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Solutions of nonlinear parabolic equations without growth restrictions on the data *

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Dedicated to the memory of our friend Philippe Bénilan

Abstract

The purpose of this paper is to prove the existence of solutions for certain types of nonlinear parabolic partial differential equations posed in the whole space, when the data are assumed to be merely locally integrable functions, without any control of their behaviour at infinity. A simple representative example of such an equation is

$$u_t - \Delta u + |u|^{s-1}u = f,$$

which admits a unique globally defined weak solution $u(x,t)$ if the initial function $u(x,0)$ is a locally integrable function in $\mathbb{R}^N$, $N \geq 1$, and the second member f is a locally integrable function of $x \in \mathbb{R}^N$ and $t \in [0,T]$ whenever the exponent s is larger than 1. The results extend to parabolic equations results obtained by Brezis and by the authors for elliptic equations. They have no equivalent for linear or sub-linear zero-order nonlinearities.

1 Introduction

In this paper we investigate the existence of solutions for a class of equations of the form

$$u_t + L(u) + h(x,t,u) = 0, \tag{1.1}$$

posed on the whole space $\mathbb{R}^N$, $N \geq 1$, with initial condition $u(x,0) = u_0(x)$. In the equation, L is an elliptic differential operator in divergence form with some structure conditions, which include the Laplacian operator, $L(u) = -\Delta u$, or the p-Laplacian operator,

$$L(u) = -\operatorname{div}(|Du|^{p-2}Du), \tag{1.2}$$

and h is a function of the variables x, t, u, which grows uniformly with u at a sufficient rate as $|x| \to \infty$. The main novelty of the problem posed here is

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that the initial condition u_0 is a locally integrable function defined for $x \in \mathbb{R}^N$ without any control of its growth as $|x| \to \infty$, and so is the dependence of h on x. We will prove existence of a solution of this problem by imposing the condition that the growth of h with respect to u is larger than the structural growth of the elliptic operator. In the case where L is the Laplace operator, this basic growth assumption says that the initial-value problem for equation (1.1) has a solution for every $u_0 \in L^1_{\rm loc}(\mathbb{R}^N)$ if

$$h(x,t,u)u \geq c\,|u|^{s+1} - f(x,t)\,u, \tag{1.3}$$

where $s > 1$ and $f \in L^1(0,T; L^1_{\rm loc}(\mathbb{R}^N))$. In the case of the p-Laplacian operator the above conditions take the form $s > p-1$. Such a condition has been shown to be necessary under certain assumptions. The main idea behind the result is the existence of a priori estimates of local type under the stated conditions, which are first established for a sequence of approximate problems and then preserved in the limit.

There is a similar phenomenon for elliptic equations, which has been studied by Brezis [8] *in the semilinear case* (i.e., when L is the Laplacian), and by the present authors, [7] for general operators (see also [14]). The main steps of the proof consist of obtaining local estimates for suitable approximate problems and then passing to the limit. There are essentially two difficulties introduced in treating nonlinear elliptic operators L instead of the Laplacian. The first one is to obtain local estimates on the solutions u and the gradients Du, since *we can only integrate once by parts* in x. It is at this stage that the growth condition on h is needed, even when the operator L is the Laplacian. The second difficulty is to pass to the limit when the nonlinearity of L depends also on Du. Though the general principle is similar to the elliptic case as is usual in many parabolic problems, the time variable makes for a shift in the results and critical exponents that we describe in some detail.

We also show a second phenomenon: for large growth rates of h with respect to u (i.e., large values of s in (1.3)) the solutions have *better regularity* both for u and Du than the one expected from the standard theory.

The paper is organized as follows. We present the problem in Section 2 and state the basic existence and regularity result in Section 3, while the improved regularity obtainable for strongly superlinear lower-order terms is stated in Section 4. The proof of these results occupies the following three sections: Section 5 contains the basic local estimates, Section 6 completes the derivation of a priori estimates, and Section 7 contains the construction of the solutions. The case $p > N$ and the variational approach are reviewed in the next section.

Finally, Section 9 deals with the question of uniqueness of the local solutions, which remains an open problem in general. However, the constructed solutions have the standard good properties of parabolic problems, and we show at the end of the paper how to construct classes of unique solutions which enjoy the Maximum Principle.

2 Statement of the problem. Conditions

In subsequent sections we will study the class of equations that we write for convenience in the form

$$u_t - \operatorname{div} A(x,t,Du) + g(x,t,u) = f(x,t)\,. \tag{2.1}$$

They are posed in $Q = \{(x,t) : x \in \mathbb{R}^N,\ 0 < t < T\}$, with initial condition

$$u(x,0) = u_0(x). \tag{2.2}$$

Here $u(x,t)$ and $f(x,t)$ are scalar functions of $x \in \mathbb{R}^N$ and $t \in (0,T)$, and Du denotes the gradient of u with respect to x. Both A and g have to satisfy certain structural assumptions. Thus, A satisfies:

(A1) $A(x,t,\xi) : Q \times \mathbb{R}^N \to \mathbb{R}^N$ is measurable in $(x,t) \in Q$ for any fixed $\xi \in \mathbb{R}^N$ and continuous in $\xi \in \mathbb{R}^N$ for a.e. $(x,t) \in Q$.

(A2) There exist two constants $p > 1$ and $c > 0$ such that for all ξ and a.e. (x,t)

$$A(x,t,\xi) \cdot \xi \geq c|\xi|^p\,,$$

where the dot is used to denote scalar product of vectors in $\mathbb{R}^N$.

(A3) There exist functions $b(x,t) \in L^{p'}_{\rm loc}(\mathbb{R}^N \times [0,T))$ ($p' = p/(p-1)$), and $d(x,t)$, locally bounded in $\mathbb{R}^N \times [0,T)$, such that for all ξ and a.e. (x,t)

$$|A(x,t,\xi)| \leq b(x,t) + d(x,t)|\xi|^{p-1}\,.$$

(A4) For a.e. $(x,t) \in Q$ and all $(\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N$, $\xi \neq \eta$, we have

$$(A(x,t,\xi) - A(x,t,\eta)) \cdot (\xi - \eta) > 0.$$

Hypotheses (A1)–(A4) are classical in the study of nonlinear operators in divergence form, see [15]. Moreover, unless mention to the contrary the structural exponent p will be taken over a critical value, $p > p_1 = (2N+1)/(N+1)$. This is done to avoid the functional difficulties related to the definition of the gradient Du which may arise when dealing with L^1 data, cf. [2] for a complete discussion of the elliptic problem in the context of entropy solutions, and [1] for the parabolic case. See also [4] for the same problem in the framework of renormalized solutions. The model example of a function satisfying (A1)-(A4) is of course $A(x,t,\xi) = |\xi|^{p-2}\xi$, which for $p = 2$ leads to the Laplace operator.

The assumptions on g are the following:

(G1) $g(x,t,\sigma) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \mathbb{R}^N$, $t \in (0,T)$ for any fixed $\sigma \in \mathbb{R}$ and continuous in σ for a.e. (x,t).

(G2) There exists an exponent $s > 0$ and a constant $c_2 > 0$, such that for all σ and almost every (x, t)

$$g(x, t, \sigma)\,\sigma \geq c_2|\sigma|^{s+1}\,.$$

(G3) For all $k > 0$ the function

$$G_k(x, t) = \sup_{|\sigma|\leq k} |g(x, t, \sigma)|$$

is locally integrable over $\mathbb{R}^N \times [0, T]$.

Let us remark that for s large enough we relax the condition $p > p_1$ and approach $p = 1$ without getting out of the standard functional framework. Indeed, we may replace it by

$$p > \frac{s+1}{s} \tag{2.3}$$

which is better than $p > p_1$ if $s > (N + 1)/N$.

Finally, we assume that $f \in L^1(0, T; L^1_{\text{loc}}(\mathbb{R}^N))$ and the initial data $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$. We look for a global weak solution to (2.1), i.e. a function $u \in L^1(0, T; W^{1,1}_{\text{loc}}(\mathbb{R}^N))$ such that both $A(x, t, Du)$ and $g(x, t, u)$ are well defined in $L^1_{\text{loc}}(\mathbb{R}^N)$ and (2.1) is satisfied in $\mathcal{D}'(Q)$ and the initial condition is also satisfied in the precise sense that we state next.

Definition 2.1 A function $u \in L^1(0, T; W^{1,1}_{\text{loc}}(\mathbb{R}^N))$ is said to be a *weak solution* of problem (2.1)-(2.2) if $|Du|^{p-1}$ and $g(x, t, u(x, t)) \in L^1_{\text{loc}}(\mathbb{R}^N \times [0, T))$, and

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} (A(x, t, Du) \cdot D\phi - u\,\phi_t)\,dx\,dt + \int_0^T \int_{\mathbb{R}^N} g(x, t, u)\phi\,dx\,dt \\ &= \int_0^T \int_{\mathbb{R}^N} f(x, t)\phi(x, t)\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\,\phi(x, 0)\,dx \end{aligned} \tag{2.4}$$

holds for every test function $\phi \in C^1_c(\mathbb{R}^N\times[0, T))$, the C^1 functions with compact support.

We will find weak solutions such that $u \in L^\infty(0, T; L^1_{\text{loc}}(\mathbb{R}^N))$. Let us remark that under these conditions u satisfies the initial condition in the following sense

$$\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}}^N u(x, t)\,\phi(x)\,dx\,dt \to \int_{\mathbb{R}^N} u_0(x)\,\phi(x)\,dx \tag{2.5}$$

for every continuous function ϕ with compact support from $\mathbb{R}^N$ to $\mathbb{R}$.

Indeed, it is easy to prove (2.5) for $\phi \in C^1(\mathbb{R}^N, \mathbb{R})$ with compact support, taking $\phi(x)(\tau-t)^+$ as test functions in (2.4) (such test functions are avalaible by regularization). Then (2.5) holds for every $\phi \in C(\mathbb{R}^N, \mathbb{R})$ with compact support since the family $(\frac{1}{\tau}\int_0^\tau u(., t)\,dt)_{0<\tau\leq T}$ is bounded in $L^1(\Omega)$ for any bounded subset Ω of $\mathbb{R}^N$.

3 Basic existence and regularity results

The main existence and regularity result is the following

Theorem 3.1 *Let $p > p_1 = (2N+1)/(N+1)$ and $s > p-1$. Then for every $u_0 \in L^1_{\rm loc}(\mathbb{R}^N)$ and every $f \in L^1(0,T; L^1_{\rm loc}(\mathbb{R}^N))$ there exists a weak solution of the Cauchy problem (2.1)-(2.2) with $u \in L^\infty(0,T; L^1_{\rm loc}(\mathbb{R}^N)) \cap L^s(0,T; L^s_{\rm loc}(\mathbb{R}^N))$ and also*

$$u \in L^r(0,T; W^{1,q}_{\rm loc}(\mathbb{R}^N)), \tag{3.1}$$

under the restrictions $1 \le r, q < p$, $q < q_e = N(p-1)/(N-1)$, $q < N$, and the critical line

$$\frac{N(p-2)+p}{r} + \frac{N}{q} > N+1. \tag{3.2}$$

Furthermore, if $u_0 \ge 0$ and $f \ge 0$ then we construct a solution such that $u \ge 0$.

In the preceding theorem, we can be more precise about the Maximum Principe (or M.P. in short). Indeed, if g is nondecreasing with respect to u, then, if $u_0 \ge v_0$ a.e. and $f \ge g$ a.e., we can construct corresponding solutions, u and v, satisfying $u \ge v$ a.e. (where v is the constructed solution of (2.1)-(2.2) with v_0 and g instead of u_0 and f). If one has uniqueness of the solution of (2.1)-(2.2) we then obtain the so called Maximum Principle. See more precise results in the final remark of Section 9.

Remarks on the conditions. The lower bound on p, $p > p_1$, is not essential. It is due to the fact that we do not want to get out of the classical weak formulation where the gradient is a locally integrable function. Exponents $1 < p \le p_1$ can be handled by using a proper definition of gradient (see [2]) but we will not include the calculations here in order to avoid complicating too much the presentation. Note that the lower limit in the elliptic theory is $p_0 < p_1$; Indeed, We have $p_1 = 2 - (1/(N+1))$, while $p_0 = 2 - (1/N)$.

The condition $s > p-1$ (which, in the case of the Laplace operator, reads $s > 1$ and makes the lower order term g "superlinear") is the essential ingredient in the existence of a priori estimates that allows for the whole local theory.

As for the conditions on q, the limit $q < p$ is natural, since $q = p$ is the variational regularity, which is a limit, in general not attained, for our conditions. The condition $q < q_e$ is the restriction found in the elliptic theory, cf. [7], and we cannot avoid it since our problem includes the elliptic case in the form of stationary solutions. We do not need it for $p \ge N$ since then $q_e \ge p$.

About the context. The regularity obtained in the theorem takes into account the bound on u in $L^s(L^s_{\rm loc})$ coming from the a priori control of the zero-order term $g(x,t,u)$ only to ensure the existence of local estimates, but it does not affect otherwise the derivation of the estimates of the local norms $L^r(W^{1,q})$. This means that all the above results describe the regularity of Dirichlet or Neumann problems without zero-order term or with suitable lower-order terms.

Analysis of the result. Let us examine the set of *admissible values* of q and r for the $L^r(W^{1,q}_{\rm loc})$ estimate. It is described by a convex polygonal region in the $(1/q, 1/r)$ plane. We observe first that the points $A = (2/p, 1/p)$ and $B = (1/q_e, 1/(p-1))$ lie in the critical line (c.l.) described by formula (3.2), and that this line is not admissible. We examine the different ranges of p separately.

• For $p \geq 2$ the admissible region is limited by the c.l. between the extremal points A and B and the lines $q = 1$, $r = 1$, $q = q_e$ and $r = p$. We may sum up the best obtained regularity as

Corollary 3.2 *For* $2 \leq p \leq N$ *we get* $u \in L^r(0, T; W^{1,q}_{\rm loc}(\mathbb{R}^N))$ *with*

$$\begin{aligned} &1 \leq r < p \quad &&\text{and } 1 \leq q < p/2, \\ &1 \leq r < p-1 \quad &&\text{and } 1 \leq q < q_e, \end{aligned} \tag{3.3}$$

as well as the interpolations along the characteristic line.

Observe that for $p = N$ the upper limit q_e becomes $q_e = p = N$, and we are in the border of the variational regularity as expected. On the other hand, in the *balanced case* $r = q$ the limiting value of q and r is

$$q_d = p - \frac{N}{N+1},$$

which is less than q_e. This value agrees with [1], page 304, see also [5].

• In the case $p < 2$ the points A and B fail to be borderline for the admissibility restrictions $q \geq 1$, $r \geq 1$, and we have

Corollary 3.3 *For* $p_1 \leq p < 2$ *the admissible region is reduced to a triangle limited by a segment of the critical line, which decreases in length with* p *until it becomes the point* $(1, 1)$ *for* $p = p_1$. *This limiting value of* p *is easily determined from the condition* $q_d = 1$.

Note that we still have $q_e > 1$ for $p > p_1$.

• For $p > N$ the point B is not valid because it violates the conditions $q < N$ and $q < p$. It must be replaced a new extremal point with $q = N$ and r determined from condition (3.2): $r = p(N + 1)/N - 2$. However, the result in this case is less interesting since we can obtain further regularity by using the Sobolev embedding of $W^{1,q}$ into C^α for $q > N$ and we get locally bounded solutions in space, not far from the variational formulation. We will devote Section 8 to discuss this issue.

Regularity for u. We shall now look briefly at the regularity $u \in L^r(0, T; L^m_{\rm loc}(\mathbb{R}^N))$ obtained from the previous one by use of the standard Sobolev embeddings. Using the rule $m = qN/(N - q)$ for $q < N$ we get the new critical line

$$\frac{N(p-2)+p}{r} + \frac{N}{m} > N. \tag{3.4}$$

From this we get for $2 \leq p < N$ the extremal point $r = p - 1$, $m = m_e$ with $m_e = N(p-1)/(N-p)$, which tends to infinity as $p \to N$. In the balanced case $r = m$ we get the admissible values

$$1 \leq r = m < p - 1 + \frac{p}{N},$$

which is a better space than $L^s(0,T; L^s_{\text{loc}}(\mathbb{R}^N))$ only if $s + 1 < p(N+1)/N$.

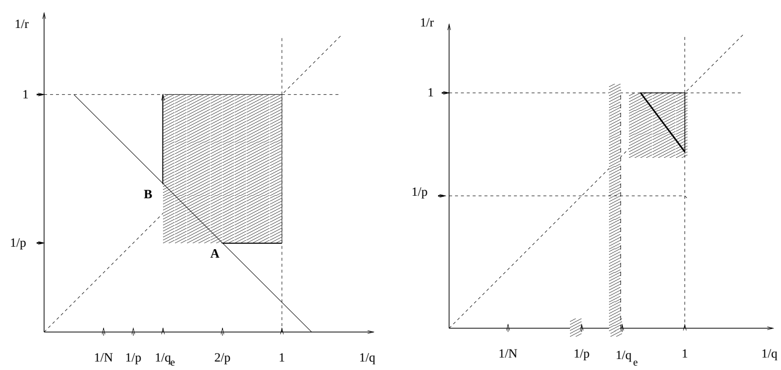

Figure 1: Admissible region for $2 < p < N$ and for $p < 2$

4 Improved Regularity

In accordance with the last observation we may expect a better regularity when s is large enough by taking into account the bound on u in $L^s(L^s_{\text{loc}})$. Indeed, there is an improvement of the previous regularity results for $s \gg 1$ which is reflected in the following result.

Theorem 4.1 *Let $p > 1$ and $s > p - 1$, $s > 1/(p-1)$. Then we can construct a weak solutions as before with the additional regularity*

$$u \in L^r(0,T; W^{1,q}_{\text{loc}}(\mathbb{R}^N)) \tag{4.1}$$

for every $r, q \geq 1$ such that $r < p$, $q < ps/(s+1)$ and

$$(s-1)\frac{p}{r} + \frac{p}{q} > s + 1. \tag{4.2}$$

The new condition $s(p-1) > 1$ is necessary for small values of p in order for the admissible set of exponents to be non-empty. Another extremal point C is now given by the coordinates $q = r = ps/(s+1)$. Taking into account the common point A and comparing the slopes of the critical lines implies that an improvement of the admissibility region takes place if $(N(p-2)+p)/N < s-1$, i.e.

Corollary 4.2 *If s is large enough, precisely for*

$$s+1 > p\frac{N+1}{N}, \tag{4.3}$$

the admissible (q,r) region of Theorem 3.1 is extended. The maximal q-regularity is improved when $N(p-1)/(N-1) < ps/(s+1)$, i.e., when

$$p < N \quad \text{and } s > \frac{N(p-1)}{N-p}, \tag{4.4}$$

and in that case the new admissible region completely contains the previous one.

The latter result is natural since the bound for s is just the Sobolev conjugate exponent of $q_e = N(p-1)/(N-1)$. In any case, it is clear that as $s \to \infty$ we approach the variational regularity $q = p$. Note also that for the Laplacian case the improvement of regularity starts from the exponent $s = (N+2)/N$.

One further improvement concerns the lower bound for p. We remark that whenever $p > (s+1)/s$ we have estimates that allow to construct a weak solution in $L^r(0,T;\, W^{1,q}_{\rm loc}(\mathbb{R}^N))$ for some $r, q \geq 1$. This means that for large enough s the restriction $p > p_1$ is overcome inside the framework of weak solutions.

5 Basic local estimates

Model equation. Basic Step. In order to present the main ideas without undue complications we begin with the model example

$$u_t - \operatorname{div}(|Du|^{p-2}Du) + |u|^{s-1}u = f. \tag{5.1}$$

We perform the analysis on the solutions of the Cauchy-Dirichlet problems posed in balls $B_R(0)$ with zero Dirichlet conditions on the lateral boundary $|x| = R$. Both u_0 and f are suitably cut into bounded functions defined for $x \in B_R$, $0 \leq t \leq T$, preserving their $L^1_{\rm loc}$ bounds. It follows from standard theory [15] that there exists a solution $u = u_R$ of the approximate problem, and

$$u \in L^p(0,T; W^{1,p}_0(B_R)) \cap C([0,T]; L^2(B_R)),$$

$$u_t \in L^{p'}(0,T; W^{-1,p'}(B_R)), \quad |u|^s \in L^1(B_R \times (0,T)).$$

Moreover, the Maximum Principle holds and, in particular, $u_0 \geq 0$ and $f \geq 0$ imply $u \geq 0$.

We want to obtain local estimates of the size of u and its gradient Du which are uniform in R. In order to do that we take a cutoff function in the space variables $\theta(x)$, which is smooth, supported in a ball $B_{2\rho}(0)$, with $0 < 2\rho < R$ and such that $0 \leq \theta \leq 1$ and $\theta = 1$ for $|x| \leq \rho$. We also take a small number $0 < m < 1$ and introduce the function ϕ defined for $u > 0$ by

$$\phi(u) = \int_0^u \frac{1}{(1+s)^{m+1}}\, ds = \frac{1}{m} - \frac{1}{m(1+u)^m}, \tag{5.2}$$

and by $\phi(u) = -\phi(-u)$ for $u < 0$. It is a bounded monotone function. We also introduce

$$\psi(u) = \int_0^u \phi(s)\, ds. \tag{5.3}$$

Now we multiply (5.1) by $\phi(u(x,t))\,\theta^\gamma(x)$, $\gamma > 1$, and integrate by parts to obtain

$$\begin{aligned}
&\int_{B_R} \psi(u(x,T))\theta(x)^\gamma\, dx + \int_0^T \int_{B_R} |Du|^p \phi'(u)\theta^\gamma\, dx\, dt \\
&+ \int_0^T \int_{B_R} |u|^{s-1} u\phi(u)\theta^\gamma\, dx\, dt + \gamma \int_0^T \int_{B_R} (Du \cdot D\theta)|Du|^{p-2}\, \phi(u)\theta^{\gamma-1}\, dx\, dt \\
&= \int_{B_R} \psi(u_0(x))\theta(x)^\gamma\, dx + \int_0^T \int_{B_R} f\phi(u)\theta^\gamma\, dx\, dt.
\end{aligned}$$

We remark that

$$\begin{aligned}
|Du|^{p-1}|D\theta|\,\phi(u)\theta^{\gamma-1} &\le \frac{1}{2\gamma}|Du|^p\phi'(u)\theta^\gamma + C(p,\gamma)|D\theta|^p\theta^{\gamma-p}\frac{\phi(u)^p}{\phi'(u)^{p-1}} \\
&\le \frac{1}{2\gamma}|Du|^p\phi'(u)\theta^\gamma + C_1(1+|u|)^{(m+1)(p-1)}\theta^{\gamma-p},
\end{aligned}$$

where $C_1 > 0$ depends on p, m, $\gamma > 1$ and $\rho > 0$. We now choose m so that $(m+1)(p-1) < s$, which is possible for small $m > 0$ since $s > p-1$ by assumption. Using Young's inequality we get

$$C_1(1+|u|)^{(m+1)(p-1)}\theta^{\gamma-p} \le \frac{1}{2}|u|^{s-1}u\phi(u)\theta^\gamma + C_2(1+\theta^{\gamma-\frac{ps}{s-(m+1)(p-1)}}). \tag{5.4}$$

In view of the last exponent we choose $\gamma > ps/(s-(m+1)(p-1))$ and then

$$\begin{aligned}
&\int_{B_R} \psi(u(x,T))\theta(x)^\gamma\, dx + \frac{1}{2}\int_0^T \int_{B_R} |Du|^p\phi'(u)\theta^\gamma\, dx\, dt \\
&+\frac{1}{2}\int_0^T \int_{B_R} |u|^{s-1}u\phi(u)\theta^\gamma\, dx\, dt \\
&\le \int_{B_{2\rho}} \psi(u_0(x))\, dx + C_3\int_0^T \int_{B_{2\rho}} |f|\, dx\, dt + C_3\, T|B_{2\rho}|,
\end{aligned} \tag{5.5}$$

where $|B_{2\rho}|$ denotes the volume of $B_{2\rho}$ in $\mathbb{R}^N$. In view of our assumptions on the data we conclude that

$$\int_0^T \int_{B_\rho} \frac{|Du|^p}{(1+|u|)^{m+1}}\, dx\, dt \le C_4, \quad \int_0^T \int_{B_\rho} |u|^s\, dx\, dt \le C_4, \tag{5.6}$$

where C_4 depends on ρ, p, s and T and not on R. The dependence on ρ takes place through the local norms of u_0 and f. The main point is that the different constants C_i appearing in this calculation do not depend on R. Besides, in view of the form of ψ, the first term in (5.5) (replacing T by t, which is possible if $0 \le t \le T$) gives, for every $0 < t \le T$, the bound

$$\int_{B_\rho} |u(x,t)|\,dx \le C_5,$$

where C_5 has the same dependence as C_4. Hence, we get a bound of u in the spaces $L^\infty(0,T;L^1_{\rm loc}(\mathbb{R}^N))$ and $L^s(0,T;L^s_{\rm loc}(\mathbb{R}^N))$.
General case. When we consider the general equation (2.1) with the structure conditions of Section 2, we also have existence of solutions for the approximate problems obtained when u_0 and f are suitably cut into bounded functions (in order to contruct these solutions, we can first approximate the function $g(x,t,u)$ as in [2]). Moreover (thanks to the construction process), $u_0 \ge 0$ and $f \ge 0$ imply $u \ge 0$. Multiplication of the equation by the same test function gives

$$\begin{aligned}
&\int_{B_R} \psi(u(x,T))\theta(x)^\gamma\,dx + \int_0^T\int_{B_R} (A(x,t,Du)\cdot Du)\phi'(u)\theta^\gamma\,dx\,dt\\
&+\int_0^T\int_{B_R} g(x,t,u)\phi(u)\theta^\gamma\,dx\,dt + \gamma\int_0^T\int_{B_R}(A(Du)\cdot D\theta)\phi(u)\theta^{\gamma-1}\,dx\,dt\\
&= \int_{B_R}\psi(u_0(x))\theta(x)^\gamma\,dx + \int_0^T\int_{B_R} f\phi(u)\theta^\gamma\,dx\,dt.
\end{aligned}$$

Using the structure conditions we get

$$\begin{aligned}
&\int_{B_R}\psi(u(x,T))\theta(x)^\gamma\,dx + c\int_0^T\int_{B_R}|Du|^p\phi'(u)\theta^\gamma\,dx\,dt\\
&+c_2\int_0^T\int_{B_R}|u|^s|\phi(u)|\theta^\gamma\,dx\,dt\\
&\quad\le \gamma\int_0^T\int_{B_R}|D\theta|(|d||Du|^{p-1}+|b|)\,\phi(u)\theta^{\gamma-1}\,dx\,dt\\
&\qquad +C\int_{B_{2\rho}}\psi(u_0)\,dx + C\int_0^T\int_{B_{2\rho}}|f|\,dx\,dt\,.
\end{aligned}$$

Remark that all integrals are performed inside the ball $B_{2\rho}(0)$. At this stage we can repeat the end of previous step to get the same list of estimates.

6 Local estimates continued

We will now combine the above estimates in a suitable way to produce the necessary final local estimates which will allow to pass to the limit in the approximate problems and prove our existence results.

Interpolation Step for $q < N$. In order to find local estimates in the spaces stated in Theorem 3.1 we proceed as follows (similarly to [5]): we choose some

q, $1 \le q < p$, $q < N$. If $q^\star$ is the Sobolev conjugate exponent, $q^\star = Nq/(N-q)$, we have

$$\Big(\int_{B_\rho} |u|^{q^\star}\,dx\Big)^{q/q^\star} \le C + C\int_{B_\rho} |Du|^q\,dx. \tag{6.1}$$

The constant C does not depend on R thanks to the previous estimate on u in $L^\infty(0,T;\, L^1_{\rm loc}(\mathbb{R}^N))$ independently of R, but it depends on ρ. Indeed, in order to obtain (6.1), we use the Sobolev imbedding, The Poincaré inequality with average and the $L^\infty(0,T;\, L^1_{\rm loc}(\mathbb{R}^N))$ estimate on u (the latter gives and estimate on the mean value of u on B_ρ). Moreover,

$$\int_{B_\rho} |Du|^q\,dx \le \Big(\int_{B_\rho} \frac{|Du|^p}{(1+|u|)^{m+1}}\,dx\Big)^{q/p}\Big(\int_{B_\rho} (1+|u|)^{(m+1)q/(p-q)}dx\Big)^{(p-q)/p}.$$

Raising to the power r/q,

$$\begin{aligned}&\Big(\int_{B_\rho} |Du|^q\,dx\Big)^{r/q}\\ &\le \Big(\int_{B_\rho} \frac{|Du|^p}{(1+|u|)^{m+1}}\,dx\Big)^{r/p}\Big(\int_{B_\rho} (1+|u|)^{(m+1)q/(p-q)}dx\Big)^{r(p-q)/pq}.\end{aligned}$$

Now we take $r < p$ and integrate in time applying Hölder

$$\begin{aligned}&\int_0^T \Big(\int_{B_\rho} |Du|^q\,dx\Big)^{r/q} dt\\ &\le \Big(\int_0^T\int_{B_\rho} \frac{|Du|^p}{(1+|u|)^{m+1}}\,dx\,dt\Big)^{r/p}\\ &\quad\times\Big(\int_0^T \Big(\int_{B_\rho} (1+|u|)^{(m+1)q/(p-q)}dx\Big)^{r(p-q)/q(p-r)} dt\Big)^{(p-r)/p}.\end{aligned}$$

The first integral of the right-hand side is bounded independently of R by (5.6). In order to estimate the last integral, we first remark that for $q/(p-q) < 1$, i.e., $q < p/2$, we may take a small $m > 0$ so that the exponent

$$\alpha = \frac{(m+1)q}{p-q} \tag{6.2}$$

is equal or less than 1, and then the right-hand side is bounded and so is the left-hand side. This leads to a bound in $L^r(0,T;W^{1,q}(B_\rho))$ with $1 \le q < p/2$ and $1 \le r < p$. This agrees exactly with inequality (3.2). Otherwise, for $q \ge p/2$ we need to interpolate again in the space variable, so that we choose $1 \le \alpha < q^\star$. This inequality is possible if and only if $q/(p-q) < q^\star$, i.e.,

$$q < \frac{N(p-1)}{N-1} = q_e, \tag{6.3}$$

which is a further condition on q. We can then write

$$\begin{aligned}&\int_{B_\rho} (1+|u|)^\alpha dx\\ &\leq \Big(\int_{B_\rho} (1+|u|)\,dx\Big)^{(q^\star-\alpha)/(q^\star-1)}\Big(\int_{B_\rho} (1+|u|)^{q^\star}\,dx\Big)^{(\alpha-1)/(q^\star-1)}.\end{aligned}$$

The intermediate integral is controlled by the $L^\infty(0,T;L^1_{\rm loc}(\mathbb{R}^N))$ estimate on u. Hence,

$$\Big(\int_{B_\rho}(1+|u|)^\alpha dx\Big)^{(p-q)r/q(p-r)} \leq C_6\Big(\int_{B_\rho}(1+|u|)^{q^\star}\,dx\Big)^{(\alpha-1)(p-q)r/(q^\star-1)q(p-r)}.$$

Putting together all these estimates we get

$$\begin{aligned}&\int_0^T \left(\int_{B_\rho} |u|^{q^\star}\,dx\right)^{r/q^\star}\\ &\leq\ C_7 + C_7\Big(\int_0^T \Big(\int_{B_\rho} (1+|u|)^{q^\star}\,dx\Big)^{(\alpha-1)(p-q)r/(q^\star-1)q(p-r)} dt\Big)^{(p-r)/p}.\end{aligned}$$

An estimate follows for the first member if the exponent in the first member is larger than the exponent in the second, namely

$$\frac{r}{q^\star} > \frac{(\alpha-1)(p-q)r}{(q^\star-1)q(p-r)}. \tag{6.4}$$

In order to fulfill this inequality we may choose a small value of α, always larger than $q/(p-q)$, by taking $m>0$ very small. Therefore, we are reduced to check the limit case $\alpha-1=(2q-p)/(p-q)$ which gives

$$\frac{q^\star-1}{q^\star} > \frac{2q-p}{q(p-r)}.$$

Working out this formula gives the relation

$$\frac{(p-2)N+p}{r} + \frac{N}{q} > N+1. \tag{6.5}$$

This relation is complemented by the restrictions on q: $1\leq q<p$, $q<N$ and $q<N(p-1)/(N-1)$, together with the restriction on r: $1\leq r<p$.

Improved regularity. We now re-do the interpolation estimates making use of the fact that

$$\int_0^T\int_{B_\rho} |u|^s\,dx\,dt \leq C_4$$

to change the way of finding a bound in $L^r(0,T;W^{1,q}_{\rm loc}(\mathbb{R}^N))$. We write again for $1\leq q<p$ with $m>0$

$$\int_{B_\rho} |Du|^q\,dx \leq \Big(\int_{B_\rho} \frac{|Du|^p}{(1+|u|)^{m+1}}\,dx\Big)^{q/p}\Big(\int_{B_\rho}(1+|u|)^\alpha dx\Big)^{(p-q)/p},$$

where $\alpha = (m+1)q/(p-q)$ as before. Rising to the power r/q with $r < p$, and integrating in time applying Hölder and the gradient bound of formula (5.6) we get

$$\int_0^T \|Du(t)\|^r_{L^q(B_\rho)} dt \le C\Big(1 + \int_0^T \|u(t)\|^{(m+1)r/(p-r)}_{L^\alpha(B_\rho)} dt\Big)^{(p-r)/p}.$$

If $q < p/2$ we may take again $\alpha \le 1$ and we are finished. Otherwise, In order to estimate the last integral we need to interpolate in the space variable, and this time we do in terms of the L^s bound, which is possible if

$$1 \le \frac{q}{p-q} < s, \tag{6.6}$$

and this occurs if $q < ps/(s+1)$. We then choose $m > 0$ so that $1 < \alpha < s$, and write

$$\|u(t)\|_{L^\alpha(B_\rho)} \le \|u(t)\|^\theta_{L^s(B_\rho)} \|u(t)\|^{1-\theta}_{L^1(B_\rho)},$$

with interpolation exponent

$$\theta = \frac{s(\alpha-1)}{\alpha(s-1)}.$$

The bound in $L^r(0,T;W^{1,q}(B_\rho))$ is found if $\theta(m+1)r/(p-r) \le s$. Putting m very small and after some manipulations, we get the condition

$$(s-1)\frac{p}{r} + \frac{p}{q} > s+1. \tag{6.7}$$

Note that this inequality is satisfied for all $q < p/2$ and $r < p$, hence for the case we had dealt with separately. Summing up, the local estimate is obtained under the restrictions of Theorem 4.1.

7 Construction of the solution

We proceed now with the construction of the solution of problem (2.1)-(2.2) using a rather standard approximation method, as in [2]. We approximate the initial function u_0 by a sequence of bounded functions $\{u_{0n}\}$ defined in $B_n(0)$ such that $|u_{0n}| \le n$, $|u_{0n}| \le |u_0|$; we also approximate f by a sequence of bounded functions $\{f_n\}$ defined in $Q_n = B_n(0) \times (0,T)$ such that $|f_n| \le n$ and $|f_n| \le |f|$, so that

$$u_{0n} \to u_0 \quad \text{in } L^1_{\rm loc}(\mathbb{R}^N), \quad f_n \to f \quad \text{in } L^1(0,T;L^1_{\rm loc}(\mathbb{R}^N)). \tag{7.1}$$

We now consider u_n solution of problem (P_n) consisting of equation (2.1) posed in Q_n with right-hand side f_n, plus zero Dirichlet data on the lateral boundary and initial data u_{0n} in $B_n(0)$. There exists a solution of this problem in the space $L^p(0,T;W^{1,p}(B_n(0)))$. One also has $u_n \in C([0,T], L^2(B_n(0)))$. Moreover, $u_0 \ge 0$ and $f \ge 0$ imply $u \ge 0$.

According to the estimates obtained before, if r and q as in the previous section, for any given $\rho > 0$ the following sequences are bounded uniformly in $n > 2\rho$:

$$\begin{gathered}\{u_n\} \quad \text{in } L^r(0,T;W^{1,q}(B_\rho(0))),\\ \{g(x,t,u_n)\} \quad \text{in } L^1(B_\rho(0)\times(0,T)),\\ \{u_n'\} \quad \text{in } L^1(0,T;W^{-1,\delta}(B_\rho(0))) + L^1(0,T;L^1(B_\rho(0))),\end{gathered}$$

for some $\delta > 1$. Moreover, since one obtains that the sequence $\{u_n'\}$ is bounded in $L^1(0,\,T;\,W^{-1,1}\,(B_\rho(0)))$, using compactness arguments (see [19]) it is easy to see that the sequence $\{u_n\}$ is relatively compact in $L^1(Q_\rho)$. By a diagonal process we may select a subsequence, also denoted by $\{u_n\}$, such that

$$u_n \to u \quad \text{a.e. and in} \quad L^1(0,T;L^1_{\text{loc}}(\mathbb{R}^N)),$$

and also $u_n \to u$ weakly in $L^r(0,T;W^{1,q}_{\text{loc}}(\mathbb{R}^N))$ for q,r as in Theorem 4.1 and Corollary 4.2.

We want to pass to the limit in the equation in order to get a solution of the original problem. We need to prove first the convergence of the sequence $\{g(x,t,u_n)\}$, and also the *a.e.* convergence (up to a subsequence) of the gradients of $\{u_n\}$, which will imply the convergence of $\{Du_n\}$ to Du in $L^r(0,T;L^q_{\text{loc}}(B_R))$, for any $R > 0$.

Let us prove the result about $g(x,t,u_n)$, which is based on an argument of local equi-integrability. We resume the notations and calculations of Section 5 with slight variations. We consider the function ϕ defined in (5.2) and we displace it by an amount $t > 0$ to get

$$\phi^t(s) = \begin{cases} \phi(s-t), & s \geq t\\ 0, & |s| < t\\ -\phi^t(-s), & s \leq -t.\end{cases}$$

Next, as in (5.3) we define

$$\psi^t(s) = \int_0^s \phi^t(\sigma)d\sigma.$$

We also take a cutoff function $\theta(x)$ as there and put $v = \psi^t(u_n)\theta^\gamma$, $\gamma > 1$. Dispensing with the superscripts t for ϕ and ψ and much as in (5.5) we have

$$\begin{aligned}&\int_{B_R} \psi(u_n(x,T))\theta(x)^\gamma\,dx + c\int_0^T\int_{B_R} |Du_n|^p\phi'(u_n)\theta^\gamma\,dx\,dt\\ &+\int_0^T\int_{B_R} g(x,t,u_n)\phi(u_n)\theta^\gamma\,dx\,dt + \gamma\int_0^T\int_{B_R}(A(Du_n)\cdot D\theta)\phi(u_n)\theta^{\gamma-1}\,dx\,dt\\ &\leq \int_{B_R}\psi(u_{0n}(x))\theta(x)^\gamma\,dx + \int_0^T\int_{B_R} f_n\phi(u_n)\theta^\gamma\,dx\,dt.\end{aligned}$$

Now we use the following inequalities for some $\varepsilon > 0$ (thanks to the boundedness of ϕ and Young's inequality as in (5.4), recall that $\frac{s}{(m+1)(p-1)} > 1$)

$$
\begin{aligned}
&|Du_n|^{p-1}|D\theta|\phi(u_n)\theta^{\gamma-1} \\
&\leq \varepsilon|Du_n|^p\phi'(u_n)\theta^\gamma + c(p,\varepsilon)|D\theta|^p\theta^{\gamma-p}\frac{\phi(u_n)^p}{\phi'(u_n)^{p-1}} \\
&\leq \varepsilon|Du_n|^p\phi'(u_n)\theta^\gamma + \varepsilon|u_n|^{s-1}u_n\phi(u_n)\theta^\gamma + c(\theta,\varepsilon,p,s)\phi(u_n)^p.
\end{aligned}
$$

Choosing $\varepsilon > 0$ small enough and working as in (5.5) this leads to

$$
\begin{aligned}
&\int_{B_R}\psi(u_n(x,T))\theta(x)^\gamma\,dx + \frac{c}{2}\int_0^T\int_{B_R}|Du_n|^p\phi'(u_n)\theta^\gamma\,dx\,dt \\
&+\frac{1}{2}\int_0^T\int_{B_R}g(x,t,u_n)\phi(u_n)\theta^\gamma\,dx\,dt \qquad (7.2)\\
&\leq \int_{B_{2R}}\psi(u_0(x))\theta^\gamma\,dx + C_3\int_0^T\int_{B_{2R}}|f|\,\phi(u_n)\,dx\,dt \\
&\quad +C_3\int_0^T\int_{B_{2R}}\phi^p(u_n)\,dx\,dt.
\end{aligned}
$$

We recall that ϕ stands for ϕ^t and ψ for ψ^t. As $t\to\infty$ the right-hand side of the last displayed formula tends to zero, hence instead of the estimates (5.6) we now conclude that

$$\int_0^T\int_{B_R}g(x,t,u_n)\phi(u_n)\theta^\gamma\,dx\,dt \to 0$$

as $t\to\infty$, uniformly in n. Since, for τ large $\phi^\tau(u_n) \geq c_0\chi_{\{|u_n|>2\tau\}}$, we get

$$\lim_{\tau\to\infty}\sup_n\int_0^T\int_{B_R}g(x,t,u_n)\theta^\gamma\chi_{\{|u_n|>2\tau\}}\,dx\,dt \to 0$$

This means that $\{g(u_n)\}$ is equi-integrable.

By Vitali's Lemma $g(u_n)\to g(u)$ in $L^1(0,T;L^1_{\rm loc}(\mathbb{R}^N)$ and a.e.

The last step is the a.e. convergence of the sequence $\{Du_n\}$ and this is done by means of the concept of convergence in measure as in the papers [6], [2]. We will prove that, for any $\rho > 0$, the sequence $\{Du_n\}$ is a Cauchy sequence in measure on $B_\rho\times[0,T]$. To prove this result, let $\rho > 0$, let $\varepsilon > 0$ and $T_\varepsilon(s) = \max\{-\varepsilon,\min\{\varepsilon,s\}\}$. Define $Q_\varepsilon = \{(x,t)\in B_\rho(0) : |u_n(x,t)-u_m(x,t)| \leq \varepsilon\}$. Taking a cutoff function θ equal to 1 on B_ρ and 0 outside $B_{2\rho}$ and $T_\varepsilon(u_n-u_m)\theta$ as test function in the equations satisfied by u_n and u_m lead to (neglecting the positive contribution of the term with time derivative)

$$
\begin{aligned}
&\int\int_{Q_\varepsilon}(A(x,t,Du_n)-A(x,t,Du_m))(Du_n-Du_m)dxdt \qquad (7.3)\\
&\leq \varepsilon\int_0^T\int_{B_{2\rho}}(|F_n|+|F_m|)dxdt,
\end{aligned}
$$

where $\{F_n\}$ is a bounded sequence in $L^1(B_{2\rho} \times (0,T))$ (its bound depends on bounds already obtained for the sequences $\{u_n\}$ and $\{Du_n\}$).

From now on, we can follow the proof of [2] and we can show that the sequence $\{Du_n\}$ is a Cauchy sequence in measure on $B_\rho \times [0,T]$.

Thus, up to a subsequence, the sequence $\{Du_n\}$ converges a.e. to Du. The passage to the limit is now standard. The improved regularity results of Theorem 4.1 and Corollary 4.2 follow in the same way from the estimates we derived in the last part of Section 5.

8 Variational framework for the case $p > N$

When $p > N$ we expect better regularity, as happens in the elliptic case. Indeed, it has been observed in [7] that an estimate for $|Du|$ in $L^q_{\rm loc}(\mathbb{R}^N)$ together with an estimate for u in $L^s_{\rm loc}(\mathbb{R}^N)$ implies an estimate for u in $L^\infty_{\rm loc}(\mathbb{R}^N)$ for the solutions of the elliptic equation (solutions of our problem with no time dependence), and then we get the regularity $|Du| \in L^p_{\rm loc}(\mathbb{R}^N)$, which makes the theory with right-hand side in $L^1_{\rm loc}(\mathbb{R}^N)$ fall into the usual variational framework. However, in the parabolic case there is an extra complication due to the presence of the time variable, so that the regularity of the gradient $|Du| \in L^1(0,T;L^q_{\rm loc}(\mathbb{R}^N))$, $q > N$, is not enough to guarantee the local boundedness of u, and the corresponding statement is not so clear. Indeed, we have the following improvement of Theorem 3.1.

Theorem 8.1 *For $s > p-1$ and $p > N$ we also get the regularity $u \in L^r(0,T;W^{1,q}_{\rm loc}(\mathbb{R}^N)) \cap L^r(0,T;L^\infty_{\rm loc}(\mathbb{R}^N))$ under the conditions $N < q < p$, $1 < r < p-1$.*

Proof. We re-do the interpolation step of Section 6 for $p > N$. In this case we may select q in the range $N < q < p$. The fact that $q > N$ implies by Morrey's inequality, cf. [11, Theorem 4.5.3.3], that we have an estimate of the form

$$\|u\|_{L^\infty(B_\rho)} \le C + C\Big(\int_{B_\rho} |Du|^q\,dx\Big)^{1/q}, \tag{8.1}$$

where C depends only on ρ, q and the L^1 norm of u in B_ρ, which is uniformly bounded independently of R thanks to the previous estimate on u in $L^\infty(0,T;L^1_{\rm loc}(\mathbb{R}^N))$. Continuing as before, we can write for $r < p$

$$\begin{aligned}
&\int_0^T \Big(\int_{B_\rho} |Du|^q\,dx\Big)^{r/q} dt \\
&\le \Big(\int_0^T\int_{B_\rho} \frac{|Du|^p}{(1+|u|)^{m+1}}\,dx\,dt\Big)^{r/p} \\
&\quad \times\Big(\int_0^T \Big(\int_{B_\rho} (1+|u|)^{(m+1)q/(p-q)} dx\Big)^{r(p-q)/q(p-r)} dt\Big)^{(p-r)/p},
\end{aligned}$$

where the first integral of the right-hand side is bounded independently of R by estimate (5.6) and the second term is also bounded if $q < p/2$. In that case we conclude as before. Otherwise, for $q \geq p/2$ we estimate this last factor by

$$\begin{aligned} & C\Big(\int_0^T (1+\|u\|_{L^\infty(B_\rho)})^{(m+1)r/(p-r)} dx)\, dt\Big)^{(p-r)/p} \\ \leq\ & CT^{(p-r)/p} + C\Big(\int_0^T \Big(\int_{B_\rho} |Du|^q\, dx\Big)^{(m+1)r/(p-r)q} dt\Big)^{(p-r)/p}, \end{aligned}$$

where we have used (8.1). Imposing the condition $r < p-1$ and choosing m small enough we can get an exponent $(m+1)r/(p-r)q < r/q$, so that after applying a suitable Hölder inequality we conclude the integral

$$\int_0^T \Big(\int_{B_\rho} |Du|^q\, dx\Big)^{r/q} dt$$

is bounded, and so is the integral we wanted to estimate. In conclusion, u admits an a priori estimate in $L^r(0,T;W^{1,q}_{\rm loc}(\mathbb{R}^N))$ for every $N < q < p$ and $r < p-1$. Then $u \in L^r(0,T;L^\infty_{\rm loc}(\mathbb{R}^N))$.

Locally bounded solutions can be obtained if f has a stronger regularity, as in the next result.

Theorem 8.2 *If $f \in L^{p'}(0,T;L^1_{\rm loc}(\mathbb{R}^N))$, $p' = p/(p-1)$, then the solutions of problem (2.1)-(2.2) are locally bounded and $|Du|$ belongs to $L^p(0,T;L^p_{\rm loc}(\mathbb{R}^N))$.*

The proof is based on the observation that $L^{p'}(0,T;L^1(\mathbb{R}^N))$ is included in the dual of the variational space $L^p(0,T;W^{1,p}(\mathbb{R}^N))$, plus a localization of the estimates. Under the general assumption $f \in L^1(0,T;L^1_{\rm loc}(\mathbb{R}^N))$ the previous methods do not produce locally bounded solutions.

9 Uniqueness, uniqueness classes and the Maximum Principle

The question of uniqueness of the solutions constructed in this paper is quite open in the general nonlinear framework, as it is the analogous question for elliptic problem with locally integrable data considered in [7]. The uniqueness question can be solved however when the second-order operator is linear and regular enough and the zero-order part depends monotonically on u. This was proved for the elliptic case by Brezis (see [8]).

Theorem 9.1 *Let u_0 and f be locally integrable functions as before and let $g(x,t,u)$ be as before and also monotone nondecreasing in u. The solution of the equation*

$$u_t - \Delta u + g(x,t,u) = f \tag{9.1}$$

under the above conditions is unique.

Proof. Let us consider two solutions, u_1 and u_2. We take the difference, $u = u_1 - u_2$, which is a weak solution of the equation

$$u_t = \Delta u - (g(x,t,u_1) - g(x,t,u_2)). \tag{9.2}$$

Since $u_t - \Delta u$ is a locally integrable function, a parabolic version of Kato's inequality gives

$$|u|_t - \Delta|u| \leq (u_t - \Delta u)\,\mathrm{sign}(u),$$

in the sense of distributions. We now multiply (9.2) by $\phi = \mathrm{sign}(u_1 - u_2)$ and use the monotonicity of g. We conclude that

$$|u|_t \leq \Delta|u| \quad \text{in} \;\; \mathcal{D}'(Q).$$

Since $|u| \geq 0$ and its initial trace is 0, standard heat equation theory (cf. e.g. [12]) implies that $|u| = 0$, hence $u_1 = u_2$.

Double monotone limits and the Maximum Principle. Let us now discuss the existence of a class of solutions which obeys the Maximum Principle when the zero-order part depends monotonically on u. First, the M. P. holds for the appoximate problems, for instance when the data are bounded and compactly supported. Next, we observe that whenever u_0 and f are assumed to be locally bounded from below we may use any uniform method of monotone approximation from below and obtain in the limit weak solutions which inherit from the approximations the M. P. property. We can call this class a class of *minimal solutions obtained as limits of approximations*, MSOLA for short. Finally, for general u_0 and f we perform a second step of monotone approximation from above with MSOLA's and obtain in the limit solutions with the M. P. property.

The same conclusions can be reached by performing the monotone limits in the converse order. The classes need not coincide if uniqueness fails.

10 Comments and extensions

We have shown the existence of a local theory that applies to parabolic equations in divergence form with certain structure conditions involving power growth. The idea is presented in the simplest representative case where a technique is needed to deal with gradient-dependent diffusivity. Naturally, the theory has a number of extensions where similar ideas will work. Let us comment on some of them.

A possible extension case would be the so-called *doubly nonlinear equations*, like

$$u_t = \mathrm{div}(u^{m-1}|Du|^{p-2}Du) + f,$$

that includes for different exponents m and p the heat equation, the porous medium equations and p-Laplacian equations. Diffusion equations with free boundaries like the Stefan free-boundary problem could also be considered.

Another line would be to consider more general growth conditions in the assumptions on A and g, but then we would lose the optimal regularity results in Sobolev spaces.

In another direction, we have chosen not to include in this presentation the cases where p is close to 1 and the concept of entropy or renormalized solution is needed along with additional technical work, cf. [1, 2, 4] for previous work.

Finally, we also point out to the interplay of our local existence results with convective terms, i.e. first-order additional terms in the equations of the form $v \cdot f(u)$, where v is a vector in $\mathbb{R}^N$ and $f = (f_1, \cdots, f_N)$ is a vector-valued function of u, and possibly x, t, cf. global results in [3, 10, 18] and listed references.

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