Monotone solutions of a nonautonomous differential equation for a sedimenting sphere *

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Abstract

We study a class of integrodifferential equations and related ordinary differential equations for the initial value problem of a rigid sphere falling through an infinite fluid medium. We prove that for creeping *Newtonian* flow, the motion of the sphere is monotone in its approach to the steady state solution given by the Stokes drag. We discuss this property in terms of a general nonautonomous second order differential equation, focusing on a decaying nonautonomous term motivated by the sedimenting sphere problem.

1 Introduction

A rigid sphere falling through a viscous medium is a classic problem in fluid dynamics, which was first solved in the steady state for the limit of vanishingly small Reynolds number in an infinite domain by G. G. Stokes in 1851 [23]. The time-dependent approach to the steady state allows the partial differential equation for the sphere and fluid to be reduced to an integrodifferential equation for the motion of the sphere in an infinite medium. The physical effects included in this equation are the buoyancy which drives the motion, the inertia of the sphere, the viscous drag, an added mass term, and a memory or history integral [7].

In the case of a *Newtonian fluid*, the main effect of the memory integral on the dynamics is to modify the approach to steady state from exponential to algebraic. The integral also makes the equation effectively second order, though it is generally accepted that no oscillations occur as the sphere reaches its steady state value [7, 25]. Physically it is clear that no oscillations can occur due to the absence of a restoring force against gravity, and a sphere released from rest in a Newtonian fluid at low Reynolds number is observed to reach its terminal velocity monotonically. However, it is not directly evident mathematically that
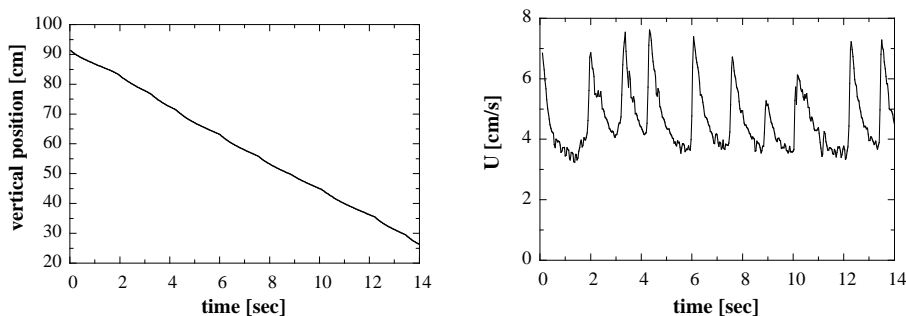

Figure 1: Motion of a 1/8" diameter teflon sphere falling through an aqueous solution of 6 mM CTAB/NaSal: (left) position vs time; (right) calculated velocity vs time [13].

oscillating solutions are precluded, particularly as the governing integrodifferential equation can be transformed to a nonautonomous second order ordinary differential equation which has the form of a harmonic oscillator [7, 24].

In a *non-Newtonian fluid*, such as a polymer solution, a falling sphere is often observed to undergo transient oscillations before reaching its terminal velocity [1, 25]. These oscillations occur due to the elasticity of the fluid, which provides a restoring force [5]. The steady state value is of primary concern in many applications, and much work focuses only on this aspect of the problem. The oscillations which occur during the approach to steady state have been reproduced in a linear viscoelastic model by King and Waters [15].

More recently, *nontransient* oscillations of falling spheres (and rising bubbles) have been observed in specific aqueous solutions of surfactants (wormlike micellar solutions) [3, 13]. These observations were initially made for a bubble in the wormlike micellar fluid CTAB/NaSal [12, 21], which showed oscillations in its position and shape. The shape oscillations included an apparent cusp which momentarily appears at the trailing end of the bubble. Such a cusplike tail is a well known property of rising bubbles in non-Newtonian fluids [11, 17], which we initially believed to play an important role in the micellar oscillations.

Subsequent observations of solid spheres which also oscillate while falling through the same solutions made it clear that the cusp is not involved in the phenomenon, and that another explanation must be sought. Unlike a sedimenting sphere in a conventional non-Newtonian fluid, these oscillations do not appear to be transient [13]. An example is given in Figure 1, which shows the motion of a 1/8" teflon sphere falling through a tube ($L = 98$ cm, $R = 3.2$ cm) filled with a 6 mM 1:1 solution of CTAB/NaSal [13].

Our attempts to model this phenomenon brought to our attention the unusual aspects of the integrodifferential equation for a falling sphere. We prove here that the equation for sedimenting sphere in a Newtonian fluid in the limit of zero Reynolds number (creeping flow) does not admit oscillating solutions, de-

spite some appearances that it does. This result is due to the special properties of the error function when multiplied by oscillating functions. It is ultimately related to the stability of nonautonomous ordinary differential equations with monotone secular terms, which is appropriately viewed as an initial value problem, and not in terms of linear stability analysis around the terminal velocity.

2   The Motion of a Sedimenting Sphere

We begin by reviewing some classical results for the equation of motion governing a falling sphere in a viscous Newtonian fluid of infinite extent (no boundaries).

2.1   Equation of Motion of the Sphere

An incompressible fluid in the absence of body forces is described by the equations

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right) = -\nabla p + \operatorname{div}\sigma, \tag{2.1}$$

$$\operatorname{div}\vec{u} = 0, \tag{2.2}$$

where $\rho(\vec{x}, t)$ is the density of the fluid, p is the pressure, $\vec{u}(\vec{x}, t)$ is the velocity field for the fluid, and $\sigma(\vec{x}, t)$ is the extra stress tensor, which measures force per unit area (other than pressure) in the present configuration of the fluid. A Newtonian fluid is a fluid for which the stress tensor σ is linearly related to the rate of strain tensor D through the relation

$$\sigma = 2\mu D, \tag{2.3}$$

where μ is the viscosity of the fluid and $D = (\nabla\vec{u} + (\nabla\vec{u})^T)/2$ is the symmetric part of the velocity gradient $\nabla\vec{u}$. From (2.1),(2.2), and (2.3) one obtains the Navier-Stokes equation:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right) = -\nabla p + \mu\Delta\vec{u}. \tag{2.4}$$

Non-Newtonian fluids are fluids for which the assumption (2.3) is invalid. For instance, polymeric and viscoelastic fluids often fail to conform to the instantaneous relation between stress and velocity gradients implicit in (2.3). In general $\sigma = \sigma(t, \vec{x}, D)$ will depend nonlinearly on D and on the past history of stress in the fluid.

By choosing an appropriate time and length scale, (2.4) can be written in a nondimensional form

$$\frac{\partial \tilde{u}}{\partial \tau} + \operatorname{Re}\,(\tilde{u}\cdot\nabla)\tilde{u} = -\nabla\tilde{p} + \Delta\tilde{u}, \tag{2.5}$$

$$\operatorname{div}\tilde{u} = 0, \tag{2.6}$$

where, $\tilde{u}$ and $\tilde{p}$ are the nondimensional velocity and pressure. The dimensionless constant Re is called the Reynolds number and it measures the relative importance of inertial effects to that of viscous effects. When the inertial effects are negligible (Re = 0), equation (2.4) is called the Stokes equation. In this paper we restrict our analysis to this situation.

Stokes solved the steady version of (2.5)-(2.6) for the case of sphere falling though the fluid for vanishing Reynolds number. The Stokes solution gives the steady state drag on the sphere of radius R falling through a fluid with a steady speed U to be $F = 6\pi\mu RU$. However, in order to solve the transient problem of falling sphere, we first solve the problem of sphere oscillating with a frequency ω and compute the drag experienced by the sphere as a function of ω. The drag experienced by a sphere falling at a arbitrary speed $U(t)$ can then be computed as a Fourier integral of this drag.

Consider a sphere of radius R and density ρ_s in a Newtonian fluid of density ρ and viscosity μ. The exterior Stokes flow driven by small oscillations of the sphere at a frequency ω can be solved exactly [2, 16], which leads to a hydrodynamic force dependent on both U and dU/dt :

$$F = 6\pi\mu R\left(1 + \frac{R}{\delta}\right)U + 3\pi R^2 \rho\delta\left(1 + \frac{2R}{9\delta}\right)\frac{dU}{dt}, \tag{2.7}$$

where $\delta = \sqrt{2\nu/\omega}$ is a diffusive lengthscale common to Stokes problems, and $\nu = \mu/\rho$ is the kinematic viscosity. Using this, the general time-dependent problem of the motion of a falling sphere can be reduced from a partial to an ordinary differential equation for the speed $U(t)$ of the sphere, an exact equation which takes into account the motion of the surrounding fluid [7, 24]. For a sphere moving with an arbitrary speed $U(t)$, the hydrodynamic drag it experiences can be calculated by representing $U(t)$ as a Fourier integral:

$$U(t) = \int_{-\infty}^{\infty} U_\omega e^{-i\omega t}\, d\omega.$$

The drag for each Fourier component is then given by (2.7). The total hydrodynamic drag on the sphere is obtained by integrating over all Fourier components, leading to

$$F_{drag} = 6\pi\mu RU(t) + \frac{1}{2}\rho\mathcal{V}\frac{dU}{dt} + 6\pi\rho R^2\sqrt{\frac{\nu}{\pi}}\int_{-\infty}^{t}\frac{U'(s)}{\sqrt{t-s}}\, ds \tag{2.8}$$

where the first term represents the steady state drag on a sphere falling with a velocity U, the second term represents the added mass term (the force required to accelerate the surrounding fluid), the third term is the Basset memory term, and $\mathcal{V} = 4\pi R^3/3$ is the volume of the sphere. If the sphere starts from rest, then the lower limit of the integral in (2.8) starts from $\tau = 0$ instead of $\tau = -\infty$. The expression for the unsteady drag force can then be substituted into the balance of force equation for the sphere:

$$\rho_s\mathcal{V}U'(t) = F_{buoy} - F_{drag}.$$

Thus the equation of motion for the sphere is

$$(\rho_s + \frac{\rho}{2})\mathcal{V}U'(t) \; + \; 6\pi\mu R U(t) \; + \; 6\pi R^2 \sqrt{\frac{\rho\mu}{\pi}} \int_0^t \frac{U'(s)}{\sqrt{t-s}} \, ds$$
$$= \; (\rho_s - \rho)\mathcal{V}g, \tag{2.9}$$

which can be rewritten in the simpler form

$$U'(t) + BU(t) + Q \int_0^t \frac{U'(s)}{\sqrt{t-s}} \, ds = M, \tag{2.10}$$

where

$$B = \frac{9\mu}{R^2 (2\rho_s + \rho)}, \tag{2.11}$$

$$Q = \frac{9\rho}{R (2\rho_s + \rho)} \sqrt{\frac{\mu}{\rho\pi}}, \tag{2.12}$$

$$M = \frac{2g\Delta\rho}{2\rho_s + \rho}, \tag{2.13}$$

and $\Delta\rho = \rho_s - \rho$ is the density difference which drives the motion. In this approach the motion of the sphere is described by an integrodifferential equation whose integral term has the same singularity as Abel's equation [14, 8]. Note that this equation is only valid in the limit of zero Reynolds number [19, 18].

Physically one expects the solution $U(t)$ to approach a terminal velocity. It is clear from (2.10) that the only steady state solution ($U' = 0$) possible is

$$U_0 = \frac{M}{B} = \frac{2\Delta\rho g R^2}{9\mu}, \tag{2.14}$$

which is the classical result of balancing the Stokes drag with the buoyancy.

2.2   Solving the Integrodifferential Equation

We first rewrite the integrodifferential equation (IDE) for the sphere in a nondimensional form using U_0 as the velocity scale, and $1/B$, the viscous diffusion time, as the time scale. The variables are

$$\tau = Bt \qquad \text{and} \qquad u(\tau) = U(\tau/B)/U_0.$$

With this rescaling (2.10) becomes

$$u'(\tau) + u(\tau) + \sqrt{\frac{\kappa}{\pi}} \int_0^\tau \frac{u'(s)}{\sqrt{\tau-s}} \, ds = 1, \tag{2.15}$$

where the control parameter κ is given by

$$\kappa = \frac{\pi Q^2}{B} = \frac{9\rho}{2\rho_s + \rho}. \tag{2.16}$$

Thus the motion of the sphere depends only on the relative densities of the sphere and the fluid through the parameter κ. The density of the sphere ρ_s can range from zero to infinity, which implies a parameter range $0 < \kappa < 9$. When the density of the sphere is equal to the density of the fluid, $\kappa = 3$ and $U_0 = 0$. Here we will only be concerned with falling spheres, for which $0 < \kappa < 3$.

Although we will solve the IDE (2.15) directly, it is of interest to connect the problem to ordinary differential equations (ODE's) and discuss some important consequences therein, especially with regard to the stability of the terminal velocity solution. Following Villat [24], we can rewrite (2.15) as an ODE using Abel's Theorem (see Appendix):

$$u'' + (2 - \kappa)u' + u = 1 - \sqrt{\frac{\kappa}{\pi\tau}}, \tag{2.17}$$

$$u(0) = 0, \quad u'(0) = 1. \tag{2.18}$$

More general initial conditions $U(0) \neq 0$ and $U'(0) = M - BU(0)$ lead to a slightly different ODE:

$$u'' + (2 - \kappa)u' + u = 1 + \sqrt{\frac{\kappa}{\pi\tau}}(u(0) - 1), \tag{2.19}$$

$$u(0) = \xi, \quad u'(0) = 1 - \xi. \tag{2.20}$$

Note that it follows from equation (2.15) that $u'(0)$ is prescribed in terms of $u(0)$, and thus the second order ODE we have obtained requires only one initial condition.

Since we are investigating the possibility of steady-state oscillations of a sedimenting sphere, we are primarily concerned with the asymptotic behavior of (2.17). Moreover, since the nonautonomous term tends to zero as $t \to \infty$, one might expect the stability of (2.17) to mimic the homogeneous problem. With this in mind, let α and β denote the roots of the characteristic equation

$$m^2 + (2 - \kappa)m + 1 = 0. \tag{2.21}$$

It is readily verified that (2.21) has complex roots for $0 < \kappa < 4$. Moreover, the roots have positive real parts for $2 < \kappa < 4$. Since the relevant range of κ for a falling sphere is $0 < \kappa < 3$, one sees that oscillations are not a priori precluded. In terms of the actual densities of the fluid and sphere the condition for complex roots corresponds to $\rho_s > (5/8)\rho$, which is true in the case of a heavy sphere falling through a lighter liquid ($\rho_s > \rho$). If additionally $\rho_s < (7/4)\rho$, then the complex roots have positive real parts. If we rewrite (2.17) in the asymptotic limit ($t \to \infty$) as a first order linear system

$$x' = y \tag{2.22}$$
$$y' = (1 - x) + (\kappa - 2)y, \tag{2.23}$$

where $x = u$, and $y = u'$, then $(x, y) = (1, 0)$ is the unique equilibrium point, which corresponds to the terminal velocity. The eigenvalues of this system are precisely α and β, whence the equilibrium point becomes unstable.

Nonetheless, as we will show, even in this range $(2 < \kappa < 3)$, the solution to the full equation (2.17) *monotonically* approaches the value 1, corresponding to the monotonic approach to the steady Stokes value (2.14) for the actual velocity U. Clearly the nonautonomous term continues to play a dominant role in the stability of (2.17), despite its algebraic approach to zero.

To solve for $u = u(\tau)$, we return to the IDE (2.15) and apply the Laplace transform in the case $u(0) = 0$:

$$\mathcal{L}\{u\}(s) = \frac{1}{s(s + \sqrt{\kappa}\sqrt{s} + 1)} \quad \text{or} \quad \mathcal{L}\{u'\}(s) = s\mathcal{L}\{u\} = \frac{1}{s + \sqrt{\kappa}\sqrt{s} + 1}.$$

Since $\alpha\beta = 1$ and $\sqrt{\alpha} + \sqrt{\beta} = \sqrt{\kappa}$, we may express this last equation in the form

$$\mathcal{L}\{u'\}(s) = \frac{\sqrt{\kappa}}{\alpha - \beta}\left[\frac{\sqrt{\alpha}}{\sqrt{s}(\sqrt{s} + \sqrt{\alpha})} - \frac{\sqrt{\beta}}{\sqrt{s}(\sqrt{s} + \sqrt{\beta})}\right].$$

Moreover, the identity

$$\mathcal{L}\{e^{\alpha t}\text{Erfc}\sqrt{\alpha t}\} = \frac{1}{\sqrt{s}(\sqrt{s} + \sqrt{\alpha})},$$

implies

$$u'(\tau) = \frac{\sqrt{\kappa}}{\alpha - \beta}\left[\sqrt{\alpha}e^{\alpha\tau}\text{Erfc}\sqrt{\alpha\tau} - \sqrt{\beta}e^{\beta\tau}\text{Erfc}\sqrt{\beta\tau}\right]. \tag{2.24}$$

Finally, since

$$\sqrt{\alpha}\int e^{\alpha t}\text{Erfc}\sqrt{\alpha t}\,dt = 2\sqrt{\frac{t}{\pi}} + \frac{1}{\sqrt{\alpha}}e^{\alpha t}\text{Erfc}\sqrt{\alpha t} + C,$$

we find the solution

$$u(\tau) = 1 + \frac{\sqrt{\kappa}}{\alpha - \beta}\left[\frac{e^{\alpha\tau}\text{Erfc}\sqrt{\alpha\tau}}{\sqrt{\alpha}} - \frac{e^{\beta\tau}\text{Erfc}\sqrt{\beta\tau}}{\sqrt{\beta}}\right]. \tag{2.25}$$

This solution to the IDE (2.15) is also the solution to the ODE (2.17)-(2.18). Applying transform methods to the more general set of equations defined by (2.19)-(2.20) one finds the solution

$$u(\tau) = (1 - \epsilon)u_0(\tau) + \epsilon \qquad \epsilon \equiv u(0), \, u'(0) = 1 - \epsilon, \tag{2.26}$$

where $u_0(\tau)$ is the solution defined by (2.25). Note that the solution for arbitrary initial velocity ϵ is a simple rescaling of the solution for the sphere initially at rest. It is not obvious from the form of u in (2.25) that the values approach 1 monotonically. Let us first investigate the asymptotic behavior of this solution.

2.3   Asymptotic Approach to the Steady Stokes Solution

Finding the asymptotic behavior of the solution u_0 is straightforward. We employ the asymptotic expansion of the error function [9]

$$e^{z^2}\text{Erfc}\,z \sim \frac{1}{\sqrt{\pi}z}\left[1 + \sum_{m=1}^{\infty}(-1)^m\frac{1\cdot 3\cdots(2m-1)}{(2z^2)^m}\right] \tag{2.27}$$

as $z \to \infty$, provided $|\arg(z)| < 3\pi/4$. Since $0 < \arg(\sqrt{\alpha t}) < \pi/2$ for $0 < \kappa < 3$, we see that asymptotically

$$\text{Erfc}\sqrt{\alpha\tau} \sim \frac{e^{-\alpha\tau}}{\sqrt{\alpha\tau}}.$$

Thus the product of the exponential term and the error function approaches zero in the limit $t \to \infty$. Using this expansion in (2.25) we obtain

$$\lim_{\tau\to\infty} u_0(\tau) = 1 \quad \text{or} \quad \lim_{t\to\infty} U(t) = U_0. \tag{2.28}$$

As the solution for any initial condition is a rescaling of u_0, we see this limit applies for all values of $u(0)$.

Although the asymptotics of this solution are clear, the transient solution has some unusual properties. Numerical simulation of the IDE (2.15), or even attempts to plot the analytic solution (2.25), eventually blow up at large τ when κ is in the unstable range. Clearly the cancellation between the exponentially growing and decreasing terms are quite sensitive to numerical errors. This is an indication that the product $e^{\alpha\tau}\text{Erfc}\sqrt{\alpha\tau}$ should be considered as a special function with its own properties.

3   Monotonicity of the Transient Solution

We begin with the transient solution to the IDE or ODE considered in the previous section. In addition to the insensitivity of the transient solution to the real part of the homogeneous roots, it is surprising that the nonzero complex part does not lead to *any* oscillations in the velocity of the sphere, although there has sometimes been confusion on this point regarding transient oscillations [24]. Experimentally the sphere in a Newtonian fluid has never been observed to oscillate, in contradistinction to most non-Newtonian (particularly elastic) fluids [1, 25]. We will show that the solution u defined by (2.25) is monotone as a function of τ for all $\kappa \in (0, 4)$. Although this may be well known, we have not yet found a reference to any proof other than the case $\alpha, \beta \in \mathbb{R}$ [26], which corresponds to $\kappa > 4$.

Let us define the function $\text{Vi} : \mathbb{C} \to \mathbb{C}$ by

$$\text{Vi}(z) \doteq e^z \text{Erfc}\sqrt{z} = \frac{2e^z}{\sqrt{\pi}} \int_{\sqrt{z}}^{\infty} e^{-s^2} \, ds. \tag{3.1}$$

We shall refer to Vi as the Villat function, since this combination appeared in the explicit solution of the differential equation for the falling sphere problem by Villat [24]. Closely related to $\text{Vi}(z)$ is the "plasma dispersion function" w, defined by [9]:

$$w(z) = e^{-z^2}\text{Erfc}(-iz).$$

In fact, $\text{Vi}(\alpha t) = w(i\sqrt{\alpha t})$. Using the Villat function we may now prove the main theorem.

Theorem 3.1 (Monotonicity) *For each $\kappa \in (0,4)$, the function*

$$u(t) = 1 + \frac{\sqrt{\kappa}}{\alpha - \beta} \left[\frac{e^{\alpha t} \, Erfc\sqrt{\alpha t}}{\sqrt{\alpha}} - \frac{e^{\beta t} \, Erfc\sqrt{\beta t}}{\sqrt{\beta}} \right] \tag{3.2}$$

approaches the limit 1 monotonically.

Proof. We have shown that $u(t) \to 1$, thus it remains to show it does so monotonically. We will demonstrate this by proving $u'(t) > 0$ for all $t > 0$. To this end, fix $t > 0$, $\kappa \in (0,4)$ and recall that α and β denote the conjugate pair of roots of the polynomial $m^2 + (2 - \kappa)m + 1$. Recall from (2.24) that

$$u'(t) = \frac{\sqrt{\kappa}}{\alpha - \beta} \left[\sqrt{\alpha} \, \mathrm{Vi}(\alpha t) - \sqrt{\beta} \, \mathrm{Vi}(\beta t) \right].$$

Since $\mathrm{Erfc}(\overline{z}) = \overline{\mathrm{Erfc}\, z}$, it follows that $\mathrm{Vi}(\beta t) = \overline{\mathrm{Vi}(\alpha t)}$ and

$$u'(t) = \sqrt{\kappa} \, \frac{\sqrt{\alpha} \, \mathrm{Vi}(\alpha t) - \overline{\sqrt{\alpha} \, \mathrm{Vi}(\alpha t)}}{\alpha - \overline{\alpha}} = \sqrt{\kappa} \, \frac{\Im\{\sqrt{\alpha} \, \mathrm{Vi}(\alpha t)\}}{\Im\{\alpha\}}. \tag{3.3}$$

Since $\Im\{\alpha\} > 0$ for each $\kappa \in (0,4)$, it is evident from (3.3) that the sign of $u'(t)$ is determined by the imaginary part of the function $\sqrt{\alpha} \, \mathrm{Vi}(\alpha t)$. For the plasma dispersion function w introduced above, the real and imaginary parts are given by (see e.g., [9, 7.4.13-7.4.14])

$$\Re(w(x + iy)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y e^{-s^2}}{(x - s)^2 + y^2} \, ds \quad (x \in \mathbb{R}, y > 0)$$

and

$$\Im(w(x + iy)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x - s) e^{-s^2}}{(x - s)^2 + y^2} \, ds \quad (x \in \mathbb{R}, y > 0).$$

It is readily verified that $|\alpha| = 1$, thus in polar form we have $\alpha = e^{i\theta}$ for some fixed $\theta \in (0, \pi)$, in which case $\mathrm{Vi}(\alpha t) = w(i\sqrt{\alpha t}) = w(x + iy)$, where

$$x = -\sqrt{t} \sin\left(\frac{\theta}{2}\right) \qquad \text{and} \qquad y = \sqrt{t} \cos\left(\frac{\theta}{2}\right).$$

Using this information we compute

$$\begin{aligned}
\Im\{\sqrt{\alpha} \, \mathrm{Vi}(\alpha t)\} &= \cos\left(\frac{\theta}{2}\right) \Im\{w(x + iy)\} + \sin\left(\frac{\theta}{2}\right) \Re\{w(x + iy)\} \\
&= \frac{1}{\pi} \cos\left(\frac{\theta}{2}\right) \int_{-\infty}^{\infty} \frac{(x - s) e^{-s^2}}{(x - s)^2 + y^2} \, ds \\
&\quad + \frac{1}{\pi} \sin\left(\frac{\theta}{2}\right) \int_{-\infty}^{\infty} \frac{y e^{-s^2}}{(x - s)^2 + y^2} \, ds \\
&= \frac{1}{\pi} \cos\left(\frac{\theta}{2}\right) \int_{-\infty}^{\infty} \frac{-s e^{-s^2}}{(x - s)^2 + y^2} \, ds.
\end{aligned} \tag{3.4} \tag{3.5}$$

In the last step we have used the fact that $x\cos(\theta/2) + y\sin(\theta/2) = 0$. Since $\sqrt{\alpha}$ lies in the first quadrant, the prefactor of the last integral above is positive and we may conclude that $\Im\{\sqrt{\alpha}\,\mathrm{Vi}(\alpha t)\} > 0$ provided

$$\int_{-\infty}^{\infty} \frac{se^{-s^2}}{(x-s)^2 + y^2}\, ds = \int_{-\infty}^{\infty} \frac{se^{-s^2}}{(s + \sqrt{t}\sin\frac{\theta}{2})^2 + t\cos^2\frac{\theta}{2}}\, ds < 0. \qquad (3.6)$$

Let us denote the integrand as

$$F(s) = \frac{se^{-s^2}}{P(s)} \quad \text{where} \quad P(s) = \left(s + \sqrt{t}\sin\frac{\theta}{2}\right)^2 + t\cos^2\frac{\theta}{2}.$$

Note that $P(s) > 0$ for $s \in \mathbb{R}$ (recall $t > 0$ is fixed). The proof is complete once the following two observations are made:

  (a) $|F(-s)| > F(s)$ for $s > 0$;

  (b) $\int_{\mathbb{R}} F\, ds = \int_0^{\infty} F(s)\, ds - \int_0^{\infty} |F(-s)|\, ds$.

To see (a), notice for $s > 0$ we have $0 < P(-s) < P(s)$, thus

$$|F(-s)| = \frac{se^{-s^2}}{P(-s)} > \frac{se^{-s^2}}{P(s)} = F(s), \qquad \text{for } s > 0.$$

Observation (b) follows from a standard change of variables

$$\int_{\mathbb{R}} F(s)\, ds = \int_{-\infty}^{0} F(s)\, ds + \int_0^{\infty} F(s)\, ds = \int_0^{\infty} F(s)\, ds - \int_0^{\infty} |F(-s)|\, ds.$$

The two observations above imply the inequality (3.6) holds, in which case by (3.3) and (3.5) we see $u'(t) > 0$. Since $t > 0$ was arbitrary, the proof is complete. $\diamondsuit$

Corollary 3.2 *The solution to the initial value problem (2.19)-(2.20) monotonically approaches its steady state value $u = 1$.*

Proof.   The proof follows from applying Theorem 3.1 to equation (2.26). $\diamondsuit$

4   Related Aspects of the Newtonian Problem

To investigate the generality of the above result, consider the nonautonomous linear damped harmonic oscillator equation for $u = u(t)$

$$u'' + bu' + u = 1 - G(t), \qquad (4.1)$$

as an initial value problem with arbitrary initial conditions $u(0)$ and $u'(0)$. We are specifically interested in the case where $G(t) \to 0$ as $t \to \infty$, as opposed to

the often studied case where $G(t)$ is periodic (see e.g. [10]). The Newtonian sphere problem (2.17) is a special case of (4.1), with $b = 2 - \kappa$ and $G(t) = \sqrt{\kappa/\pi t}$. Making the change of variables $v = u - 1$, we may simplify the equation to

$$v'' + bv' + v = -G(t) \tag{4.2}$$

so that $v = 0$ solves the homogeneous equation. Note however that if $G(t) \neq 0$, then $v = 0$ is not a solution to (4.2) for any $t > 0$. We are interested in the following question: what conditions on $G(t)$ and b are necessary for a solution $v(t)$ to remain monotone, even within the regime of instability for the homogeneous equation.

As a first step in this direction we consider the following initial value problem for $t \geq 0$:

$$v'' + b\,v' + v = -\frac{A}{\sqrt{\pi(t + t_0)}}, \tag{4.3}$$

where $b, A \in \mathbb{R}$ and $t_0 \geq 0$ are constants. The motivation for this form is to test the necessity of the singularity at $t = 0$ in the monotonicity result of Section 3. To ensure complex roots, we assume $b \in (-2, 2)$.

Using variation of parameters one finds a particular solution of (4.3) to be

$$
\begin{aligned}
v_p(t) \;=\; & \frac{A}{\beta - \alpha} \left\{ \sqrt{\beta} e^{\alpha(t + t_0)} \left(\mathrm{Erfc}\sqrt{\alpha t_0} - \mathrm{Erfc}\sqrt{\alpha(t + t_0)} \right) \right. \\
& \left. - \sqrt{\alpha} e^{\beta(t + t_0)} \left(\mathrm{Erfc}\sqrt{\beta t_0} - \mathrm{Erfc}\sqrt{\beta(t + t_0)} \right) \right\}, \quad (4.4)
\end{aligned}
$$

where α and β are the roots of the characteristic polynomial $m^2 + bm + 1$. Employing the Villat function we may express equation (4.4) as

$$v_p(t) = \frac{A}{\beta - \alpha} \left\{ \sqrt{\beta}\mathrm{Vi}(\alpha t_0)e^{\alpha t} - \sqrt{\alpha}\mathrm{Vi}(\beta t_0)e^{\beta t} \right\} + A\,M(t + t_0),$$

where the function M is defined by

$$M(t) = \frac{1}{\alpha - \beta} \left\{ \sqrt{\beta}\mathrm{Vi}(\alpha t) - \sqrt{\alpha}\mathrm{Vi}(\beta t) \right\}. \tag{4.5}$$

In Section 3 we proved M approaches 0 monotonically for all $b \in (-2, 2)$. The general solution to (4.3) may be expressed as

$$
\begin{aligned}
v(t) \;=\; & C_1 e^{\alpha t} + C_2 e^{\beta t} + v_p(t) \tag{4.6} \\
=\; & \left\{ C_1 + \frac{\sqrt{\beta}A\mathrm{Vi}(\alpha t_0)}{\beta - \alpha} \right\} e^{\alpha t} + \left\{ C_2 - \frac{\sqrt{\alpha}A\mathrm{Vi}(\beta t_0)}{\beta - \alpha} \right\} e^{\beta t} \\
& + A\,M(t + t_0). \tag{4.7}
\end{aligned}
$$

This equation clearly demonstrates how the long term dynamics of $v(t)$ depend on the solution of the homogeneous problem. In particular, it shows that the solution will retain the stability properties of the homogeneous solution unless

the coefficients C_1 and C_2 are chosen to zero out the first two terms in (4.7). The unique choice of C_1 and C_2 for this to happen is

$$C_1 = -\frac{A\sqrt{\beta}}{\beta - \alpha}\mathrm{Vi}(\alpha t_0) \qquad \text{and} \qquad C_2 = \frac{A\sqrt{\alpha}}{\beta - \alpha}\mathrm{Vi}(\beta t_0). \qquad (4.8)$$

Moreover, it is clear from (4.7) that the solution in this case is $v(t) = A\,M(t + t_0)$, with $v(0) = A\,M(t_0)$ and $v'(0) = A\,M'(t_0)$. Thus, in this case, the solution is a translate of the monotone solution. The coefficients C_1 and C_2 are related to the initial conditions $v(0)$ and $v'(0)$ via

$$C_1 = \frac{\beta\,v(0) - v'(0)}{\beta - \alpha} \qquad \text{and} \qquad C_2 = \frac{v'(0) - \alpha\,v(0)}{\beta - \alpha}. \qquad (4.9)$$

The values of $v(0) = A\,M(t_0)$ and $v'(0) = A\,M'(t_0)$ may also be obtained by solving equations (4.8) and (4.9).

In summary, given $b \in (-2, 2)$, $A > 0$, and $t_0 \geq 0$, for the equation

$$v'' + b\,v' + v = -\frac{A}{\sqrt{\pi(t + t_0)}}, \qquad (4.10)$$

there exists a unique choice of initial values $v(0) = A\,M(t_0)$, $v'(0) = A\,M'(t_0)$ such that the solution $v(t)$ remains monotone for all $t > 0$. Therefore the presence of a singularity at $t = 0$ in the nonhomogeneous term is not necessary to obtain a monotone solution.

In light of the above analysis it becomes clear how the solution for the sedimenting sphere remains monotone in its approach to terminal velocity for *all* relevant values of κ (i.e., sphere densities). From equations (2.17)-(2.18) we see that for each value of κ, equation (4.10) describes the dynamics for the dimensionless velocity $v = u - 1$, with $b = 2 - \kappa$, $t_0 = 0$, and $A = \sqrt{\kappa}$. Moreover, for each $\kappa \in (0, 4)$ we have demonstrated that equation (4.10) with $t_0 = 0$, $b = 2 - \kappa$, and $A = \sqrt{\kappa}$, has a unique initial value for which the solution remains monotone, namely,

$$v(0) = AM(0) = \sqrt{\kappa}\,\frac{\sqrt{\beta} - \sqrt{\alpha}}{\alpha - \beta} = -\frac{\sqrt{\kappa}}{\sqrt{\alpha} + \sqrt{\beta}}, \qquad (4.11)$$

where α and β denote the roots of the polynomial $m^2 + (2 - \kappa)m + 1$. However, since $\sqrt{\alpha}$ lies in the first quadrant, $\sqrt{\alpha} + \sqrt{\beta} > 0$, and the computation

$$(\sqrt{\alpha} + \sqrt{\beta})^2 = \alpha + \beta + 2 = -b + 2 = \kappa,$$

together with (4.11), implies

$$v(0) = -1.$$

In other words, the particular relation between the parameters A and b decouples $v(0)$ from all parameters, so that one obtains a monotone solution for all values of the sphere density.

We conclude this section with a geometric interpretation of the monotonicity result. In particular, we focus on the interesting case of (4.3) when the parameter $b \in (-2, 0)$. For these parameter values the solution $v = 0$ of the homogeneous problem is unstable. We have shown that there is a unique set of initial conditions that defines a solution to the nonhomogeneous problem which approaches the unstable fixed point $v = 0$ monotonically, despite the surrounding instabilities. Thus we return to the fundamental puzzle posed in Section 2.3: How is it that the nonautonomous term in (4.3), which decays to zero as $t \to \infty$, can "stabilize" a trajectory for all $t > 0$, in the sense that this solution approaches 0 while all other trajectories diverge due to the instability of the linearized problem? The following observation resolves the puzzle. First, recall that the unique initial conditions for which the nonhomogeneous problem remains monotone are defined by

$$v(0) = A\,M(t_0) \qquad \text{and} \qquad v'(0) = AM'(t_0).$$

Second, note that as the amplitude A of the nonhomogeneous term tends to zero, the initial conditions $(A\,M(t_0), A\,M'(t_0))$ approach $(0, 0)$. This corresponds to the initial condition starting on the unstable equilibrium point, which is the unique initial condition for the homogeneous problem whose solution does not diverge. In other words we have a correspondence between the trajectories of the homogeneous equation and the nonhomogeneous equation, which is continuous with respect to the parameter A. The monotone solution is then the image of the unstable fixed point under this map.

Summary and Conclusion

In this paper we have studied the ODE model for a sphere falling through a Newtonian fluid. We have proven that the equations do not admit oscillations, even in the transient, in agreement with general experimental observations.

From our analysis it appears that the lack of oscillations is due to a delicate balance of terms. It is tempting to conclude that an oscillating motion could be produced with only a slight modification to the equations. However it is important that the solution still remain bounded, and as we have shown there is only one trajectory which is insensitive to the linear instability ($\Re(\alpha) > 0$) of the homogeneous equation. Transient oscillations of a falling sphere have been successfully modeled by King & Waters using an elastic constitutive model [15], for which a final steady state velocity is approached.

In principle, however, one cannot simply modify the differential equation (2.17) or even (2.15) to address the oscillations of a sedimenting sphere in a micellar fluid [3, 13]; one must return to the full time-dependent partial differential equation. This was indeed how King & Waters obtained their result for a linear viscoelastic constitutive model [15], but it is not clear that this approach will continue to be fruitful as the complexity of the problem increases. Self-assembling wormlike micellar solutions are thought to have a nonmonotonic stress/shear rate relation [22, 20], based on the existence of an apparently inaccessible range of shear rates [20, 6]. It may be that the dynamics of such a

nonlinear fluid requires the spatial information inherent in the PDEs, and that the ODE reduction discussed here is practically limited to linear models.

5 Appendix: Derivation of Eqns. (2.17)-(2.18)

The equation describing the transient motion of a falling sphere is

$$u' + u + \sqrt{\frac{\kappa}{\pi}} \int_0^t \frac{u'(s)}{\sqrt{t-s}} ds = 1, \tag{5.1}$$

where $u(t)$ is the velocity of the sphere and κ is a non-dimensional parameter which depends on the relative densities of the sphere and the fluid. This integro-differential equation can be converted to a second order ODE through the following procedure. If

$$F(t) = \int_0^t \frac{u'(s)}{\sqrt{t-s}} ds,$$

then Abel's theorem (see e.g., [14, §3.7]) implies

$$\int_0^t \frac{F(\tau)}{\sqrt{t-\tau}} d\tau = \pi \left[u(t) - u(0) \right]. \tag{5.2}$$

Multiplying (5.1) by $1/\sqrt{t-\tau}$, integrating, and using (5.2) yields the equation

$$\int_0^t \frac{u'(\tau)}{\sqrt{t-\tau}} d\tau + \int_0^t \frac{u(\tau)}{\sqrt{t-\tau}} d\tau + \pi\sqrt{\frac{\kappa}{\pi}} \left[u(t) - u(0) \right] = \int_0^t \frac{1}{\sqrt{t-\tau}} d\tau. \tag{5.3}$$

From (5.1) one observes

$$\int_0^t \frac{u'(\tau)}{\sqrt{t-\tau}} d\tau = \sqrt{\frac{\pi}{\kappa}} \left(1 - u - u' \right). \tag{5.4}$$

Substituting this into (5.3) and rewriting yields

$$u' = \left(1 - 2\sqrt{\frac{\kappa t}{\pi}} \right) + (\kappa - 1) u - \kappa u(0) + \sqrt{\frac{\kappa}{\pi}} \int_0^t \frac{u(\tau)}{\sqrt{t-\tau}} d\tau. \tag{5.5}$$

The desired second order differential equation is now obtained by differentiating (5.5). In this regard, note that the substitution $\tau = t - x^2$ implies

$$I(t) = \sqrt{\frac{\kappa}{\pi}} \int_0^t \frac{u(\tau)}{\sqrt{t-\tau}} d\tau = 2\sqrt{\frac{\kappa}{\pi}} \int_0^{\sqrt{t}} u(t - x^2) \, dx, \tag{5.6}$$

thus

$$\begin{aligned}
\frac{dI}{dt} &= \sqrt{\frac{\kappa}{\pi}} \frac{u(0)}{\sqrt{t}} + 2\sqrt{\frac{\kappa}{\pi}} \int_0^{\sqrt{t}} u'(t - x^2) \, dx \\
&= \sqrt{\frac{\kappa}{\pi}} \frac{u(0)}{\sqrt{t}} + \sqrt{\frac{\kappa}{\pi}} \int_0^t \frac{u'}{\sqrt{t-\tau}} d\tau \\
&= \sqrt{\frac{\kappa}{\pi}} \frac{u(0)}{\sqrt{t}} + (1 - u - u'),
\end{aligned} \tag{5.7}$$

where again we have used (5.4).

Therefore differentiating (5.5) and using (5.7) yields the second order equation

$$u'' = (\kappa - 2)u' - u + \left[1 + \sqrt{\frac{\kappa}{\pi t}}(u(0) - 1)\right]. \tag{5.8}$$

Note that from (5.1) the initial value of u' is determined by the initial value of u, i.e., $u'(0) = 1 - u(0)$. Therefore, the equation describing the transient motion of the sphere is

$$u'' + (2 - \kappa)u' + u = 1 + \sqrt{\frac{\kappa}{\pi t}}(u(0) - 1), \tag{5.9}$$

$$u(0) = \xi, \quad u'(0) = 1 - \xi. \tag{5.10}$$

If the sphere starts from rest (i.e., $u(0) = 0$) then the system reduces to

$$u'' + (2 - \kappa)u' + u = 1 - \sqrt{\frac{\kappa}{\pi \tau}},$$

$$u(0) = 0, \quad u'(0) = 1,$$

which is precisely (2.17)-(2.18).

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