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A THEOREM OF ROLEWICZ'S TYPE FOR MEASURABLE EVOLUTION FAMILIES IN BANACH SPACES

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Abstract. Let φ be a positive and non-decreasing function defined on the real half-line and $\mathcal{U}$ be a strongly measurable, exponentially bounded evolution family of bounded linear operators acting on a Banach space and satisfing a certain measurability condition as in Theorem 1 below. We prove that if φ and $\mathcal{U}$ satisfy a certain integral condition (see the relation 1 from Theorem 1 below) then $\mathcal{U}$ is uniformly exponentially stable. For φ continuous and $\mathcal{U}$ strongly continuous and exponentially bounded, this result is due to Rolewicz. The proofs uses the relatively recent techniques involving evolution semigroup theory.

Let X be a real or complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators on X. Let $\mathbf{T} = \{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ be a strongly continuous semigroup on X and $\omega_0(\mathbf{T}) = \lim_{t\to\infty} \frac{\ln(\|T(t)\|)}{t}$ be its growth bound. The Datko-Pazy theorem ([4] , [10]) states that $\omega_0(\mathbf{T}) < 0$ if and only if for all $x \in X$ the maps $t \longmapsto \|T(t)x\|$ belongs to $L^p(\mathbb{R}_+)$ for some $1 \leq p < \infty$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that $\varphi(t) > 0$ for all $t > 0$. If for each $x \in X, ||x|| \leq 1$ the maps $t \mapsto \varphi(||T(t)x||)$ belongs to $L^1(\mathbf{R}_+)$ then $\mathbf{T}$ is exponentially stable, i.e. $\omega_0(\mathbf{T})$ is negative. This later result is due to J. van Neerven [8, Theorem 3.2.2.]. The Datko-Pazy Theorem follows from this by taking $\varphi(t) = t^p$ $(t \geq 0)$. Moreover, it is easily to see that the above Neerven's result remain true if we replace the strongly continuity assumption about $\mathbf{T}$ with strongly measurability and exponentially boundedness assumptions about $\mathbf{T}$.

A family $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\} \subset \mathcal{L}(X)$ is called an *evolution family* of bounded linear operators on X if $U(t,t) = I$ (the identity operator on X) and $U(t,\tau)U(\tau,s) = U(t,s)$ for all $t \geq \tau \geq s \geq 0$. Such a family is said to be *strongly continuous* if for every $x \in X$, the maps

$$(t,s) \mapsto U(t,s)x : \{(t,s) : t \geq s \geq 0\} \to X. \tag{1}$$

are continuous, and *exponentially bounded* if there are $\omega > 0$ and $K_\omega > 0$ such that

$$\|U(t,s)\| \leq K_\omega e^{\omega(t-s)} \text{ for all } t \geq s \geq 0. \tag{2}$$

If $\mathbf{T} = \{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ is a strongly continuous semigroup on X, then the family $\{U(t,s) : t \geq s \geq 0\}$ given by $U(t,s) = T(t-s)$ is a strongly continuous

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and exponentially bounded evolution family on X. Conversely, if $\mathcal{U}$ is a strongly continuous evolution family on X and $U(t,s) = U(t-s,0)$ for all $t \geq s \geq 0$ then the family $\mathbf{T} = \{T(t) : t \geq 0\}$ is a strongly continuous semigroup on X. For more details about the strongly continuous semigroups and other references we refer to [10], [7]. The Datko-Pazy theorem can be also obtained from the following result given by S. Rolewicz ([13], [14]).

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and nondecreasing function such that $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. If $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\} \subset \mathcal{L}(X)$ is a strongly continuous and exponentially bounded evolution family on the Banach space X such that

$$\sup_{s\geq 0} \int_s^\infty \varphi(\|U(t,s)x\|)\,dt = M_\varphi < \infty, \text{ for all } x \in X,\ \|x\| \leq 1, \tag{3}$$

then $\mathcal{U}$ is uniformly exponentially stable, that is (2) *holds with some $\omega < 0$.*

A shorter proof of the Rolewicz theorem was given by Q. Zheng [18] who removed the continuity assumption about φ. Other proofs (the semigroup case) of Rolewicz's theorem were offered by W. Littman [5] and J. van Neervan [8, pp. 81-82]. Some related results have been obtained by K.M. Przyłuski [12], G. Weiss [16] and J. Zabczyk [17]. In a very recent paper R. Schnaubelt gives a nice proof of a nonautonomous version of Datko Theorem using evolution semigroup (see [15, Theorem 5.4]). Also a very general Datko-Pazy type result in the autonomous case can be found in [9]. A family $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\} \subset \mathcal{L}(X)$ is called *strongly measurable* if for all $x \in X$ the maps given in (1) are measurable.

In this note we prove the following:

Theorem 1. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $\varphi(t) > 0$ for all $t > 0$ and $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\} \subset \mathcal{L}(X)$ be a strongly measurable and exponentially bounded evolution family of operators on the Banach space X. If $\mathcal{U}$ satisfies the conditions:*

- **(i)** *there exists $M_\varphi > 0$ such that*

$$\int_\xi^\infty \varphi(||U(t,\xi)x||)dt \leq M_\varphi < \infty, \quad \forall x \in X, ||x|| \leq 1, \forall \xi \geq 0, \tag{4}$$

- **(ii)** *for all $f \in L^1(\mathbb{R}_+, X)$ the maps*

$$t \mapsto U(\cdot, \cdot - t)f(\cdot) : \mathbb{R}_+ \to L^1(\mathbb{R}_+, X) \tag{5}$$

are measurable, then $\mathcal{U}$ is uniformly exponentially stable.

Firstly we prove the following Lemma which is essentially contained in [1, Theorem 2.1].

Lemma 1. *Let $\mathcal{U}$ be a strongly continuous and exponentially bounded evolution family of operators on X such that*

$$\sup_{s\geq 0} \int_s^\infty \varphi(\|U(t,s)x\|)\,dt = M_\varphi(x) < \infty, \text{ for all } x \in X \tag{6}$$

Then $\mathcal{U}$ is uniformly bounded, that is,

$$\sup_{t\geq\xi\geq 0} \|U(t,\xi)\| = C < \infty.$$

Proof of Lemma 1. Let $x \in X$ and $N(x)$ be a positive integer such that $M_\varphi(x) < N(x)$ and let $s \geq 0$, $t \geq s+N$. For each $\tau \in [t-N, t]$, we have

$$\begin{aligned} e^{-\omega N} 1_{[t-N,t]}(u) \|U(t,s)x\| &\leq e^{-\omega(t-\tau)} 1_{[t-N,t]}(u) \|U(t,\tau) U(\tau,s) x\| \\ &\leq K_\omega \|U(u,s)x\|, \end{aligned} \tag{7}$$

for all $u \geq s$. Here K_ω and ω are as in (2).

From (6) follows that $\varphi(0) = 0$. Then from (7) we obtain

$$\begin{aligned} N(x)\varphi\left(\frac{\|U(t,s)x\|}{K_\omega e^{\omega N}}\right) &= \int_s^\infty \varphi\left(\frac{1_{[t-N,t]}(u)\|U(t,s)x\|}{K_\omega e^{\omega N}}\right) du \\ &\leq \int_s^\infty \varphi(\|U(u,s)x\|)\,du = M_\varphi(x). \end{aligned} \tag{8}$$

We may assume that $\varphi(1) = 1$ (if not, we replace φ be some multiple of itself). Moreover, we may assume that φ is a strictly increasing map. Indeed if $\varphi(1) = 1$ and $a := \int_0^1 \varphi(t)\,dt$, then the function given by

$$\bar{\varphi}(t) = \begin{cases} \int_0^t \varphi(u)\,du, & \text{if} \quad 0 \leq t \leq 1 \\ \dfrac{at}{at+1-a}, & \text{if} \quad t > 1 \end{cases}$$

is strictly increasing and $\bar{\varphi} \leq \varphi$. Now φ can be replaced by some multiple of $\bar{\varphi}$. From (8) it follows that

$$\|U(t,s)\| \leq K_\omega e^{\omega N(x)}, \quad \text{for all } x \in X.$$

Now, it is easy to see that

$$\sup_{t \geq \xi \geq 0} \|U(t,\xi)x\| \leq 2K_\omega e^{\omega N(x)} := C(x) < \infty, \qquad \text{for all } x \in X. \tag{9}$$

The assertion of Lemma 1 follows from (9) and the Uniform Boundedness Theorem. □

It is clear that (3) follows by (6), but isn't clear if them are equivalent. In the proof of Theorem 1 we also use the following variant of Jensen inequality, see e.g. [11, Theorem 3.1].

Lemma 2. *Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex function and $w : \mathbb{R}_+ \to \mathbb{R}_+$ be a locally integrable function such that $0 < \int_0^\infty w(t)dt < \infty$. If $w\Phi \in L^1(\mathbb{R}_+)$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is such that the map $t \mapsto \Phi(f(t))$ belongs to $L^1(\mathbb{R}_+)$ then*

$$\Phi\Big(\frac{\int_0^\infty w(t)f(t)dt}{\int_0^\infty w(t)dt}\Big) \leq \frac{\int_0^\infty w(t)\Phi(f(t))dt}{\int_0^\infty w(t)dt}. \tag{10}$$

Proof of Theorem1. Let $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be a evolution family of bounded linear operators on X. We consider the evolution semigroup associated to $\mathcal{U}$ on $L^1(\mathbb{R}_+, X)$. This semigroup is defined by

$$(\mathfrak{T}(t)f)(s) := \begin{cases} U(s, s-t)f(s-t), & \text{if} \quad s \geq t \geq 0 \\ 0, & \text{if} \quad 0 \leq s \leq t \end{cases}, \tag{11}$$

for all $f \in L^1(\mathbb{R}_+, X)$. Firstly we will prove that $\mathfrak{T}(t)$ acts on $L^1(\mathbb{R}_+, X)$ for each $t \geq 0$. Indeed, if f_n are simple functions and f_n converges punctually almost everywhere to f on $\mathbb{R}_+$ when $n \to \infty$, then using the measurability of the functions given in (1) it follows that for all $n \in \mathbf{N}$ the maps $\mathfrak{T}(t)f_n$ are measurable. From (4) and Lemma 1 it follows that

$$||(\mathfrak{T}(t)f_n)(s) - (\mathfrak{T}f)(s)|| \leq K||f_n(s-t) - f(s-t)|| \to 0 \text{ as } n \to \infty$$

almost everywhere for $s \in [t, \infty)$, that is, the function in (11) is measurable.

On the other hand

$$\int_0^\infty ||(\mathfrak{T}f)(s)||ds = \int_t^\infty ||U(s, s-t)f(s-t)||dt \leq C||f||_{L^1(\mathbb{R}_+, X)} < \infty,$$

i.e. the function $\mathfrak{T}(t)f$ belongs to $L^1(\mathbb{R}_+, X)$. The functions defined in (5) are measurable for each $f \in L^1(\mathbb{R}_+, X)$, hence the maps

$$t \mapsto \mathfrak{T}(t)f : \mathbb{R}_+ \to L^1(\mathbb{R}_+, X) \tag{12}$$

are also measurable. We do not know at this stage if the measurability of the function in (12) can be obtained using only the measurability of functions in (1). We may suppose that $\varphi(1) = 1$ (if not, we replase φ by some multiple of itself). The function

$$t \mapsto \Phi(t) := \int_0^t \varphi(u)du : \mathbb{R}_+ \to \mathbb{R}_+$$

is convex, continuous on $(0, \infty)$ and strictly increasing. Moreover

$$\Phi(t) \leq \varphi(t) \quad \text{for all } t \in [0, 1].$$

Without loss of generality we may assume that

$$\sup_{t \geq 0} \|\mathfrak{T}(t)\| \leq 1.$$

Let $f \in C_c((0, \infty), X)$, the space of all continuous, X-valued functions defined on $\mathbb{R}_+$ with compact support in $(0, \infty)$, such that

$$||f||_\infty := \sup\{||f(t)|| : t \in (0, \infty)\} \leq 1.$$

Using Lemma 2 (in inequality (10) we replce $w(\cdot)$ by $\exp_{-1}$) and the Fubini Theorem it follows that

$$\begin{aligned}
&\int_0^\infty \Phi\left(\left\|\mathfrak{T}(t) \exp_{-1} \cdot f\right\|_{L^1(\mathbb{R}_+, X)}\right) dt \\
&= \int_0^\infty \Phi\left(\int_t^\infty e^{-(s-t)} \|U(s, s-t) f(s-t)\| ds\right) dt \\
&= \int_0^\infty \Phi\left(\int_0^\infty e^{-\xi} \|U(t+\xi, \xi) f(\xi)\| d\xi\right) dt \\
&\leq \int_0^\infty \left(\int_0^\infty e^{-\xi} \Phi(\|U(t+\xi, \xi) f(\xi)\|) dt\right) d\xi \\
&\leq \int_0^\infty \left(\int_0^\infty e^{-\xi} \varphi(\|U(t+\xi, \xi) f(\xi)\|) dt\right) d\xi \\
&\leq M_\varphi \int_0^\infty e^{-\xi} \|f(\xi)\| d\xi \\
&\leq M_\varphi < \infty.
\end{aligned}$$

Let $g \in L^1(\mathbb{R}_+, X)$ with $\|g\|_{L^1(\mathbb{R}_+, X)} \leq 1$ and $f_n \in C_c((0, \infty), X)$ such that $||f_n||_\infty \leq 1$ and $\exp_{-1} \cdot f_n \to g$ in $L^1(\mathbb{R}_+, X)$. Let (f_{n_k}) be a subsequence of (f_n) such that $exp_{-1} \cdot f_{n_k}$ converges at g, punctually almost everywhere for $t \in \mathbb{R}_+$, when

$k \to \infty$. Using the above estimates with f replaced by f_{n_k}, and the Dominated Convergence Theorem, it follows that

$$\int_0^\infty \Phi(\|\mathfrak{T}(t)g\|_{L^1(\mathbb{R}_+,X)})dt \le M_\varphi < \infty.$$

The assertion of Theorem 1, follows now, using a variant of Neerven's result (see the begining of our note). We recall that $\mathcal{U}$ is uniformly exponentially stable if and only if $\omega_0(\mathfrak{T})$ is negative [3, Theorem 2.2]. □

See also [2], [6] and the references therein for more details about the evolution semigroups on half line and their connections with asymptotic behaviour of evolution families of bounded linear operators acting on Banach spaces.

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