & + \int_{I_+} \big(M(|f|^q)(y)\big)^{p/q} dy \Big\} = C\|a\|_*^p \{J_1 + J_2\}.\end{aligned}$$

It is easy to see that

$$\begin{aligned}(R|f|(x))^q &= \left(\int_{\mathbb{D}_+^{n+1}} \frac{|f(y)|}{\rho(\tilde{x}-y)^{n+2}} dy \right)^q \\ &\le \left(\int_{\mathbb{D}_+^{n+1}} \frac{dy}{\rho(\tilde{x}-y)^{n+2}} \right)^{q/q'} \left(\int_{\mathbb{D}_+^{n+1}} \frac{|f(y)|^q}{\rho(\tilde{x}-y)^{n+2}} dy \right) \le CR(|f|^q)(x),\end{aligned}$$

whence

$$\begin{aligned}J_1 &\le \int_{I_+} |M(R(|f|^q))(x)|^{p/q}\, dx \le r^\lambda \|M(R(|f|^q))\|_{p/q,\lambda;\mathbb{D}_+^{n+1}}^{p/q} \\ &\le r^\lambda \|R(|f|^q)\|_{p/q,\lambda;\mathbb{D}_+^{n+1}}^{p/q} \\ &\le r^\lambda \||f|^q\|_{p/q,\lambda;\mathbb{D}_+^{n+1}}^{p/q} \le r^\lambda \|f\|_{p,\lambda;\mathbb{D}_+^{n+1}}^p\end{aligned}$$

as follows from Lemma 2.8 and (4.5). Analogous arguments allow us to estimate J_2. Using the sharp inequality, we get

$$\|\widetilde{\mathcal{C}}_{sm}[a,f]\|_{p,\lambda;\mathbb{D}_+^{n+1}} \le Cm^{(n-1)/2}\|a\|_*\|f\|_{p,\lambda;\mathbb{D}_+^{n+1}}.$$

The representation of the commutator $\widetilde{\mathcal{C}}[a,f]$ as Fourier series gives

$$\|\widetilde{\mathcal{C}}[a,f]\|_{p,\lambda;\mathbb{D}_+^{n+1}} \le C\|a\|_*\|f\|_{p,\lambda;\mathbb{D}_+^{n+1}} \sum_{m=1}^{\infty} m^{(n-1)/2-2l+(n-1)}$$

and the series converges for $l > (3n-1)/4$ which proves (4.3). □

Corollary 4.2. *Let I_r be a parabolic cylinder in $\mathbb{R}_+^{n+1}$, $a \in VMO \cap L^\infty(I_r)$ with VMO modulus $\gamma_a(r)$. Then for every $\varepsilon > 0$ there exists a positive number $r_0(\varepsilon, \gamma_a)$, such that for every $f \in L^{p,\lambda}(I_r)$, $r < r_0$ is fulfilled*

$$\|\widetilde{\mathcal{C}}[a,f]\|_{p,\lambda;I_r} \le C(p,\lambda,\Lambda)\varepsilon\|f\|_{p,\lambda;I_r}.$$

The proof is analogous to that of Theorem 3.7.

5. A priori estimates, existence and uniqueness

Theorem 5.1. *Suppose $a^{ij} \in VMO(Q_T)$ and conditions (2.2), (2.3) to be fulfilled. Then for every $f \in L^{p,\lambda}(Q_T)$, $1 < p < \infty$, $0 < \lambda < n+2$, the problem (2.1) has a unique solution $u \in W_{p,\lambda}^{2,1}(Q_T)$. Moreover, it satisfies*

$$\|u\|_{W_{p,\lambda}^{2,1}(Q_T)} \le C\|f\|_{p,\lambda;Q_T} \tag{5.1}$$

where the constant depends on $n, p, \lambda, \Lambda, T, \partial\Omega$ and the VMO-moduli of a^{ij}.

Proof. We begin with the establishment of the a priori estimate (5.1). Let $u \in W_p^{2,1}(Q_T)$ be a solution of (2.1). Its existence follows from [29] having in mind that $L^{p,\lambda}(Q_T)$ is a subspace of $L^p(Q_T)$ for every $p \in (1,\infty)$. To estimate the $L^{p,\lambda}$ norms of the derivatives $D_{ij}u$ of this solution we use their representation inside the cylinder (cf. [4]) and near the boundary (cf. [29]).

Step 1: Interior estimate. By density arguments, we consider $u \in C_0^\infty(\mathbb{R}^{n+1}_+)$ and $u(x',0)=0$. For $x \in \operatorname{supp} u$ the following *interior representation formula* holds (cf. [4])

$$\begin{aligned} D_{ij}u(x) =& P.V.\int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x;x-y)\left\{\left(a^{hk}(y)-a^{hk}(x)\right)D_{hk}u(y)+f(y)\right\}dy \\ &+ f(x)\int_{\Sigma_{n+1}} \Gamma_j(x;y)n_i d\sigma_y, \end{aligned} \tag{5.2}$$

where Γ_{ij} are the derivatives of the fundamental solution (2.5) with respect to the second variable, and n_i is the $i-th$ component of the outer normal of the surface Σ_{n+1}.

As it is shown in [15], Γ^0_{ij} is a constant PCZ kernel. Later on, from the boundedness of the fundamental solution (cf. [21])

$$\sup_{y\in\Sigma_{n+1}}\left|\left(\frac{\partial}{\partial y}\right)^\beta \Gamma(x;y)\right| \le C(\beta,\Lambda)$$

it follows that $\Gamma_{ij}(x;y)$ is a variable PCZ kernel. Define

$$\begin{aligned} \mathcal{K}_{ij}f(x) &= P.V.\int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x;x-y)f(y)dy, \\ \mathcal{C}_{ij}[a,f](x) &= P.V.\int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x;x-y)[a(y)-a(x)]f(y)dy \\ &= \mathcal{K}_{ij}(af)(x) - a(x)(\mathcal{K}_{ij}f)(x). \end{aligned}$$

Hence for $x \in \operatorname{supp} u$

$$D_{ij}u(x) = \sum_{h,k=1}^n \mathcal{C}_{ij}[a^{hk}, D_{hk}](x) + \mathcal{K}_{ij}f(x) + f(x)\int_{\Sigma_{n+1}} \Gamma_j(x;y)n_i d\sigma_y.$$

Consider parabolic cylinder I with radius r. From Theorems 3.1 and 3.7 it follows

$$\|D^2u\|_{p,\lambda;I} \le C\left(\gamma_a(r_\alpha)\|D^2u\|_{p,\lambda;I} + \|f\|_{p,\lambda;I}\right). \tag{5.3}$$

Choosing r smaller, if necessary, such that $C\gamma_a(r)<1$, we get

$$\|D^2u\|_{p,\lambda;I} \le C\|f\|_{p,\lambda;I} \le C\|f\|_{p,\lambda;Q_T},$$

where the constant depends on $n,p,\lambda,\gamma_a(r),\|D\Gamma\|_\infty$. To estimate u_t we employ the equation

$$u_t = a^{ij}(x)D_{ij}u + f(x)$$

and the boundedness of the coefficients. Hence,

$$\begin{aligned} \|u_t\|_{p,\lambda;I} &\le C\|a\|_\infty\|D^2u\|_{p,\lambda;I} + \|f\|_{p,\lambda;I} \\ &\le C\|f\|_{p,\lambda;Q_T}, \end{aligned}$$

where $C = C(n,p,\lambda,\Lambda,\|D\Gamma\|_\infty,\gamma_a(r),\|a\|_{\infty,Q_T})$, where $\|a\|_{\infty,Q_T} = \max\|a^{ij}\|_{\infty,Q_T}$ where the maximum is taken over $i,j=1,\dots,n$.

The estimate of the solution follows from the representation $u(x) = \int_0^t u_s(x', s)ds$ and the Jensen inequality, which give

$$\|u\|_{p,\lambda;I} \le Cr^2 \|f\|_{p,\lambda;Q_T}$$

where the constant depends on the same quantities. Combining the estimates above we get that for any parabolic cylinder I, such that $I \cap S_T = \emptyset$, we have

$$\|u\|_{W^{2,1}_{p,\lambda}(I)} \le C\|f\|_{p,\lambda;Q_T} \tag{5.4}$$

Considering a cylinder $Q' = \Omega' \times (0, T)$ with $\Omega' \Subset \Omega$, making a covering of Q' by parabolic cylinders I_α, $\alpha \in \mathcal{A}$, considering a partition of the unit subordinated to this covering, applying (5.4) for each I_α and using the interpolation inequality to lower order terms we get

$$\|u\|_{W^{2,1}_{p,\lambda}(Q')} \le C\big(\|f\|_{p,\lambda;Q_T} + +\|u\|_{p,\lambda;Q''}\big), \tag{5.5}$$

where $Q'' = \Omega'' \times (0, T)$ and $\Omega' \Subset \Omega'' \Subset \Omega$ and the constant depends on n, p, λ, Λ, T, $\|D\Gamma\|_\infty$, $\gamma_a(r)$, $\|a\|_{\infty,Q_T}$.

Step 2: Boundary estimate. Let us suppose now $u \in C_0^\infty(\mathbb{D}^{n+1}_+)$. Define the semycilinder

$$B_+ = \{x \in \mathbb{D}^{n+1}_+ : |x'| < R,\ x_n > 0,\ 0 < t < R^2\}$$

with a base $\Omega_+ = \{|x'| < R,\ x_n > 0\}$ and $S_+ = \{|x''| < R,\ x_n = 0, 0 < t < R^2\}$. Consider the problem

$$\begin{aligned} \mathcal{P}u &\equiv u_t - a^{ij}(x)D_{ij}u = f(x) \quad \text{a.e. in } B_+,\\ \mathcal{I}u &\equiv u(x', 0) = 0 \text{on } \Omega_+,\\ \mathcal{B}u &\equiv \ell^i(x)D_iu = 0 \text{on } S_+. \end{aligned} \tag{5.6}$$

Than we have the following *boundary representation formula* for the second spatial derivatives of the solution of (5.6) (cf. [29])

$$D_{ij}u(x) = I_{ij}(x) - J_{ij}(x) + H_{ij}(x)$$

where

$$I_{ij}(x) = P.V. \int_{B_+} \Gamma_{ij}(x; x-y)F(x;y)dy + f(x) \int_{\Sigma_{n+1}} \Gamma_j(x;y)n_i d\sigma_y,$$
$$i, j = 1, \dots, n;$$

$$J_{ij}(x) = \int_{B_+} \Gamma_{ij}(x; T(x) - y', t-\tau)F(x;y)dy,\ i,j = 1, \dots, n-1;$$

$$J_{in}(x) = J_{ni}(x) = \int_{B_+} \sum_{l=1}^n \Gamma_{il}(x; T(x) - y', t-\tau) \left(\frac{\partial T(x)}{\partial x_n}\right)^l F(x;y)dy$$
$$i = 1, \dots, n-1;$$

$$J_{nn}(x) = \int_{B_+} \sum_{l,s=1}^n \Gamma_{ls}(x; T(x) - y', t-\tau) \left(\frac{\partial T(x)}{\partial x_n}\right)^l \left(\frac{\partial T(x)}{\partial x_n}\right)^s F(x;y)dy;$$

$$\begin{aligned} H_{ij}(x) =& P.V. \int_{S_+} G_{ij}(x; x'' - y'', x_n, t-\tau) g(y'', \tau) dy'' d\tau \\ &+ g(x'', t) \int_{\Sigma_n} G_j(x; y'', x_n, \tau) n_i d\sigma_{(y'',\tau)}. \end{aligned}$$

In the above expressions $T(x)$ is given by (4.1) and

$$\frac{\partial T(x)}{\partial x_n} = \left(-2\frac{a^{n1}(x)}{a^{nn}(x)}, \dots, -2\frac{a^{nn-1}(x)}{a^{nn}(x)}, -1 \right),$$

$$g(y'',\tau) = \Big[\ell^k(0) - \ell^k(y'',\tau)\Big] D_k u(y'',\tau) - \ell^k(0)\Big[(\Gamma_k * F)\Big|_{y_n=0}\Big](y'',\tau),$$

$$F(x;y) = f(y) + [a^{hk}(y) - a^{hk}(x)]D_{hk}u(y)$$

and $G = \Gamma\mathcal{Q}$ where $\mathcal{Q}$ is a regular bounded function.

We will use in the sequel the following notations

$$\widetilde{\mathcal{K}}_{ij} f(x) = \int_{\mathbb{D}^{n+1}_+} \Gamma_{ij}(x; T(x) - y', t-\tau) f(y) dy$$

$$\widetilde{\mathcal{C}}_{ij}[a,f](x) = \int_{\mathbb{D}^{n+1}_+} \Gamma_{ij}(x; T(x) - y', t-\tau)[a(y) - a(x)] f(y) dy.$$

Hence

$$\begin{aligned} I_{ij}(x) &= \mathcal{K}_{ij} f(x) + \mathcal{C}_{ij}[a^{hk}, D_{hk}u](x), \quad i,j = 1,\dots,n; \\ J_{ij}(x) &= \widetilde{\mathcal{K}}_{ij} f(x) + \widetilde{\mathcal{C}}_{ij}[a^{hk}, D_{hk}u](x), \quad i,j = 1,\dots,n. \end{aligned}$$

Note that the components of the vector $\frac{\partial T(x)}{\partial x_n}$ are bounded so the integrals J_{in} and J_{nn} can be presented as a sum of nonsingular integral operators, exactly as J_{ij}, $i,j \neq n$. ¿From Theorems 3.1 and 4.1 it follows

$$\begin{aligned} \|I_{ij}\|_{p,\lambda;B_+} &\le C\left(\|f\|_{p,\lambda;B_+} + \gamma_a(R)\|D^2u\|_{p,\lambda;B_+}\right), \\ \|J_{ij}\|_{p,\lambda;B_+} &\le C\left(\|f\|_{p,\lambda;B_+} + \gamma_a(R)\|D^2u\|_{p,\lambda;B_+}\right). \end{aligned} \tag{5.7}$$

To estimate the $L^{p,\lambda}$ norm of H_{ij} we suppose that the vector field ℓ is extended in B_+ preserving its Lipschitz regularity. This automatically leads also to extension in B_+ of the function g

$$g(x) = \Big[\ell^k(0) - \ell^k(x)\Big] D_k u(x) - \ell^k(0)(\Gamma_k * F)(x). \tag{5.8}$$

Moreover, since G is a product of the fundamental solution Γ and a regular function $\mathcal{Q}$, the derivatives G_{ij} possess properties similar to that of Γ_{ij}. Now using [29, Theorem 1] we write

$$G_{ij} *_2 g = P.V. \int_{S_+} G_{ij}(x; x'' - y'', x_n, t-\tau) g(y'',\tau) dy'' d\tau$$

$$\begin{aligned} \int_{B_+\cap I} |G_{ij} *_2 g|^p dx &\le C\Big(\int_{B_+\cap I} |g|^p dx + \int_{B_+\cap I} |Dg|^p dx\Big) \\ &\le Cr^\lambda\Big(\frac{1}{r^\lambda}\int_{B_+\cap I} |g|^p dx + \frac{1}{r^\lambda}\int_{B_+\cap I} |Dg|^p dx\Big), \end{aligned}$$

where I is a parabolic cylinder with radius r. Taking the supremum with respect to r we get

$$\|G_{ij} *_2 g\|_{p,\lambda;B_+} \le C\left(\|g\|_{p,\lambda;B_+} + \|Dg\|_{p,\lambda;B_+}\right).$$

The second integral in H_{ij} is a product of g and bounded surface integral, hence

$$\|H_{ij}\|^p_{p,B_+\cap I} \le Cr^\lambda \left(\frac{1}{r^\lambda}\|g\|^p_{p,B_+\cap I} + \frac{1}{r^\lambda}\|Dg\|^p_{p,B_+\cap I}\right).$$

Taking the supremum with respect to r we get

$$\|H_{ij}\|_{p,\lambda;B_+} \le C\big(\|g\|_{p,\lambda;B_+} + \|Dg\|_{p,\lambda;B_+}\big).$$

An immediate consequence of (5.8) is the bound

$$\|g\|_{p,\lambda;B_+} \le \|[\ell^k(0) - \ell^k(y)]D_k u(y)\|_{p,\lambda;B_+} + C\|\Gamma_k * F\|_{p,\lambda;B_+}.$$

Denoting by $\|\ell\|_{\mathrm{Lip}(S_T)}$ the Lipschitz constant of ℓ, we have

$$\|[\ell^k(0) - \ell^k(y)]D_k u(y)\|_{p,\lambda;B_+} \le CR^2\|\ell\|_{\mathrm{Lip}(S_T)}\|Du\|_{p,\lambda;B_+}.$$

Later,

$$\|\Gamma_k * F\|_{p,\lambda;B_+} \le \|\Gamma_k * f\|_{p,\lambda;B_+} + \|\Gamma_k * [a^{hk}(\cdot) - a_{hk}(x)]D_{hk}u(\cdot)\|_{p,\lambda;B_+}.$$

The convolution $\Gamma_k * f$ can be considered as Riesz potential [16, Lemma 7.12] and the estimate is achieved as in [29, Theorem 1]

$$\int_{B_+\cap I} |\Gamma_k * f|^p dx \le CR^p \int_{B_+\cap I} |f|^p dx \le CR^p r^\lambda \|f\|_{p,\lambda;B_+}.$$

Taking again the supremum with respect to r, we get

$$\|\Gamma_k * f\|_{p,\lambda;B_+} \le CR\|f\|_{p,\lambda;B_+}.$$

It is known from the properties of the fundamental solution that $\Gamma_k \in L^1_{\mathrm{loc}}$ and it behaves like $\rho(x)^{-(n+1)}$. Multiplying and dividing by $\rho(x-y)$ we can apply the theorems for integral operators with singular kernels. Note that $\rho(x-y)^{-(n+2)}$ is a non-negative measurable function and we can apply [3, Theorem 0.1]. By the same technique as above, one gets

$$|\Gamma_k * [a^{hk}(\cdot) - a^{hk}(x)]D_{hk}u(\cdot)| \le CR\int_{B_+} \frac{|a^{hk}(y) - a^{hk}(x)||D_{hk}u(y)|}{\rho(x-y)^{n+2}}dy,$$

$$\|\Gamma_k * F\|_{p,\lambda;B_+} \le CR\left(\|f\|_{p,\lambda;B_+} + \gamma_a(R)\|D^2u\|_{p,\lambda;B_+}\right),$$

$$\|g\|_{p,\lambda;B_+} \le C\big(R^2\|Du\|_{p,\lambda;B_+} + R\|f\|_{p,\lambda;B_+} + R\gamma_a(R)\|D^2u\|_{p,\lambda;B_+}\big).$$

Further, the Rademacher theorem asserts existence almost everywhere of the derivatives $D_h\ell^k \in L^\infty(Q_T)$. Thus,

$$D_h g(x) = -D_h\ell^k(x)D_k u(x) + [\ell^k(0) - \ell^k(x)]D_{kh}u - \ell^k(0)(\Gamma_{kh} * F).$$

The L^p norm of the last term is estimated according to Theorem 3.1 while the others two are treated as above. Hence

$$\|Dg\|_{p,\lambda;B_+} \le C\left(\|Du\|_{p,\lambda;B_+} + R^2\|D^2u\|_{p,B_+} + \|f\|_{p,\lambda;B_+} + \gamma_a(R)\|D^2u\|_{p,\lambda;B_+}\right).$$

Finally, applying the Gagliardo-Nirenberg interpolation inequality to $\|Du\|_{p,B_+}$, we obtain

$$\|Du\|_{p,\lambda;B_+} \le C\left(\frac{1}{\varepsilon}\|u\|_{p,\lambda;B_+} + \varepsilon\|D^2u\|_{p,\lambda;B_+}\right).$$

Choosing $\varepsilon = R(R+1)$ for R small enough we get

$$\|H_{ij}\|_{p,\lambda;B_+} \le C\left(\frac{1}{R}\|u\|_{p,\lambda;B_+} + \|f\|_{p,\lambda;B_+} + (R + \gamma_a(R))\|D^2u\|_{p,\lambda;B_+}\right).$$

Combining the last inequality with (5.7), we get

$$\|D^2u\|_{p,\lambda;B_+} \le C\left(\frac{1}{R}\|u\|_{p,\lambda;B_+} + \|f\|_{p,\lambda;B_+} + \big(R + \gamma_a(R)\big)\|D^2u\|_{p,\lambda;B_+}\right),$$

whence, taking R small enough (recall $\gamma_a(R) \to 0$ as $R \to 0$), one obtains

$$\|D^2u\|_{p,\lambda;B_+} \le C\left(\|f\|_{p,\lambda;B_+} + \frac{1}{R}\|u\|_{p,\lambda;B_+}\right).$$

Expressing u_t from the equation we get

$$\|u_t\|_{p,\lambda;B_+} \le C\left(\|f\|_{p,\lambda;B_+} + \frac{1}{R}\|u\|_{p,\lambda;B_+}\right). \tag{5.9}$$

Now, writing $u(x) = \int_0^t u_s(x',s)ds$ and making use of Jensen's inequality and (5.9) we obtain

$$\|u\|_{p,\lambda;B_+} \le CR^2\|u_t\|_{p,\lambda;B_+} \le C\left(R^2\|f\|_{p,\lambda;B_+} + R\|u\|_{p,\lambda;B_+}\right).$$

Hence, choosing R smaller, if necessary, we get $\|u\|_{p,\lambda;B_+} \le C\|f\|_{p,\lambda;B_+}$ and therefore

$$\|u\|_{W^{2,1}_{p,\lambda}(B_+)} \le C\|f\|_{p,\lambda;B_+} \le C\|f\|_{p,\lambda;Q_T} \tag{5.10}$$

for each solution to the problem (5.6). Making a covering $\{B_\alpha\}$, $\alpha \in \mathcal{A}$ of the boundary S_T such that $Q_T \setminus Q' \subset \bigcup_{\alpha\in\mathcal{A}} B_\alpha$, considering a partition of the unit subordinated to this covering and applying the estimate (5.10) for each B_α we get

$$\|u\|_{W^{2,1}_{p,\lambda}(Q_T\setminus Q')} \le C(n,p,\lambda,\Lambda,T,\|D\Gamma\|_\infty,\gamma_a(r),\|a\|_\infty)\|f\|_{p,\lambda;Q_T}. \tag{5.11}$$

The estimate (5.1) follows from (5.5) and (5.11).

Step 3: Existence and uniqueness. The uniqueness of the solution $u \in W^{2,1}_{p,\lambda}(Q_T)$ of the problem under consideration follows trivially from the a priori estimate (5.1).

The existence of the solution can be proved by the *method of continuity,* as it is done in [29]. For this goal we connect the solvability of (2.1) with the solvability in $W^{2,1}_{p,\lambda}(Q_T)$ of the problem

$$\begin{aligned} \mathcal{H}u &\equiv u_t - \Delta u = f(x) \quad \text{a.e. in} \quad Q_T,\\ \mathcal{I}u &\equiv u(x',0) = 0 \quad \text{on} \quad \Omega,\\ \mathcal{B}u &\equiv \ell^i(x)D_iu = 0 \quad \text{on} \quad S_T. \end{aligned} \tag{5.12}$$

Obviously, for any $f \in L^{p,\lambda}(Q_T)$ the above problem is uniquely solvable in $W^{2,1}_p(Q_T)$ (cf. [21]). In the representation formula of the solution of (5.12) the commutators disappear, so it is not difficult to establish the appropriate $L^{p,\lambda}(Q_T)$ estimates which ensure solvability in $W^{2,1}_{p,\lambda}(Q_T)$ of (5.12). □

Let us recall that when $u \in W^{2,1}_p$ its derivatives D_iu are Hölder continuous functions for $p > n+2$ (see [21], [29, Corollary 1]). It is worth noting that the Morrey regularity of the solution implies a Hölder regularity of D_iu for values of p smaller than $n+2$.

Corollary 5.2. *Suppose* $a^{ij} \in VMO(Q_T)$, $f \in L^{p,\lambda}(Q_T)$, $p \in (1,\infty)$, $\lambda \in (0,n+2)$ *and the conditions* (2.2) *and* (2.3) *to be fulfilled. Let* $u \in W^{2,1}_{p,\lambda}(Q_T)$ *be a solution of the problem* (2.1)*. Then*

i) $u \in C^{0,\alpha}(\overline{Q}_T)$ *with* $\alpha = \frac{1}{n+1} + \frac{\lambda-(n+2)}{p}$ *for* $p > (n+1)(n+2-\lambda)$;

ii) $Du \in C^{0,\beta}(\overline{Q}_T)$ *with* $\beta = 1 + \frac{\lambda}{p} - \frac{n+2}{p}$ *for* $\lambda > \max\{0, n+2-p\}$.

Proof. The assertion $i)$ follows directly from Theorem 5.1 and [12, Theorem 4.1].

The second assertion could be achieved by the parabolic Poincaré inequality [7, Lemma 2.2]

$$\int_{Q_T\cap I} \left|Du-(Du)_{Q_T\cap I}\right|^p dxdt \le r^p \int_{Q_T\cap I} \left(|u_t|^p+|D^2u|^p\right) dxdt$$
$$\le Cr^{p+\lambda}\|u\|_{W^{2,1}_{p,\lambda}(Q_T)}.$$

This implies that the gradient Du belongs to the space of Campanato $\mathcal{L}^{p,p+\lambda}(Q_T)$. It is well known (cf. [12, Theorem 3.1], [23, § 3.3.2]) that for $p+\lambda\in(n+2,n+2+p)$ the space $\mathcal{L}^{p,p+\lambda}(Q_T)$ coincides with $C^{0,\beta}(\overline{Q}_T)$ with $\beta=(p+\lambda-(n+2))/p$. □

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